The Complexity of Hybrid Logics

Thomas Schneider

School of Computer Science, University of Manchester, UK

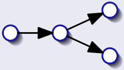
Part of this work has been done jointly with
Arne Meier, Martin Mundhenk, Michael Thomas, Volker Weber and Felix Weiß

18 June 2010

Overview

Starting point

expressive convenient well-behaved

Hybrid logics $\mathsf{E}{\downarrow}x.\Diamond{\downarrow}y.@_x\Diamond\neg y$ 

often undecidable ☹

Question

Question Under which restrictions can we get decidability back?

Answer Allow only certain classes of frames!
Restrict combinations of operators! ☺

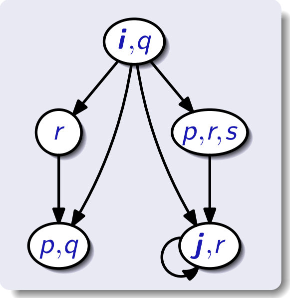
And now . . .

What is hybrid logic?

"Definition"

$$\text{Hybrid logic} = \underbrace{\text{prop. logic} + \Diamond\Box}_{\text{modal logic}} + \text{nominals} + @{\downarrow}\exists\forall\mathsf{E}\mathsf{A}\ldots$$

HL speaks about frames and models.

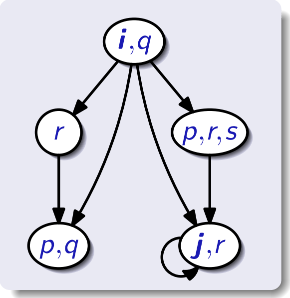


What is hybrid logic?

"Definition"

$$\text{Hybrid logic} = \underbrace{\text{prop. logic} + \Diamond\Box}_{\text{modal logic}} + \text{nominals} + @{\downarrow}\exists\forall\mathsf{E}\mathsf{A}\ldots$$

$\Diamond\varphi$	in *some* successor, φ
$\Box\varphi$	in *all* successors, φ

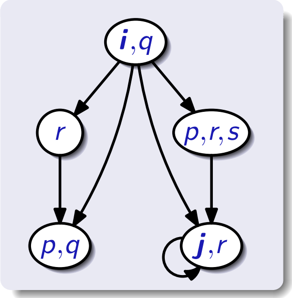


What is hybrid logic?

"Definition"

$$\text{Hybrid logic} = \underbrace{\text{prop. logic} + \Diamond\Box}_{\text{modal logic}} + \text{nominals} + @{\downarrow}\exists\forall\mathsf{E}\mathsf{A}\ldots$$

$\boldsymbol{i}$	name for a state
$@_i\varphi$	at state named $\boldsymbol{i}$, φ

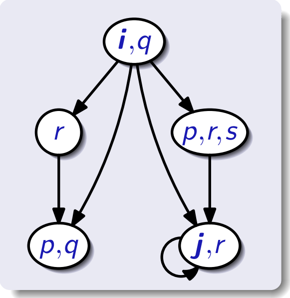


What is hybrid logic?

"Definition"

$$\text{Hybrid logic} = \underbrace{\text{prop. logic} + \Diamond\Box}_{\text{modal logic}} + \text{nominals} + @{\downarrow}\exists\forall\mathsf{E}\mathsf{A}\ldots$$

$\downarrow x.\varphi$	with x bound to *current* state, φ
$\exists x.\varphi$	with x bound to *some* state, φ
$\forall x.\varphi$	with x bound to *any* state, φ

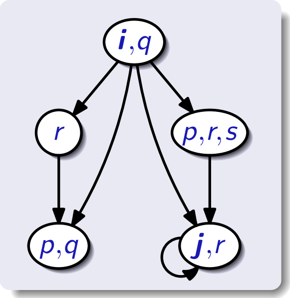


What is hybrid logic?

