



Representing Proof-Planning in the ρ -Calculus

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What is Proof Planning?



- Theorem proving process that
 - ▶ conceives proofs at an abstract mathematics-oriented level
 - ▶ makes use of mathematical knowledge
 - ▶ uses AI planning technologies for proof search
 - ▶ postpones verification of abstract proof steps until an abstract proof has been found
- ⇒ Differs considerably from traditional calculus-level proof search in resolution/tableaux/matrix provers
- Different approaches embodied in the systems CLAM, PPLANNER, λ -CLAM, ISAPLANNER, MULTI

Motivation



- We want to model proof planning in a uniform framework such that we can
 - ▶ compare/relate the proof planning approach with other proof search strategies
 - ▶ compare different proof planning approaches among each other
- Modeling in the ρ -Calculus (already used to model ELAN strategies)

Proof Planning in Ω MEGA



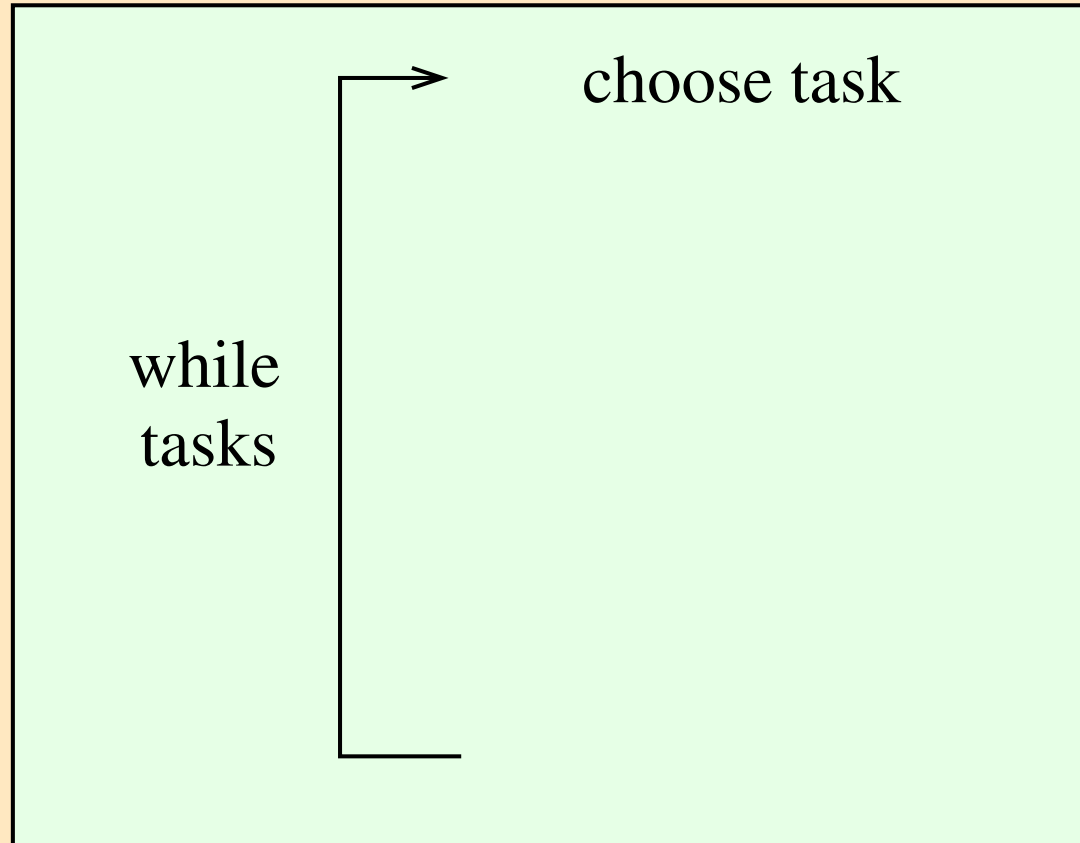
- Recent planner: MULTI [A.Meier 2004]
- Uses **mathematical proof knowledge** in
 - ▶ methods to encode abstract proof steps
 - ▶ control rules declaratively encode heuristic control knowledge
 - ▶ strategies encode proof plan refinement services
- Major domains considered:
 ϵ - δ -proofs, residue class problems, permutation group problems, homomorphism theorems, Irrationality of $\sqrt[k]{k}$

