The Complexity of Monotone Hybrid Logics over Linear Frames and the Natural Numbers

Stefan Göller[1] Arne Meier[2] Martin Mundhenk[3]
Thomas Schneider[1] Michael Thomas[4] Felix Weiß[3]

[1]Dept. of Computer Science, University of Bremen, Germany

[2]Inst. of Theoretical Comp. Science, University of Hannover, Germany

[3]Inst. of Computer Science, University of Jena, Germany

[4]TWT GmbH, Germany

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And now . . .

Hybrid logic

... has already been introduced today

We're looking at the extension of standard modal logic with

- **nominals** $i, j, \ldots$

  name single states in models

- the **binder** $\downarrow$

  $\downarrow x.\varphi$ binds variable x dynamically to the current state;
  x in φ is treated as a nominal

- the **satisfaction operator** $@_x$

  jumps to the state named by (the nominal or variable) x

The satisfiability problem for $\mathcal{HL}$

Definition

1. A formula φ is **satisfiable** if there is

   a model $\mathcal{M} = (W, R, V)$ based on a frame $\mathcal{F} = (W, R)$
   an assignment $g : \mathrm{SVAR} \to W$
   and a state $s \in W$

   such that $\mathcal{M}, g, s \models \varphi$

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Let $O \subseteq \{\Diamond \Box \downarrow @\}$.

2. $\mathcal{HL}(O)$ = set of all $\mathcal{HL}$-formulas with operators from O
3. $\mathrm{SAT}(O) = \{\varphi \in \mathcal{HL}(O) \mid \varphi \text{ is satisfiable}\}$

Complexity of satisfiability for $\mathcal{HL}$

Theorem

$\mathrm{SAT}(\Diamond\Box)$	is PSPACE-complete.	(Ladner '77)
$\mathrm{SAT}(\Diamond\Box@)$	is PSPACE-complete.	(Areces et al. '99)
$\mathrm{SAT}(\Diamond\Box\downarrow)$	is coRE-complete.	(Areces et al. '99)

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$\rightsquigarrow$

Tame $\downarrow$?

$\mathcal{HL}$ over restricted frame classes

$\mathfrak{F}$	condition on frames $(W, R) \in \mathfrak{F}$
trans	R is transitive
equiv	R is an equivalence relation
lin	R is a linear order (transitive, irreflexive, $\forall xy(xRy$ or $x = y$ or $yRx)$
$\mathbb{N}$	$(W, R) = (\mathbb{N}, <)$
$\vdots$	

Definition

$\mathfrak{F}$-SAT(O) =

$\{\varphi \in \mathcal{HL}(O) \mid \varphi$ is sat. in a model *based on a frame from* $\mathfrak{F}\}$

$\mathcal{HL}$ satisfiability over restricted frame classes

Theorem

trans-SAT($\Diamond\Box\downarrow$)	is NExpTime-complete.	(Mundhenk et al.
equiv-SAT($\Diamond\Box\downarrow$)	is NExpTime-complete.	" '05)
trans-SAT($\Diamond\Box\downarrow@$)	is coRE-complete.	"
lin-SAT($\Diamond\Box\downarrow$)	is NP-complete.	(Areces et al. '00)
$\mathbb{N}$-SAT($\Diamond\Box\downarrow$)	is NP-complete.	"
lin-SAT($\Diamond\Box\downarrow@$)	is nonelementary.	(Franceschet et al.
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$\rightsquigarrow$

Tame $\downarrow$ further?

Propositional fragments of $\mathcal{HL}$

$\rightsquigarrow$ **Restrict the set of *propositional* operators!** Why?

- **Propositional SAT** is tractable if $\not\to$ is disallowed (Lewis '79)
- **LTL-SAT** is tractable if $\not\to$ is disallowed (Bauland et al. '07)
- **SAT for $\mathcal{ML}(\Diamond\Box)$** is tractable if $\not\to$ and $\wedge$ are disallowed (Bauland et al. '06)
- **for $\mathcal{HL}$: all-SAT($\Diamond\downarrow$@)** is tractable if $\not\to$ and some self-dual operators are disallowed (Meier et al. '09)
- **SAT for certain sub-Boolean description logics** is tractable (Baader et al. '98/05/08, Calvanese et al. '05–07)