"Definition"

$$\text{Hybrid logic} = \underbrace{\text{prop. logic} + \Diamond\Box}_{\text{modal logic}} + \text{nominals} + @{\downarrow}\exists\forall\mathsf{E}\mathsf{A}\ldots$$

$\mathsf{E}\varphi$	in *some* state, φ
$\mathsf{A}\varphi$	in *all* states, φ

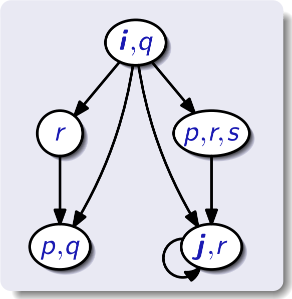


Hybrid temporal logic

"Definition"

Hybrid temporal logic = hybrid logic $- \Diamond\Box +$ FG PH US . . .

Future	$F = \Diamond$
Going to	$G = \Box$
Past	$P = \Diamond^{-1}$
Has been	$H = \Box^{-1}$



Hybrid temporal logic

"Definition"

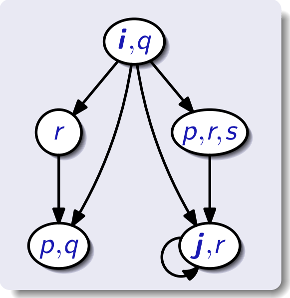
Hybrid temporal logic = hybrid logic − $\Diamond\Box$ + FG PH US . . .

Until

$\varphi \mathsf{U} \psi$ in *some* successor, ψ,
and from here until there, φ

Since

$\mathsf{S} = \mathsf{U}^{-1}$



A bit of notation

- Relevant operators: $\mathsf{F}\ \mathsf{P}\ \mathsf{U}\ \mathsf{S}\ @\ \downarrow\ \exists\ \mathsf{E}$
  (These are just duals: $\mathsf{G}\ \mathsf{H}$ $\forall\ \mathsf{A}$)
- Consider languages containing F
  and arbitrary combinations of $\mathsf{P}\ \mathsf{U}\ \mathsf{S}\ @\ \downarrow\ \exists\ \mathsf{E}$

A bit of notation

- Relevant operators: F P U S @ $\downarrow$ $\exists$ E
  (These are just duals: G H $\forall$ A)

- Consider languages containing F
  and arbitrary combinations of P U S @ $\downarrow$ $\exists$ E

- Write languages as follows

$\mathcal{ML}(\mathsf{F})$	basic modal language K
$\mathcal{HL}(\mathsf{F}, @)$	basic hybrid language
$\mathcal{HL}(\mathsf{F}, \downarrow, \mathsf{E})$	a very expressive hybrid language

Decision problems

Satisfiability problem $\mathcal{HL}(\cdot)$-SAT

Input $\varphi \in \mathcal{HL}(\cdot)$

Question Are there $\mathcal{M}, g, m \in \mathcal{M}$ such that $\mathcal{M}, g, m \models \varphi$?

Model-checking problem $\mathcal{HL}(\cdot)$-MC

Input $\varphi \in \mathcal{HL}(\cdot)$, $\mathcal{M}$, g

Question Is there $m \in \mathcal{M}$ such that $\mathcal{M}, g, m \models \varphi$?

We will focus on SAT here.

Complexity classes

Complexity classes

Name	Meaning	Examples
L	logarithmic space	graph accessibility
P	polynomial time	(model checking)
NP	nondeterministic pol. time	prop. logic SAT
PSPACE	polynomial space	modal logic SAT
EXP	exponential time	
NEXP	nondeterministic exp. time	
N2EXP	nondeterm. 2$\times$ exp. time	HL SAT
n.d.	nonelementarily decidable	
coRE	undecidable	FOL SAT