Proof Planning in Ω MEGA

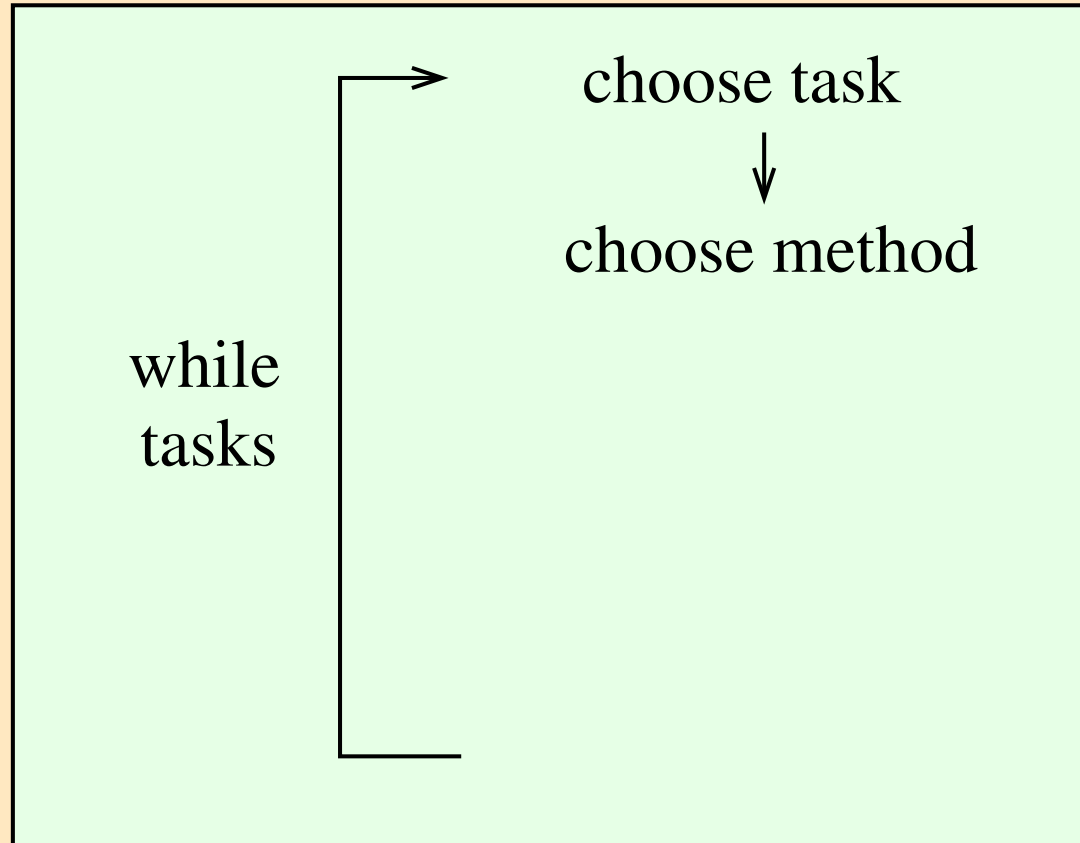


- Henceforth:
restrict to a simple proof planning algorithm at the level of
methods only