Goal

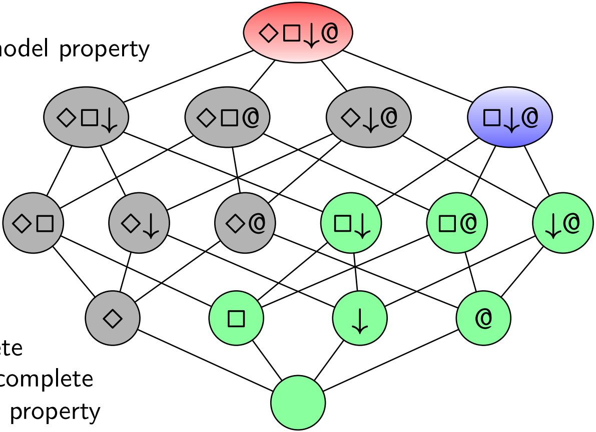
- **Consider SAT for $\mathcal{HL}$**
  - with modal/hybrid operators $O \subseteq \{\Diamond\Box{\downarrow}@\}$
  - with only monotone Boolean operators $\wedge\vee\bot\top$
  - over linear frames and $\mathbb{N}$
- **Notation:** $\mathcal{MHL}(O)$, lin-MSAT(O), $\mathbb{N}$-MSAT(O)
- **Why?**
  - $\mathcal{HL}$ over linear frames and $\mathbb{N}$ is an extension of LTL
  - Negation-freeness leads to lower complexity in other logics
- **Observation**

  with monotone operators, we can forgo propositional variables (replace them with $\top$)

And now . . .

Overview

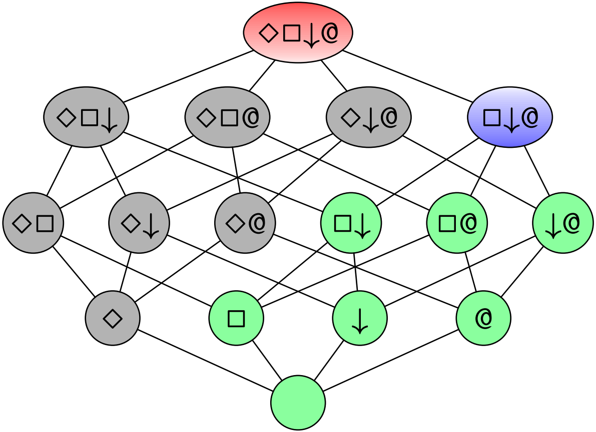
- lin: decidable, non-elementary
  $\mathbb{N}$: PSPACE-complete
- NP-complete
  quasi-polysize model property



- lin: NC^1-complete
  $\mathbb{N}$: LOGSPACE-complete
  canonical model property
- NC^1-complete
  canonical model property

The hard cases

- lin: decidable, non-elementary
  $\mathbb{N}$: PSPACE-complete



A nonelementary lower bound

Theorem

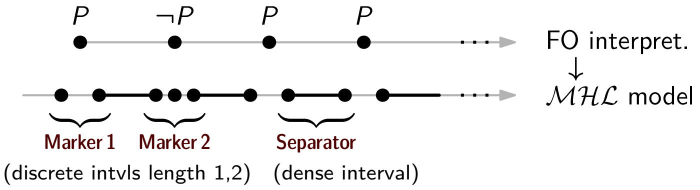
lin-MSAT($\Diamond\Box{\downarrow}@$) is decidable and nonelementary.

Proof sketch.

- Decidability from lin-SAT($\Diamond\Box{\downarrow}@$) (Franceschet et al. '03)
- Reduce from $\mathcal{FOL}$-SAT over $\mathbb{N}$ with predicates (Stockmeyer'74)
  - $<$ (natural "less-than" on $\mathbb{N}$)
  - P (one arbitrary unary predicate)
- Encode
  - $\mathcal{FOL}(P,<)$-interpretations over $\mathbb{N}$, using no propos. variables
  - formulas from $\mathcal{FOL}(P,<)$ as monotone formulas

Details of the encoding

- Encode FO interpretations as sequences of intervals:



Use $\mathcal{MHL}(\Diamond\Box{\downarrow}@)$ to enforce this structure in a hybrid model

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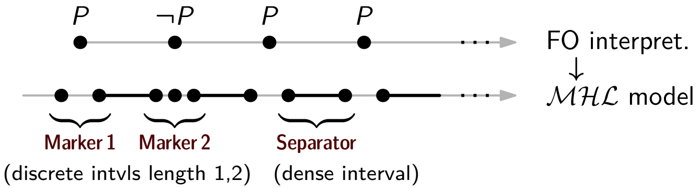


