Semantik-basierte Autorenwerkzeuge für mathematische Dokumente

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Bonn, Germany
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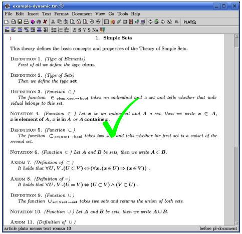
Sichere Kognitive Systeme

Motivation

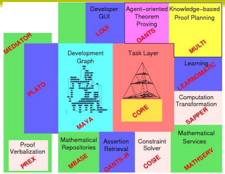
- Support “Workflow” of preparing documents with math. content
- . . . for mathematicians/scientists/software engineer for developing theories and especially proofs
- Why?
  - Verifiable formalisations and proofs
  - Added values through machine support
    - Take over routine checks/tasks in proofs
    - Organisation of large complex proofs (z.B. Kepler’ conjecture proof by T. Hales)
    - Semantic-based search for math. concepts
    - . . .



Approaches

Bottom-up Approach:

- Logic, calculus, and components build upon (proof assistants)
- Hope that eventually targeted users will use it
- Classical approach (a.o. Ω*mega*)



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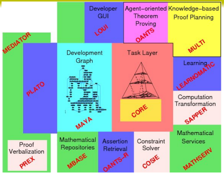

Top-down Approach:

- Start with systems that users already use (Texteditors)
- Add functionality, e.g. those provide by proof assistants and more
- Analogy: Grammar-checker, but interactive
- Working hypotheses:
  - Authors of documents should not have to learn peculiarities of proof assistance systems in order to get their support
  - Support system should adapt to user, not vice versa
  - Author always has full control over the text (layout, formulation)

The Vision

Planned Functionality

Current State

Spectrum between Text, Notations and Formal Representations

Writing the Lecture Notes

Introduction to Algebra
Thomas H.

1 Logic
2 Classes and Sets
3 Functions
4 Relations and Partitions

Introduction to Algebra
Thomas H.

1 Logic
We adopt the logical foundations developed in the lecture notes [SS08, David H.] and agree on the following notational conventions:

[SS08, David H.]	This course
$\neg P$	"not P"
$P \wedge Q$	"P and Q"
$P \vee Q$	"P or Q"
$P \supset Q$	"P implies Q", "$P \Rightarrow Q$"
$P \equiv Q$	"$P \Leftrightarrow Q$"
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2 Classes and Sets
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2 Classes and Sets
3 Functions
4 Relations and Partitions

In Gödel-Bernays form of axiomatic set theory, which we shall follow, the primitive (undefined) notions are **class, membership**, and **equality**. Intuitively, we consider a class to be a collection A of objects (elements) such that given any object x if it is possible to determine whether or not x is a member (or element) of A. We write $x \varepsilon A$ for "x is an element of A" and $x \not\varepsilon A$ for "x is not an element of A".

[. . .]

The **axiom of extensionality** asserts that two classes with the same elements are equal (formally, $[x \varepsilon A \Leftrightarrow x \varepsilon B] \supset A = B$).

A class A is defined to be a **set** if and only if there exists a class B such that $A \varepsilon B$. Thus a set is a particular kind of class. A class that is not a set is called a **proper class**. Intuitively the distinction between sets and proper classes is not too clear. Roughly speaking a set is a "small" class and a proper class is exceptionnally "large". The **axiom of class formation** asserts that for any statement $P(y)$ in the first-order predicate calculus involving a variable y, there exists a class A such that $x \varepsilon A$ if and only if x is a set and the statement $P(x)$ is true. We denote this class A by $\{x \mid P(x)\}$.

[. . .]

A class A is a **subclass** of a class B (written $A \subseteq B$) provided:

$$\text{for all } x \varepsilon A, x \varepsilon A \supset x \varepsilon B.$$

By the axioms of extensionality and the properties of equality

$$A = B \Leftrightarrow A \subseteq B \text{ and } B \subseteq A$$

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3 Functions

4 Relations and Partitions

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1 Logic

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3 Functions

Given classes A and B, a **function** (or **map** or **mapping**) f from A to B (written $f: A \to B$) assigns to each $a \in A$ exactly one element $b \in B$; b is called the value of the function at a or the image of a and is usually written $f(a)$. A is the **domain** of the function (sometimes written Dom_f) and B is the **range** or **codomain**. Sometimes it is convenient to denote the effect of the function f on an element of A by $a \mapsto f(a)$. Two functions are **equal** if they have the same domain and range and have the same value for each element of their common domain.

[. . .]