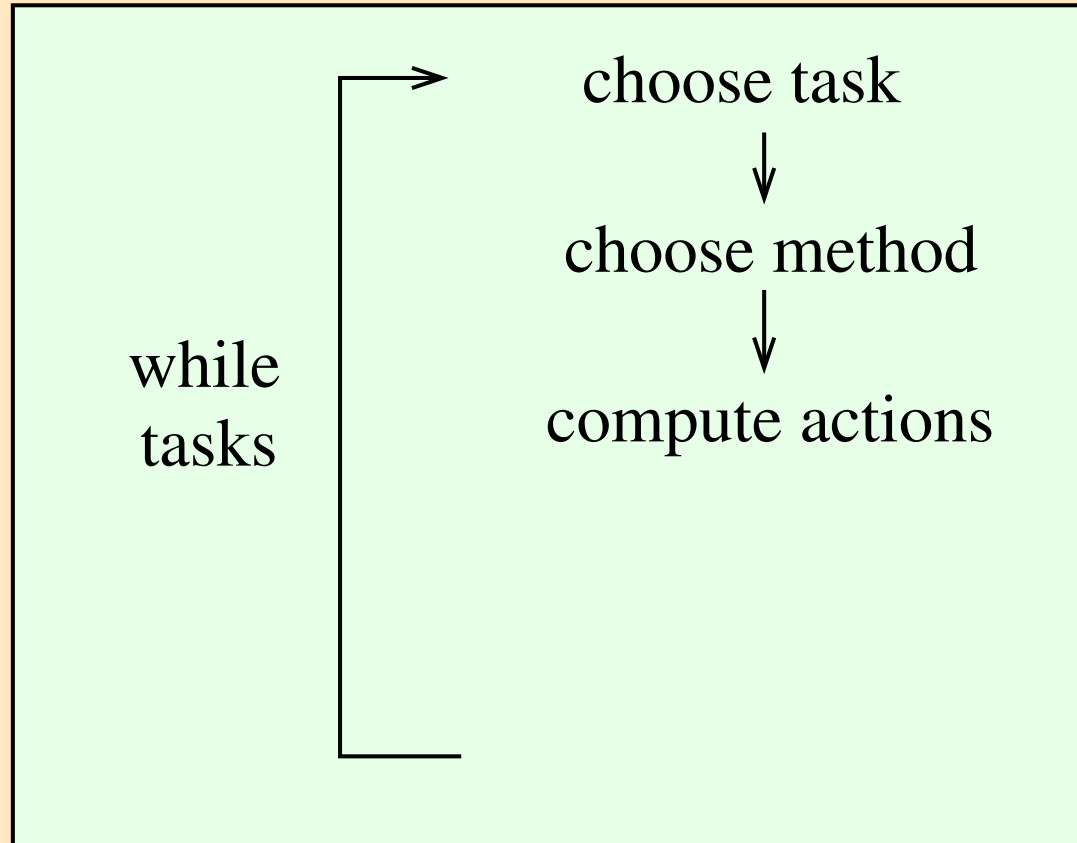
Simple Proof Planner



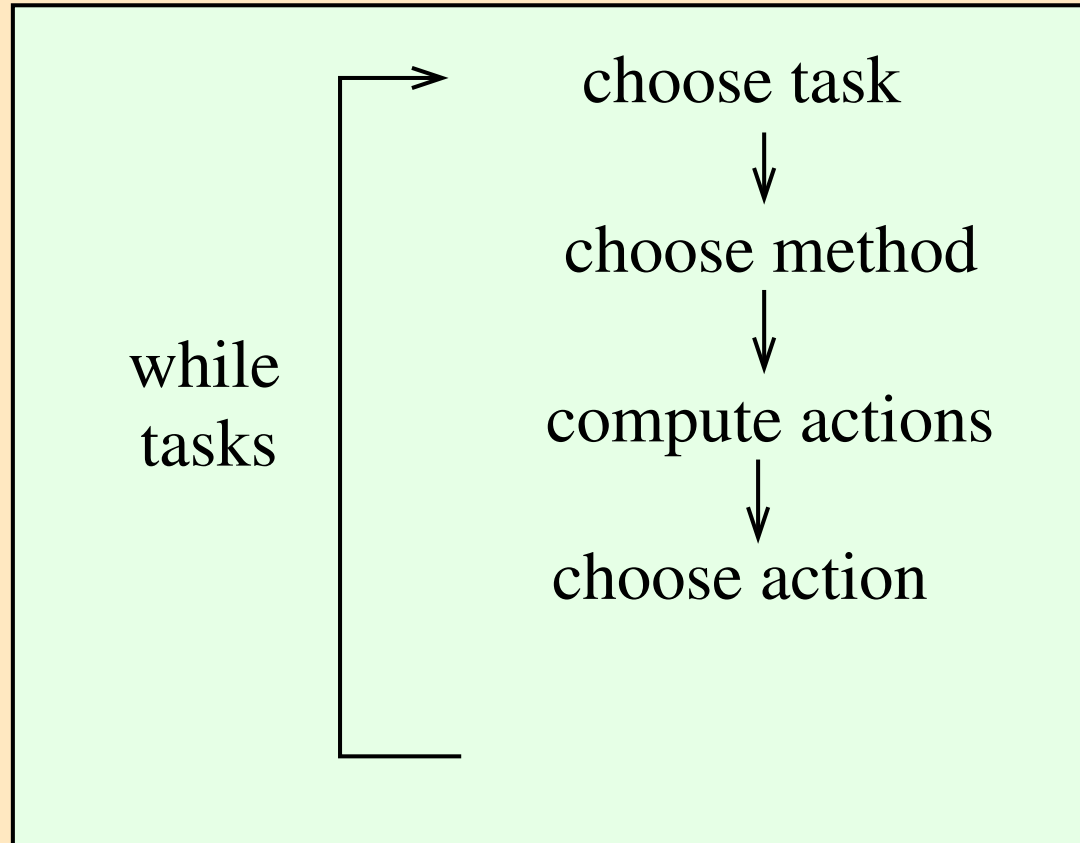
Simple Proof Planner