Use $\mathcal{MHL}(\Diamond\Box{\downarrow}@)$ to enforce this structure in a hybrid model

- Encoding of formulas (example):
  - $\forall x\big(Px \rightarrow \exists y(x < y \wedge \neg Py)\big)$ becomes

    $\Box_{\mathrm{m}}{\downarrow}x.\big(1(x) \rightarrow \Diamond_{\mathrm{m}}{\downarrow}y.2(y)\big)$; without implication:

    $\Box_{\mathrm{m}}{\downarrow}x.\big(2(x) \vee \Diamond_{\mathrm{m}}{\downarrow}y.2(y)\big)$
  - $\Diamond_{\mathrm{m}}\psi =$ "in some future state that starts a marker, ψ holds"

    $\Box_{\mathrm{m}}\psi =$ "all future states start no marker or satisfy ψ"

A PSPACE upper and lower bound

- Over $\mathbb{N}$, we can no longer use dense-discrete alternation to encode unary predicates.
- SAT for $\mathcal{FOL}(<)$ over $\mathbb{N}$ is PSPACE-complete (Ferrante, Rackoff '79)

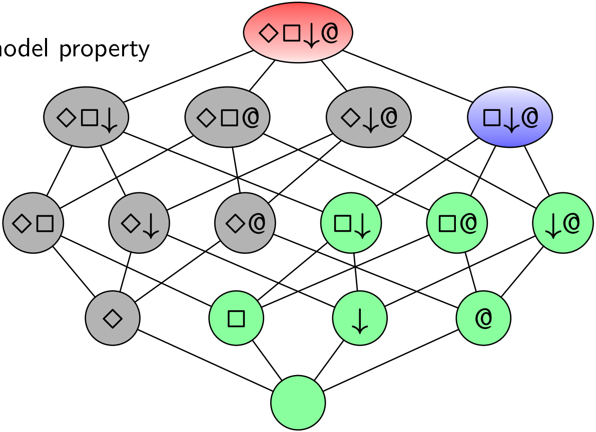
Theorem

$\mathbb{N}$-MSAT($\Diamond\Box\downarrow@$) is PSPACE-complete.

- Hardness via straightforward encoding of QBF-SAT
- Membership via reduction to SAT for $\mathcal{FOL}(<)$ over $\mathbb{N}$

The intermediate cases

- NP-complete
  quasi-polysize model property



NP-completeness

Theorem

$\Diamond \in O \subsetneq \{\Diamond\Box\downarrow@\} \quad \Rightarrow \quad$ lin- and $\mathbb{N}$-MSAT(O) are NP-complete.

- **Lower bound:** straightforward reduction from 3-SAT

  uses nominals: one per variable; 2 for "true" and "false"

- **Upper bound:**
  - lin- and $\mathbb{N}$-MSAT($\Diamond\Box@$): in NP (Areces et al. '00)
  - lin- and $\mathbb{N}$-MSAT($\Diamond\Box\downarrow$): obvious reduction to $\mathbb{N}$-MSAT($\Diamond\Box$)
  - lin- and $\mathbb{N}$-MSAT($\Diamond\downarrow@$):

    without $\Box$, $\downarrow$ binds state variables "existentially"

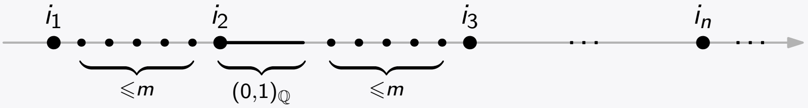
    $\rightsquigarrow$ replace with fresh nominals

    $\rightsquigarrow$ straightforward reduction to $\mathbb{N}$-MSAT($\Diamond@$)

A quasi-polysize model property (QPMP)

Theorem

Every $\varphi \in$ lin-MSAT($\Diamond\Box@$) of modal depth m has a model which,



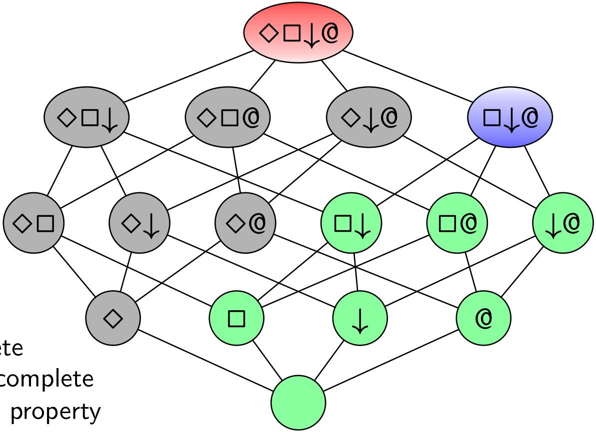
between two successive nominal states, has $\leqslant m$ further states, possibly preceded by one copy of the dense interval $(0,1)_{\mathbb{Q}}$.