4 Relations and Partitions

The **axiom of pair formation** states that for any two sets [elements] a, b there is a set $P = \{a, b\}$ such that $x \in P$ if and only if $x = a$ or $x = b$; if $a = b$ then P is the **singleton** $\{a\}$. The **ordered pair** (a, b) is defined to be the set $\{\{a\}, \{a, b\}\}$; its **first component** is a and its **second component** is b. It is easy to verify that $(a, b) = (a', b')$ if and only if $a = a'$ and $b = b'$. The **Cartesian product** of classes A and B is the class

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

A subclass R of $A \times B$ is called a **relation** on $A \times B$. For example, if $f: A \to B$ is a function, the **graph** of f is the relation $R = \{(a, f(a)) \mid a \in A\}$.

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. . . Verification Details on Demand

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By the axioms of extensionality and the properties of equality*Details*

$$A = B \Leftrightarrow A \subset B \text{ and } B \subset A$$

*Details*We first prove $A = B \Rightarrow A \subset B$ and $B \subset A$: Assume (h) $A = B$, then we have to prove (1) $A \subset B$ and (2) $B \subset A$: For (1), assuming $x \in A$, we conclude $x \in B$ from (h) and properties of equality. For (2), assuming $x \in B$, we conclude $x \in A$ from (h) and properties of equality. Conversely, we prove $A \subset B$ and $B \subset A \Rightarrow A = B$: By Definition of $\subset$ we know from $A \subset B$ and $B \subset A$ that $x \in A \Rightarrow x \in B$ and $x \in B \Rightarrow x \in A$ for all x. Hence, $x \in A \Leftrightarrow x \in B$ for all x and by extensionality follows $A = B$. □

3 Relations

4 Functions

The Vision

Planned Functionality

Current State

Spectrum between Text, Notations and Formal Representations

Planned Functionality

General

- Are all used concepts defined? Uniquely? Unambiguously?
- Have all introduced notations been followed?
- Management/Maintenance of notational information
- Use of theories from other documents (semantic citation/copy&paste)

Specifically for proofs

- In a subproof: What are possible next steps?
- Apply automatic proof procedures (verification or subtasks)
- Automatically found (sub-)proofs
  - integrated into document: readable, e.g. for inspection, explanation
  - use introduced notations
- Is a proof complete? Is it verified?

The Vision

Planned Functionality

Current State

Spectrum between Text, Notations and Formal Representations

Current State: Verimathdoc I

- Connect $\TeX_{macs}$ with Ω*mega* via Mediator *Plat*Ω

- Fully annotated (manually) $\TeX_{macs}$ document

```
\begin{definition}[Function $\in$]
The predicate \concept{\in}{elem \times set \rightarrow bool}
takes an individual and a set and tells whether that
individual belongs to this set.
\end{definition}
```

  Use Macros to indicate semantics: begin/end of theories, definitions, theorems, proofs, proof steps

Current State: Verimathdoc I

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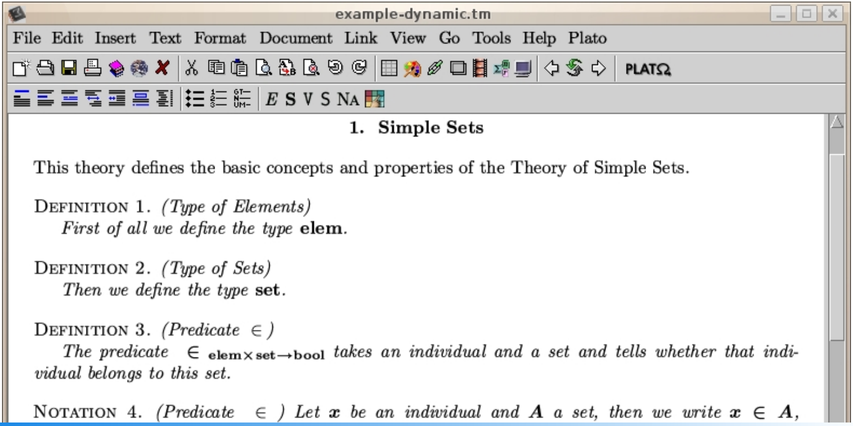