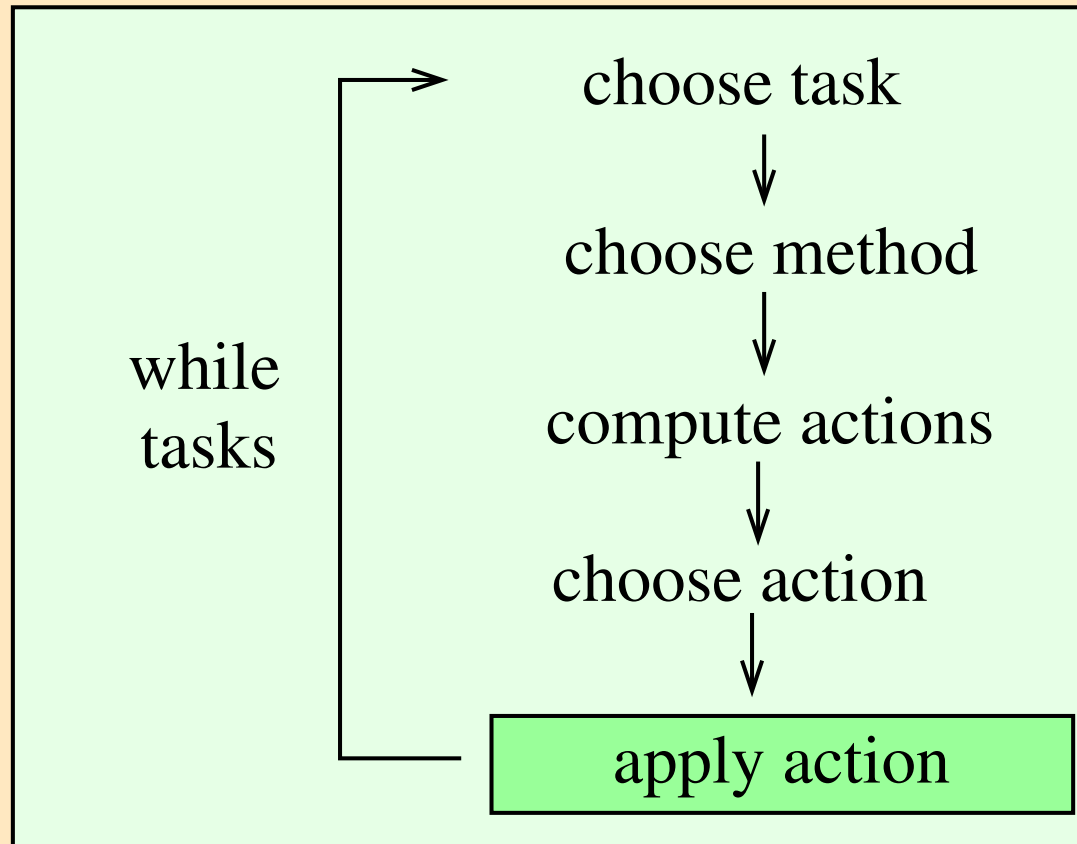
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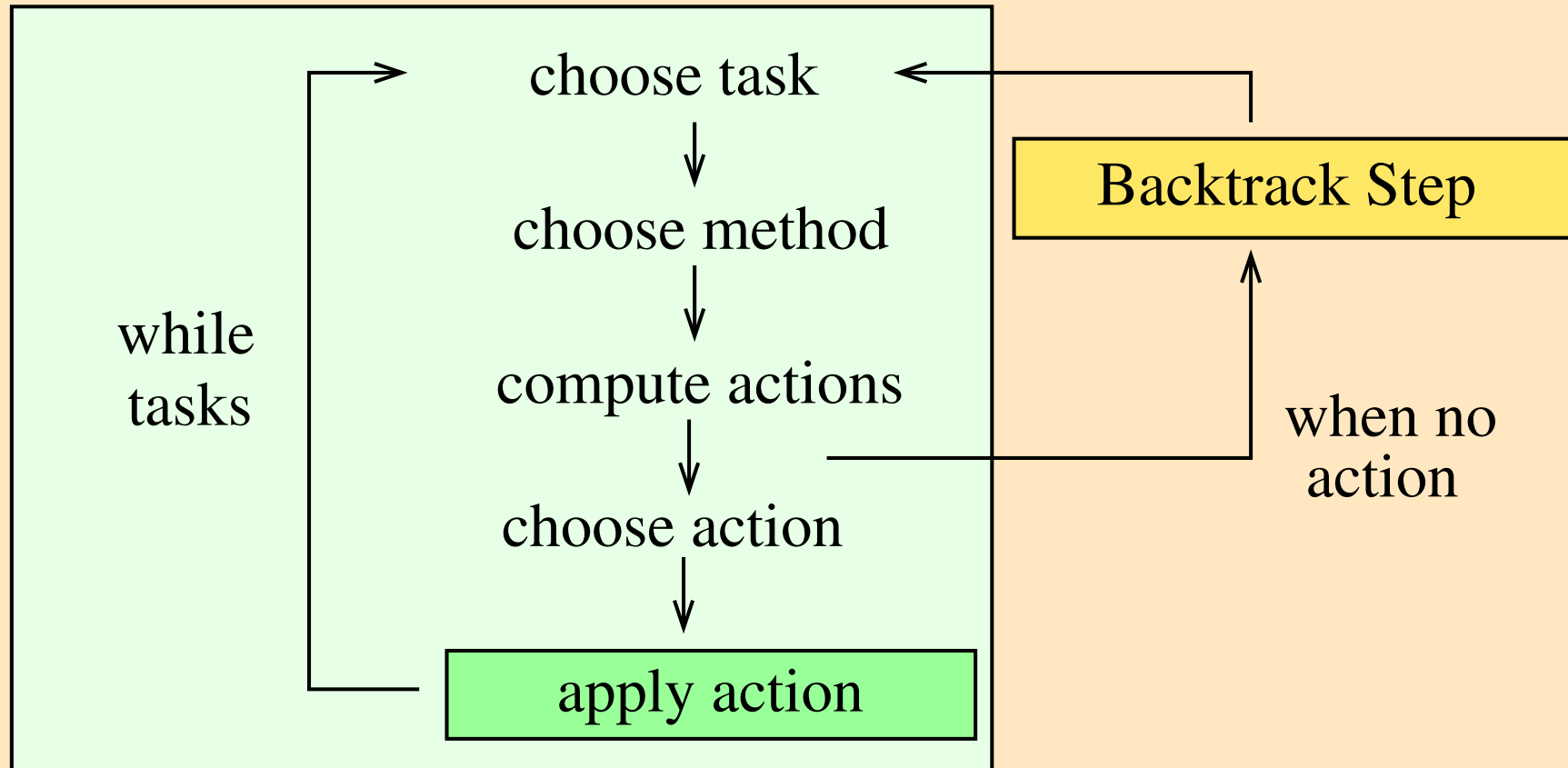