"Traditional" complexity results for SAT

Language	Completeness	Source
$\mathcal{ML}(\mathsf{F})$	**PSPACE**	Ladner 77
$\mathcal{ML}(\mathsf{F}, \mathsf{P})$	**PSPACE**	Spaan 93
$\mathcal{HL}(\mathsf{F}, @)$	**PSPACE**	Areces et al. 99
$\mathcal{HL}(\mathsf{F}, \mathsf{P})$	**EXP**	Areces et al. 99
$\mathcal{HL}(\mathsf{F}, \mathsf{P}, @)$	**EXP**	Areces et al. 99
$\mathcal{HL}(\mathsf{U}, \mathsf{S}, \mathsf{E})$	**EXP**	Areces et al. 99
$\mathcal{HL}(\mathsf{F}, \downarrow)$	**coRE**	Blackburn et al. 95, Goranko 96, Areces et al. 99

"Traditional" complexity results for SAT

Language	Completeness	Source
$\mathcal{ML}(\mathsf{F})$	**PSPACE**	Ladner 77
$\mathcal{ML}(\mathsf{F}, \mathsf{P})$	**PSPACE**	Spaan 93
$\mathcal{HL}(\mathsf{F}, @)$	**PSPACE**	Areces et al. 99
$\mathcal{HL}(\mathsf{F}, \mathsf{P})$	**EXP**	Areces et al. 99
$\mathcal{HL}(\mathsf{F}, \mathsf{P}, @)$	**EXP**	Areces et al. 99
$\mathcal{HL}(\mathsf{U}, \mathsf{S}, \mathsf{E})$	**EXP**	Areces et al. 99
$\mathcal{HL}(\mathsf{F}, \downarrow)$	**coRE**	Blackburn et al. 95, Goranko 96, Areces et al. 99

How can we tame $\downarrow$?

And now . . .

1. Hybrid Logic
2. $\mathcal{HL}$ over restricted frame classes
3. $\mathcal{HL}$ with restricted Boolean operators
4. Outlook

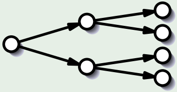
Restricted frame classes

Applications often require frames with certain properties.

Example: temporal logic

States	$\hat{=}$	points in time
mRm'	$\hat{=}$	"m' is in the future of m"
$\Diamond\varphi$ ($F\varphi$)	$\hat{=}$	"at some time in the future, φ"
$\Box\varphi$ ($G\varphi$)	$\hat{=}$	"always in the future, φ"

Relevant classes of frames:

- linear orders 
- transitive trees 

Restricted frame classes

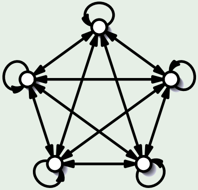
Applications often require frames with certain properties.

Example: epistemic logic

States	$\widehat{=}$	possible worlds of an agent
mRm'	$\widehat{=}$	"being in world m, the agent thinks m' possible"
$\Diamond\varphi$ ($\hat{K}\varphi$)	$\widehat{=}$	"the agent considers φ possible"
$\Box\varphi$ ($K\varphi$)	$\widehat{=}$	"the agent knows that φ"

Relevant classes of frames:

- frames with equivalence relations
- superclasses thereof,
  e.g., transitive frames



SAT over restricted frames

Satisfiability problem $\mathcal{HL}(\cdot)$-$\mathfrak{F}$-SAT

Input	$\varphi \in \mathcal{HL}(\cdot)$
Question	Are there $\mathcal{M} \in \mathfrak{F}$, g, $m \in \mathcal{M}$ with $\mathcal{M}, g, m \models \varphi$?

SAT over restricted frames

Satisfiability problem $\mathcal{HL}(\cdot)$-$\mathfrak{F}$-SAT

Input $\varphi \in \mathcal{HL}(\cdot)$

Question Are there $\mathcal{M} \in \mathfrak{F}$, g, $m \in \mathcal{M}$ with $\mathcal{M}, g, m \models \varphi$?