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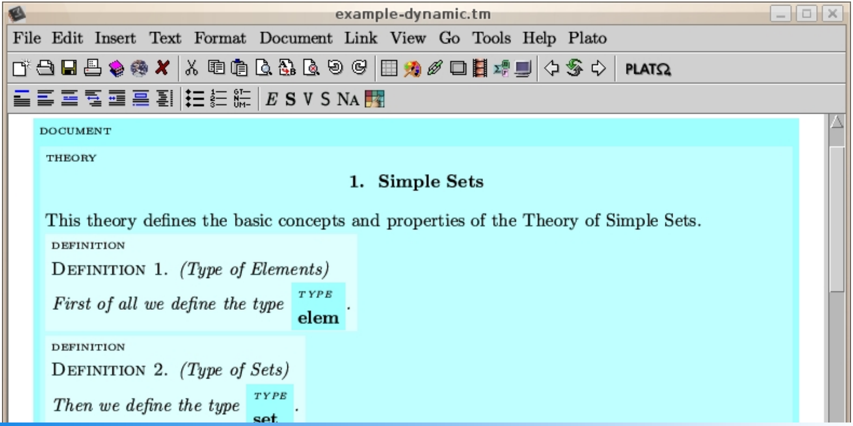

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  Use Macros to indicate semantics: begin/end of theories, definitions, theorems, proofs, proof steps
- Except for formulas.
- Provides type and proof checking, interactive proof construction
- Incremental: only changes are passed around
- Basic document change management

Current State: Verimathdoc I

- Connect *TeXmacs* with Ω*mega* via Mediator *Plat*Ω

- Fully annotated (manually) *TeXmacs* document

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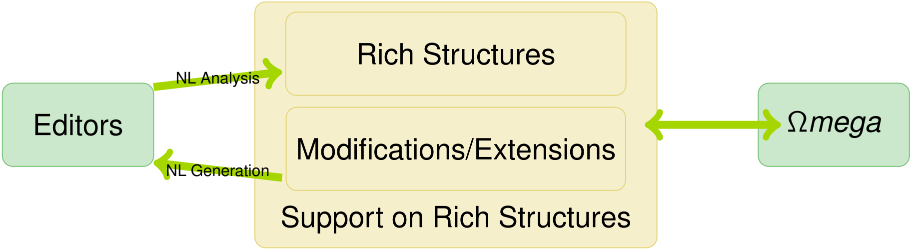
Marc's Part

Next Phase: Verimathdoc II

- Support independent from editors ($\TeX_{macs}$, Word 2007, $\LaTeX$)
- Start from Document as is
- Use NL-Analysis to obtain richer structures on which support can be offered (consistency checks, reorganisation, proof support, etc.)
- Use NL-Generation to map changes and modifications back into the document.



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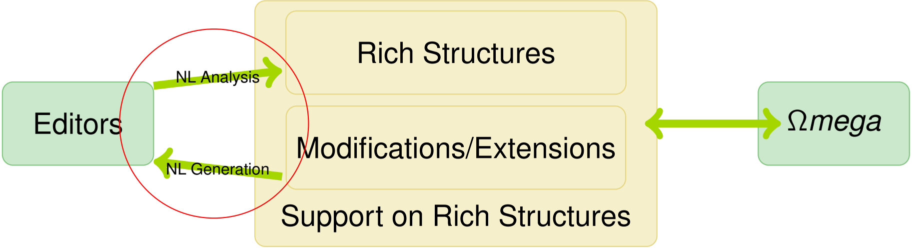

Stephan's Part: NL Research Questions & First Ideas

The Vision

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Gödel-Bernays Set Theory

Sichere Kognitive Syste

Natural Language Text

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Formal Counterpart

```
axiom "extensionality" ! A,B. class(A) /\ (class(B)
      /\ (! x.  membership(x,A) <=>  membership(x,B))) => A = B;
define set:object->o;
axiom "Set_Ax_89" ! A,a. set(a) <=> class(a) /\ (? B. class(B) /\ membership(a,B));
define properclass:object*object->o;
axiom "Set_Ax_90" ! A,a. class(a) => (properclass(a) <=> ~ set(a));
```

Definitions

Natural Language Text

A class *A* is a **subclass** of a class *B* (written $A \subset B$) provided:

$$\text{for all } x \in A, x \in A \Rightarrow x \in B.$$

Formal Counterpart

```
define  subclass : object*object->o;

axiom "Class_Ax_122" ! A,B. class(A) /\ class(B) =>
      ( subclass(A,B) <=> (! x. membership(x,A) => membership (x,B)));
```

Notation

```
notation subclass(#1, #2) <-> #1 ⊂ #2
```

Definitions

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Formal Counterpart

```
axiom "class formation" ! P:object->o . ? A. class(A) /\
      ! x . membership(x,A) <=> (set(x) /\ P(x));

define setconstr:(object->o)->object;
axiom "" ! P:object->o . class(setconstr(P)) /\
         ! x . membership(x,setconstr(P)) <=> (set(x) /\ P(x));
```