Proof idea: States with distance $> m$ from nominal states satisfy the same modal formulas of modal depth $\leqslant m$

Gain:

- Such structures can be represented polynomially
- With little extra effort, QPMP yields NP upper bounds for SAT over lin, $\mathbb{N}$, $\mathbb{Q}$

The easy cases



- lin: NC^1-complete
  $\mathbb{N}$: LogSpace-complete
  canonical model property
- NC^1-complete
  canonical model property

A canonical model property

Theorem

(1) Every $\varphi \in$ lin-MSAT$(\Box\downarrow@)$ is satisfiable

- in a one-state structure
- under an assignment g that maps all SVARs to the only state.

(2) Every $\varphi \in \mathbb{N}$-MSAT$(\Box\downarrow@)$ is satisfiable

- in $(\mathbb{N}, <)$
- under an assignment g that maps all SVARs to 0.

Main observation:
without $\Diamond$, we cannot control the order of two states

Consequence:
With (1), we can reduce lin-MSAT$(\Box\downarrow@)$ to propositional MSAT
$\rightsquigarrow$ NC^1-completeness (Schnoor '07)

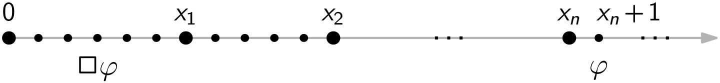
A LogSpace result over $\mathbb{N}$

Theorem

$\mathbb{N}$-MSAT($\Box\downarrow$@) is LogSpace-complete.

Proof sketch.

- Lower bound: reduction from "Order between vertices"
- Upper bound:
  - Despite $\Box$, every subformula has a **unique assignment and state of evaluation (UASE)**



  - Use UASEs to replace all SVARs with 0 or 1; relevant information can be computed on-the-fly in LogSpace
  - Evaluate remaining propositional formula (in NC^1)

And now . . .

Result overview

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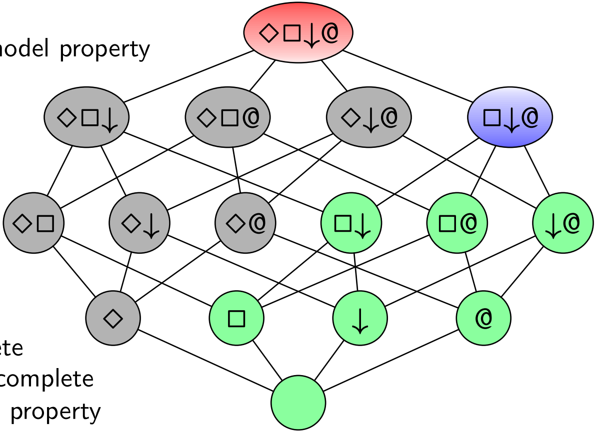

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Summary

We have established ...

- the **computational complexity** of SAT for all fragments of $\mathcal{HL}$
  - with monotone Boolean operators $\wedge \vee \bot \top$
  - with modal/hybrid operators $O \subseteq \{\Diamond \Box \downarrow @\}$
  - over linear frames and $\mathbb{N}$
- **small-model properties** for all intermediate and easy cases
- made an interesting **observation:**
  - Fragment $(\Diamond \Box \downarrow @)$ is **harder** over lin than over $\mathbb{N}$
  - Fragment $(\Box \downarrow @)$ is **easier** over lin than over $\mathbb{N}$
  - All other fragments have the **same** complexity over lin and $\mathbb{N}$

Outlook

- Does $\mathcal{MHL}(\Diamond\Box{\downarrow}@)$ have a small-model property over $\mathbb{N}$?
- Which of our results can be carried over to **strictly dense frame classes**, e.g., $(\mathbb{Q},<)$?
- Complexity of $\mathcal{HL}$-fragments with **other combinations of Boolean operators** over acyclic frame classes?

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Thank you.