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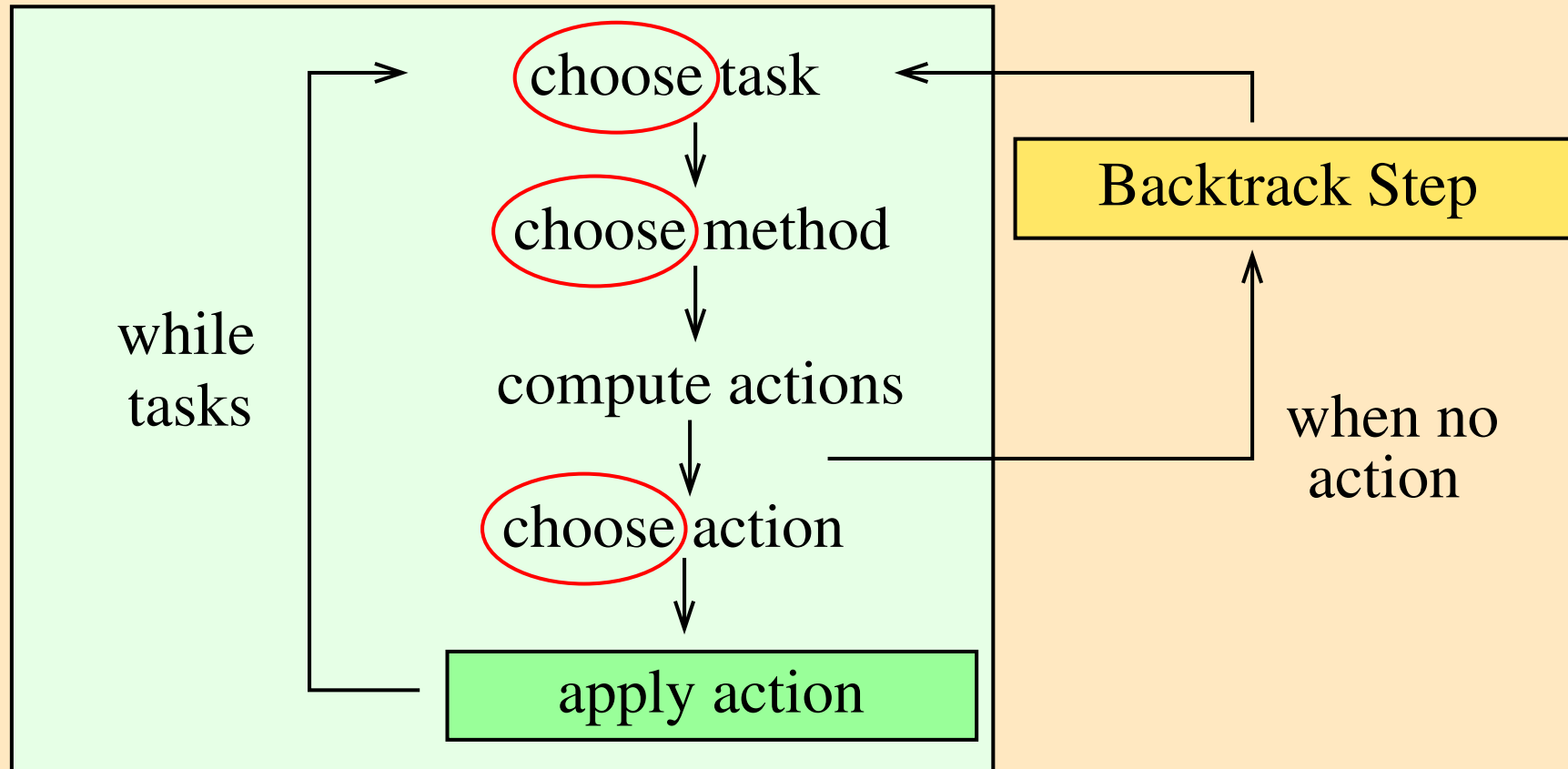
Simple Proof Planner