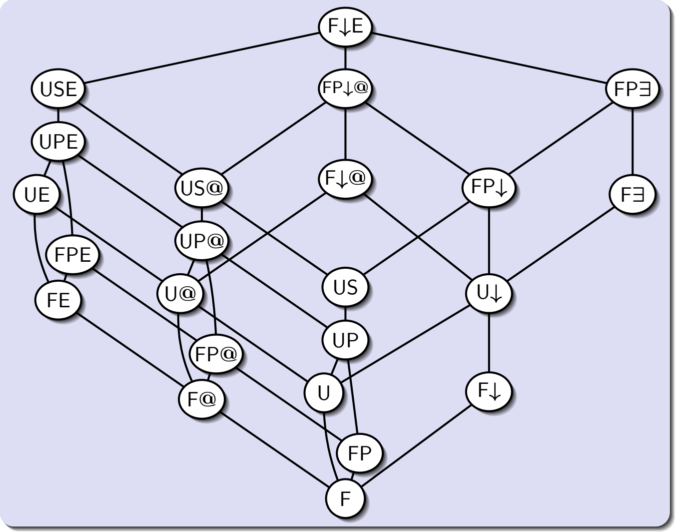
Language	Frame class	Completeness	Source
$\mathcal{ML}(\mathsf{F})$	equiv	**NP**	Ladner 77
$\mathcal{ML}(\mathsf{F}, \mathsf{P})$	lin	**NP**	Ono, Nakamura 80
$\mathcal{HL}(\mathsf{F}, \mathsf{P}, \mathsf{E})$	lin	**NP**	Areces et al. 00
$\mathcal{HL}(\mathsf{U}, \mathsf{S}, \mathsf{E})$	$(\mathbb{N}, <)$	**PSPACE**	Areces et al. 00
$\mathcal{HL}(\mathsf{F}, \downarrow, \mathsf{E})$	lin	**n.d.**	Franceschet et al. 03
$\mathcal{HL}(\mathsf{F}, \downarrow)$	trans, equiv	**NEXP**	Mundhenk et al. 05

A more systematic approach

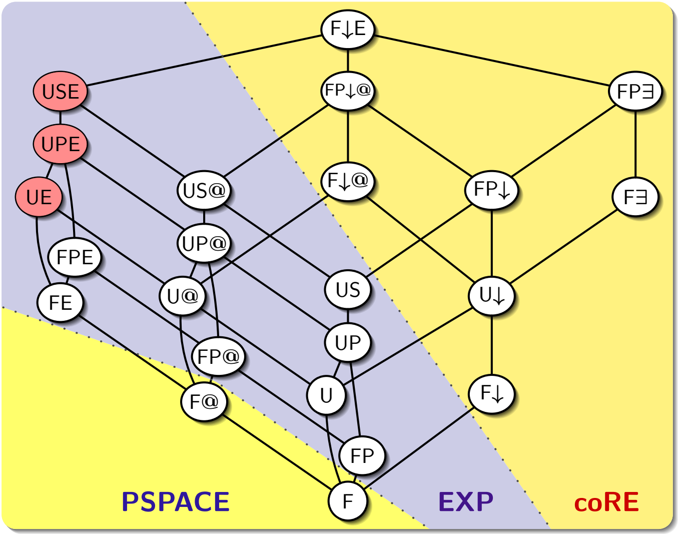
Examine complexity of SAT for all hybrid languages with F and arbitrary combinations of P U S @ $\downarrow$ $\exists$ E over

- all frames
- transitive frames
- transitive trees
- linear orders
- $(\mathbb{N}, <)$
- frames with equivalence relations