Notation

```
notation setconstr(lam #1 . #2) <-> {#1 | #2}
```

Theorems and Proofs

Natural Language Text

By the axioms of extensionality and the properties of equality

$$A = B \Leftrightarrow A \subset B \text{ and } B \subset A$$

Formal Counterpart

```
conjecture "Class_Th_124" ! A,B. class(A) /\ class(B) =>
     (A = B <=>  subclass(A,B) /\  subclass(B,A));

proof
 fact ! A,B. class(A) /\ class(B) => (A = B <=>
      subclass(A,B) /\  subclass(B,A)) by "Extensionality", "Properties of ="
 trivial
end
```

Theorems, Proofs and Checking Information

Natural Language Text

By the axioms of extensionality and the properties of equality

$$A = B \Leftrightarrow A \subset B \text{ and } B \subset A$$

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conjecture "Class_Th_124" ! A,B. class(A) /\ class(B) =>
     (A = B <=>  subclass(A,B) /\  subclass(B,A));
proof
 fact ! A,B. class(A) /\ class(B) =>  (A = B <=>
    subclass(A,B) /\  subclass(B,A)) by "Extensionality", "Properties of ="
    proof
     assume class(a), class(b);
     subgoal "1": a = b =>  subclass(a,b) /\  subclass(b,a);
       proof assume "hyp" : a = b;
         subgoal "1a": subclass(a,b)
           proof assume "hyp1" membership(x,a); subgoal membership(x,b);
                 trivial by "hyp", "hyp1" and "=" end
         subgoal "1b": subclass(b,a)
           proof assume "hyp2" membership(x,b); subgoal membership(x,a);
                 trivial by "hyp", "hyp2" and "=" end
```

Theorems, Proofs and Checking Information

Formal Counterpart

```
    subgoal "2": subclass(a,b) /\  subclass(b,a) => a = b;
      proof assume subclass(a,b), subclass(b,a);
            fact ! x . membership(x,a) => membership(x,b) by "subclass";
            fact ! x . membership(x,b) => membership(x,a) by "subclass";
            fact ! x . membership(x,b) <=> membership(x,a) by "logic";
            trivial by "Extensionality"; end
   end; trivial; end
```

Natural Language Text

By the axioms of extensionality and the properties of equality *Details*

$$A = B \Leftrightarrow A \subset B \text{ and } B \subset A$$

Details We first prove $A = B \Rightarrow A \subset B$ and $B \subset A$: Assume (h) $A = B$, then we have to prove (1) $A \subset B$ and (2) $B \subset A$: For (1), assuming $x \in A$, we conclude $x \in B$ from (h) and properties of equality. For (2), assuming $x \in B$, we conclude $x \in A$ from (h) and properties of equality. Conversely, we prove $A \subset B$ and $B \subset A \Rightarrow A = B$: By Definition of $\subset$ we know from $A \subset B$ and $B \subset A$ that $x \in A \Rightarrow x \in B$ and $x \in B \Rightarrow x \in A$ for all x. Hence, $x \in A \Leftrightarrow x \in B$ for all x and by extensionality follows $A = B$. □

More to come…

Current state (Verimathdoc I)

TeXmacs ↔ PlatΩ ↔ Ωmega

Marc's Part: Document management, Ambiguities, Verification

Natural language text processing/generation (Verimathdoc II)

Editors — NL Analysis → Rich Structures

Editors ← NL Generation — Modifications/Extensions

Support on Rich Structures ↔ Ωmega

Stephan's Part: NL Research Questions & First Ideas

Gödel-Bernays Set Theory

Sichere Kognitive Syste

Natural Language Text

In Gödel-Bernays form of axiomatic set theory, which we shall follow, the primitive (undefined) notions are **class, membership**, and **equality**. Intuitively, we consider a class to be a collection A of objects (elements) such that given any object x if it is possible to determine whether or not x is a member (or element) of A. We write $x \in A$ for "x is an element of A" and $x \notin A$ for "x is not an element of A".

Formal Counterpart

```
define class:object->o;
define membership:object*object->o;

axiom !x,a . membership(x, a) or ˜ membership(x, a);
```

Notation

notation $\#1 \in \#2$ `<-> membership(\#1, \#2)`
notation $\#1 \notin \#2$ `<-> ˜ membership(\#1, \#2)`