Simple Proof Planner



Simple Proof Planner



Guidance by control rules

Abstract simple proof planner



While $get\text{-}tasklist(A) \neq \emptyset$ do

1. Choose task T from A using the control rules CL ;
2. Choose method M from ML using the control rules CL ;
3. Apply method M to T which results in a list of actions, i.e., instantiated methods, denoted by $M(\vec{p}_1), \dots, M(\vec{p}_n)$;
4. Choose an action $M(\vec{p}_i)$ using control rules from CL and apply it to T
to obtain a list of new tasks TL ;
5. $A := A - \{T\} \cup TL$.

Tasks



- Task = Single conclusion sequent with a label (proof line) and a justification

$$\text{Lbl} : \langle G \triangleleft P_1, \dots, P_n \rangle (\text{Just})$$

- Premises: P_1, \dots, P_n
- Goal: G
- Label: Lbl
- Justification Just : the name of a method and a list of labels
 $\text{Open}, \forall I, \text{ComplexEstimate}, \text{Otter}, \text{Maple}, \dots$

Agenda



- Simple approach: The agenda is a set of tasks
- Operations on the agenda:
 - ▶ Selection of a task

$$(X, P, X') \rightarrow P$$

selects all the members of the list that match the pattern P .
If no member matches P , the result is \perp .

- ▶ removal of a task with label ℓ is achieved by

$$X, \ell : T(J), X' \rightarrow X, X'$$

where T and J are variables.

Computations and External Systems



- Methods often uses external procedures, for computing a witness or a side condition.
- Example: a unification procedure *Unify*(·, ·), [Call to Automated Theorem Provers](#), or [Computer Algebra System](#).
- Calls to a procedure will be denoted *Proc_name*(arg₁, ..., arg_n).
- Until the procedure is defined (or if it is a call to an external system), the name *Proc_name* is considered as a constant with some matching theory defining its behaviour.
- For instance *Unify* is defined by the axioms:

$$\begin{aligned} \textit{Unify}(t, t') &= \sigma && \text{if } \sigma t = \sigma t' \text{ and } \forall \theta (\theta t = \theta t' \Rightarrow \exists \theta', \theta = \theta' \circ \sigma) \\ &= \perp && \text{otherwise} \end{aligned}$$

Encoding Methods



- A method takes a task, checks if it fulfills some conditions (some of which can be described by pattern matching), makes some computations and returns a list of new tasks.
- Thus its general shape is:

$$\langle G \triangleleft P \rangle \rightarrow (X \rightarrow TL) \text{Comp}(G, P)$$

- Example: A method for sideways application of $<$ -transitivity $\frac{a < c \quad c < b}{a < b}$

$$\langle a < b \triangleleft X, a' < c, X' \rangle$$

$$\rightarrow \text{first} \left(\begin{array}{l} \perp \rightarrow \text{fail}; \\ \sigma \rightarrow \langle \text{Apply}(\sigma, c) < \text{Apply}(\sigma, b) \triangleleft X, \text{Apply}(\sigma, a') < \text{Apply}(\sigma, c), X' \rangle \end{array} \right)$$

$$\text{Unify}(a, a')$$

Encoding control rules



- Control rules modify a list of tasks, methods or actions in order to rank them by priority (and possibly add or remove some from the list).
- Their application is guided by conditions on the current agenda plan.
- Their general shape is:

$$CR(\text{Kind}, A \rightarrow L \rightarrow ((\text{True} \rightarrow \text{Reorder}(L); \text{False} \rightarrow L)\text{Cond}(A)))$$

where $\text{Kind} \in \{\text{Task}, \text{Meth}, \text{Action}\}$.