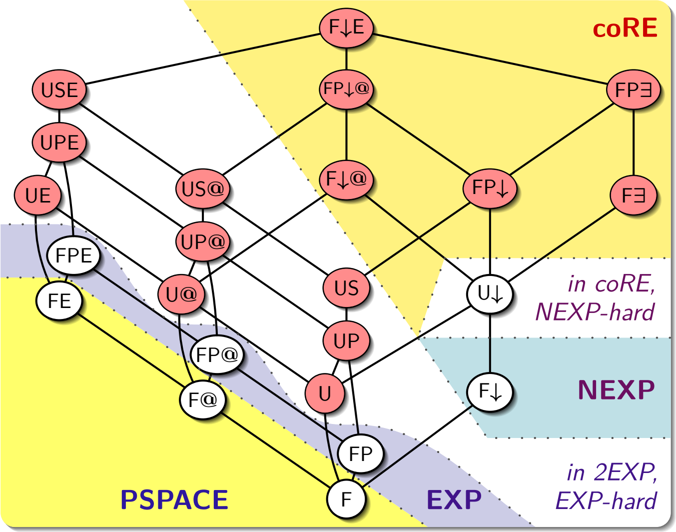
The lattice of languages



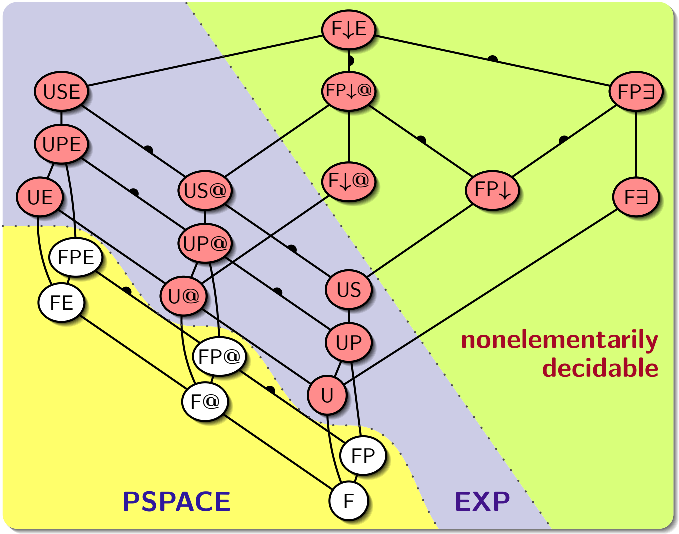
Complexity results over arbitrary frames



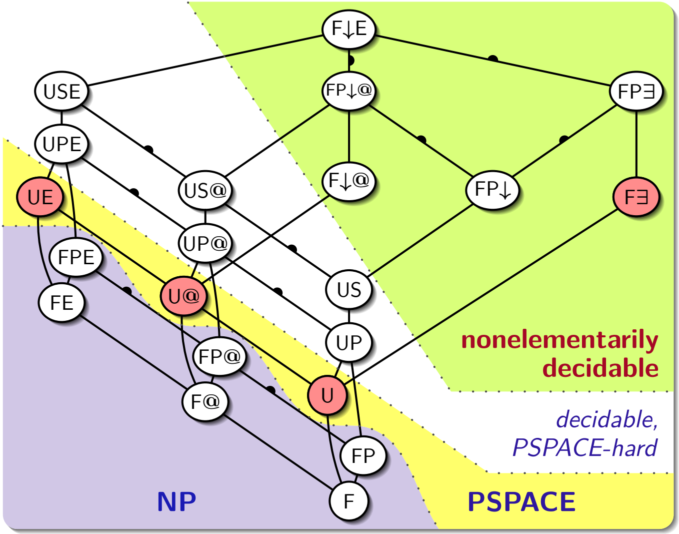
Complexity results over transitive frames



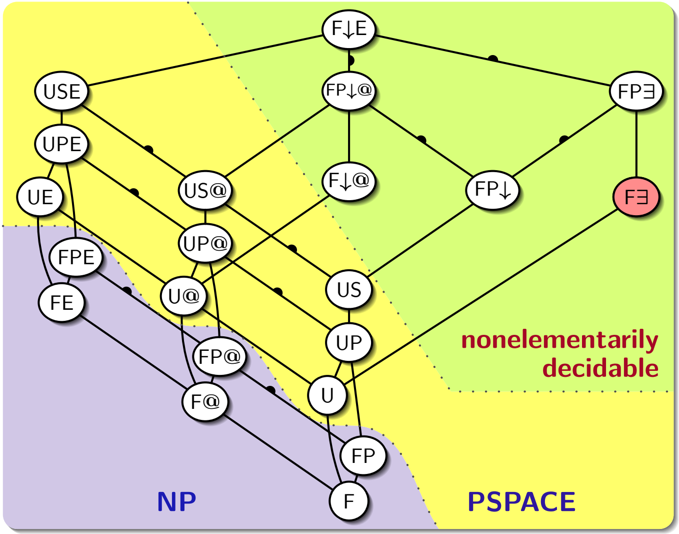
Complexity results over transitive trees