- Cond is a (TRS computing a) boolean predicate on an agenda.
- Reorder is a modification to be applied on the list. . .

Rordering Lists



Reorder is a modification to be applied on the list from

- *Selection* of a subset, performed by $X, P, X' \rightarrow P$

where P and Q are patterns describing which group of *Kind* are manipulated

Rordering Lists



Reorder is a modification to be applied on the list from

- *Selection* of a subset, performed by $X, P, X' \rightarrow P$
- *Rejection* of a subset, performed by repeated application of $X, P, X' \rightarrow X, X'$

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- *Deference* of a subset, performed by repeated application of $X, P, X' \rightarrow X, X', P$
- *Reordering* of a subset in front of another, performed by repeated application of $X, P, X', Q, X'' \rightarrow X, Q, P, X', X''$

where **P** and **Q** are patterns describing which group of **Kind** are manipulated

Rordering Lists



Reorder is a modification to be applied on the list from

- *Selection* of a subset, performed by $X, P, X' \rightarrow P$
- *Rejection* of a subset, performed by repeated application of $X, P, X' \rightarrow X, X'$
- *Preference* of a subset, performed by repeated application of $X, P, X' \rightarrow P, X, X'$
- *Deference* of a subset, performed by repeated application of $X, P, X' \rightarrow X, X', P$
- *Reordering* of a subset in front of another, performed by repeated application of $X, P, X', Q, X'' \rightarrow X, Q, P, X', X''$
- *Insertion* of a new element e , performed by $X \rightarrow e, X$

where P and Q are patterns describing which group of **Kind** are manipulated

Example Control Rule

A control rule promoting $\forall I$ for agendas fulfilling condition C

$CR(\text{Meth},$

$A \rightarrow ML \rightarrow$

$\left(\begin{array}{l} \text{True} \rightarrow (X, \forall I, X' \rightarrow \forall I, X, X')ML; \\ \text{False} \rightarrow ML \end{array} \right)_{C(A)}$

Encoding the Simple Proof Planner



```
A → CL → ML →  
( T →  
  ( M →  
    ( TL-Set →  
      ( TL → justify(T, A, justification(T, M, TL)) )select-tasklist(TL-Set, CL)  
    )apply-method(M, T)  
  )select-method(ML, T, CL)  
)select-task(get-tasklist(A), CL)
```

Encoding the Simple Proof Planner



```
A → CL → ML →  
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  )select-method(ML, T, CL)  
)select-task(get-tasklist(A), CL)
```

where the following procedures are used:

get-tasklist is simply the identity in this case, since the agenda is already a list of tasks.

Encoding the Simple Proof Planner



```

A → CL → ML →
( T →
  ( M →
    ( TL-Set →
      ( TL → justify(T, A, justification(T, M, TL)) )select-tasklist(TL-Set, CL)
    )apply-method(M, T)
  )select-method(ML, T, CL)
)select-task(get-tasklist(A), CL)

```

where the following procedures are used:

select-task applies successively all the task control rules over the task list and finally takes the first:

$$TL \rightarrow \text{first} \left(\begin{array}{l} \text{nil} \rightarrow TL; \\ CR(\text{ Task, R}) , CL' \rightarrow \text{select-task}((R TL), CL'); \\ C, CL' \rightarrow \text{select-task}(TL, CL') \end{array} \right)$$

Encoding the Simple Proof Planner



```

A → CL → ML →
( T →
  ( M →
    ( TL-Set →
      ( TL → justify(T, A, justification(T, M, TL)) )select-tasklist(TL-Set, CL)
    )apply-method(M, T)
  )select-method(ML, T, CL)
)select-task(get-tasklist(A), CL)

```

where the following procedures are used:

select-method applies successively all the method control rules over the method list and then tries every method on the current task. Keeps first successful method

$$ML \rightarrow T \rightarrow \text{first} \left(\begin{array}{l} \text{nil} \rightarrow \text{first}(ML)T; \\ CR(\text{Meth}, R) , CL' \rightarrow \text{select-method}((R \ ML), T, CL'); \\ C, CL' \rightarrow \text{select-method}(ML, T, CL') \end{array} \right)$$

Encoding the Simple Proof Planner



```
A → CL → ML →  
( T →  
  ( M →  
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      ( TL → justify(T, A, justification(T, M, TL)) )select-tasklist(TL-Set, CL)  
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  )select-method(ML, T, CL)  
)select-task(get-tasklist(A), CL)
```

where the following procedures are used:

select-tasklist applies successively all the action control rules over the list of task lists and finally takes the first:

$$TL\text{-Set} \rightarrow \text{first} \left(\begin{array}{l} \text{nil} \rightarrow TL\text{-Set}; \\ CR(\text{ Action, R}) , CL' \rightarrow \text{select-tasklist}((R \text{ TL-Set}), CL'); \\ C, CL' \rightarrow \text{select-tasklist}(TL\text{-Set}, CL') \end{array} \right)$$

Encoding the Simple Proof Planner



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A → CL → ML →  
( T →  
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      ( TL → justify(T, A, justification(T, M, TL)) )select-tasklist(TL-Set, CL)  
    )apply-method(M, T)  
  )select-method(ML, T, CL)  
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```

where the following procedures are used:

justify replaces an open task with the selected justification and adds the list of new open tasks to the current agenda.

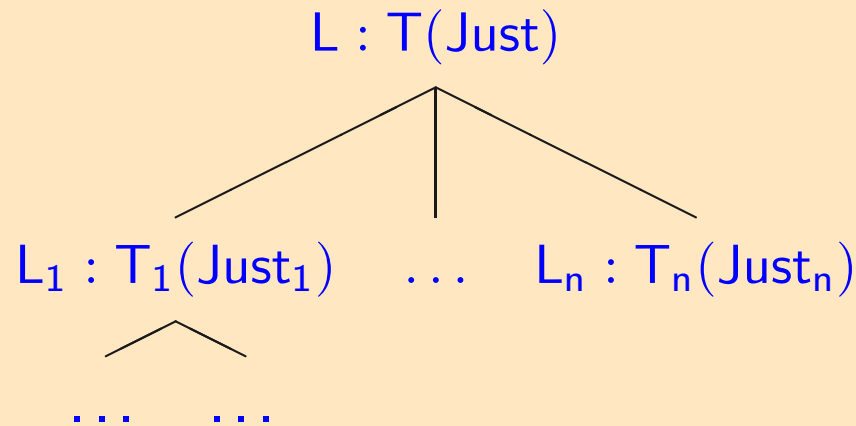
A more complex agenda



- How to backtrack can also be defined by control rules (defining a *backtracking strategy*)
- Independently *where* the backtracking strategies decide to prune the search tree, they all do dependency-based pruning of subtrees
- This is not supported by using an agenda composed only of the current list of tasks
- Need to track dependencies by using a tree data structure which is closer to the proof datastructure (PDS)
- A node of the tree is still a labelled task with a justification, but
...

Agenda as Dependency Tree

- A node of the tree is still a labelled task with a justification, but the justification can contain new nodes (instead of labels)



is represented as

$$L : T(\text{Just}(L_1 : T_1(\text{Just}_1(\dots, \dots)), \dots, L_n : T_n(\text{Just}_n)))$$

- The search of a given node now requires a traversal function

Examples



- Example: Selection of all the nodes that match the pattern P .

$$(S \rightarrow \left(\begin{array}{l} P \rightarrow P; \\ L : T(J(X, N, X')) \rightarrow S S N \end{array} \right)) (S \rightarrow \left(\begin{array}{l} P \rightarrow P; \\ L : T(J(X, N, X')) \rightarrow S S N \end{array} \right))$$

- Pruning the subtree at a given node (once this node is found) is achieved by the simple term $L : T(J) \rightarrow L : T(\text{Open})$.

Comparison with Elan



PPlanning	Rho	Elan
Agenda	List term	
Agenda	Tree term	Algebraic term
Precondition	Pattern	Conditional rewriting
Method	Abstraction	Rule
Main loop	Fixed points	repeat*
Control rules	List management	
	Structures + \mapsto_{δ}	dk / dc
Dependency-based pruning	Pruning	
Try + Backtracking Control Rules (implicit first)	first / stk	try / first

Future

Modelling Proof Planning Procedures

- Include expansion of proof planning steps
- Treatment of Metavariables
- MULTI: Describe the level of *Strategies*
 - ▶ Strategy = Set of methods + Control rules
 - ▶ MULTI selects strategy for the simple proof planner
- λ -CLAM, ISAPLANNER: Model continuations explicitly
 - ▶ For ISAPLANNER: Support
 - access to continuation by simple planner
 - manipulation of continuation by the planner

Compare with other search procedures

- How well can (specific) search procedures, control, backtracking be encoded in the different formalisms?

Conclusion



- So far modelling in the ρ -Calculus is (mostly) straightforward
- Allows for a very concise representation of the main aspects
- Even for the simple planner we have already initial results by comparing it with other search strategies
Language constructs provided/used for modelling search
- We are at the beginning but it looks promising that we can get nice results