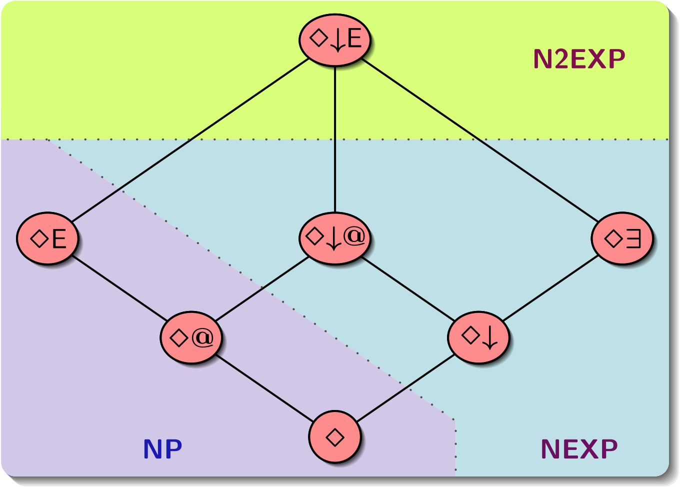
Complexity results over linear orders



Complexity results over $(\mathbb{N}, <)$



Complexity results over equivalence relations



Complexity results for multi-modal languages

For these six frame classes,
$\mathcal{HL}(\Diamond_1, \Diamond_2, \downarrow)$-$\mathfrak{F}$-SAT is already **coRE**-complete.

And now . . .

1. Hybrid Logic
2. $\mathcal{HL}$ over restricted frame classes
3. $\mathcal{HL}$ with restricted Boolean operators
4. Outlook

Propositional fragments of $\mathcal{HL}$

Restrict the set of *propositional* operators!

- Why?

  Propositional SAT becomes tractable, e.g., without negation. (Lewis '79)

  SAT for $\mathcal{ML}$ or LTL becomes tractable for certain restrictions. (Bauland et al. '06/07)

  SAT for many sub-Boolean description logics is tractable. (Baader et al. '98/05/08, Calvanese et al. '05–07)

- 3 parameters:

$$\left.\begin{array}{l}\text{frame class F}\\ \text{set } O \text{ of modal/hybrid operators}\\ \text{set } B \text{ of Boolean operators}\end{array}\right\} \quad \rightsquigarrow \quad \mathcal{HL}(O, B)\text{-}\mathfrak{F}\text{-SAT}$$

New goal

Classify $\mathcal{HL}(O, B)$-$\mathfrak{F}$-SAT for decidability and complexity w.r.t.

- *all* B
- O with $\{\Diamond, \downarrow\} \subseteq O \subseteq \{\Diamond, \Box, \downarrow, @\}$
- $\mathsf{F} \in \{\text{all}, \text{trans}, \text{equiv}, \text{serial}\}$

- Find border between decidable and undecidable fragments
- Find tight complexity bounds

Complexity results over arbitrary frames

$\mathcal{HL}(O, B)$-all-SAT is ...

coRE-compl.	if	B can express $x \wedge \neg y$ or all self-dual functions
coNP-hard	if	B contains $\wedge$ and $\Box \in O$
in **L**	if	B can express only $\wedge, \vee, \top, \bot$ and $\Box \notin O$ or B can express only $\vee, \top, \bot$ or only $\neg, \top, \bot$
trivial	in almost all other cases	

Complexity results over arbitrary frames

$\mathcal{HL}(O, B)$-all-SAT is . . .

coRE-compl.	if	B can express $x \wedge \neg y$ or all self-dual functions
coNP-hard	if	B contains $\wedge$ and $\Box \in O$
in **L**	if	B can express only $\wedge, \vee, \top, \bot$ and $\Box \notin O$ or B can express only $\vee, \top, \bot$ or only $\neg, \top, \bot$
trivial	in almost all other cases	

Almost the same classification for $\mathcal{HL}(O, B)$-trans-SAT

Complexity results over serial frames

$\mathcal{HL}(O, B)$-serial-SAT is . . .

coRE-compl.	if	B can express $x \wedge \neg y$ or all self-dual functions
in **L**	if	B can express only monotone functions or B can express only $\neg, \top, \bot$
trivial	in almost all other cases	

Complexity results over equivalence relations

$\mathcal{HL}(O, B)$-equiv-SAT is ...

NEXP-compl.	if	B contains $x \wedge \neg y$ or B all self-dual functions
in **L**	if	B can express only monotone functions or B can express only $\neg, \top, \bot$
trivial	in almost all other cases	

And now . . .

1. Hybrid Logic
2. $\mathcal{HL}$ over restricted frame classes
3. $\mathcal{HL}$ with restricted Boolean operators
4. Outlook

Modularity of specifications

Specification $\varphi_1 \wedge \varphi_2$ *refines* φ_1 if:
for every ψ that uses only symbols from φ_1:
if $\varphi_1 \wedge \varphi_2 \models \psi$, then $\varphi_1 \models \psi$.

$\rightsquigarrow$ If we're only interested in the part of a theory that speaks about a certain subsignature, we can "forget" unnecessary conjuncts.

Modularity of specifications

Specification $\varphi_1 \wedge \varphi_2$ *refines* φ_1 if:
for every ψ that uses only symbols from φ_1:
if $\varphi_1 \wedge \varphi_2 \models \psi$, then $\varphi_1 \models \psi$.

$\rightsquigarrow$ If we're only interested in the part of a theory that speaks about a certain subsignature, we can "forget" unnecessary conjuncts.

We also say $\varphi_1 \wedge \varphi_2$ is a *conservative extension* of φ_1.

Deciding and approximating conservativity

- Deciding conservativity is
  - at least as hard as satisfiability [Ghilardi at al. 06]
  - **coNEXP**-complete for $\mathcal{ML}(\Diamond)$ [Ghilardi at al. 06]
  - undecidable for description logics (DLs) with nominals [Lutz et al. 07]
- Sufficient conditions for conservativity in expressive DLs exist

  $\rightsquigarrow$ efficient module extraction algorithms

Deciding and approximating conservativity

- Deciding conservativity is
  - at least as hard as satisfiability [Ghilardi at al. 06]
  - **coNEXP**-complete for $\mathcal{ML}(\Diamond)$ [Ghilardi at al. 06]
  - undecidable for description logics (DLs) with nominals [Lutz et al. 07]
- Sufficient conditions for conservativity in expressive DLs exist
  $\rightsquigarrow$ efficient module extraction algorithms

Carry over insights to hybrid logics:

- Devise module notions for HL similar to locality
- Find efficient algorithms for refinement test, module extraction

Deciding and approximating conservativity

- Deciding conservativity is
  - at least as hard as satisfiability [Ghilardi at al. 06]
  - **coNEXP**-complete for $\mathcal{ML}(\Diamond)$ [Ghilardi at al. 06]
  - undecidable for description logics (DLs) with nominals [Lutz et al. 07]
- Sufficient conditions for conservativity in expressive DLs exist
  $\rightsquigarrow$ efficient module extraction algorithms

Carry over insights to hybrid logics:

- Devise module notions for HL similar to locality
- Find efficient algorithms for refinement test, module extraction

Thank you.