Systeme Hoher Sicherheit und Qualität
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Lecture 10 (14.12.2015)

Foundations of Software Verification

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Where are we?

- 01: Concepts of Quality
- 02: Legal Requirements: Norms and Standards
- 03: The Software Development Process
- 04: Hazard Analysis
- 05: High-Level Design with SysML
- 06: Formal Modelling with SysML and OCL
- 07: Detailed Specification with SysML
- 08: Testing
- 09: Program Analysis
- 10: Foundations of Software Verification
- 11: Verification Condition Generation
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- 13: Model-Checking
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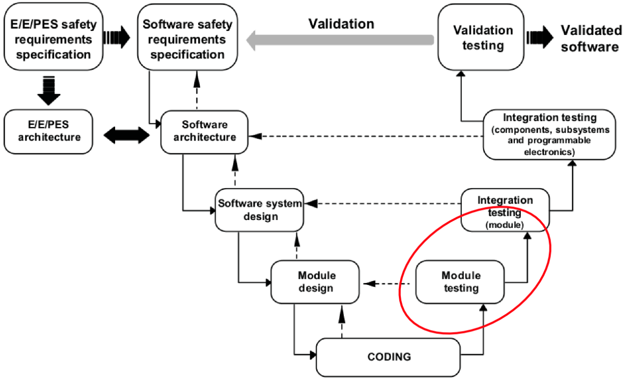
Today: Software Verification using Floyd-Hoare logic



- The Floyd-Hoare calculus proves properties of imperative programs.
- Thus, it is at home in the lower levels of the verification branch, much like the static analysis from last week.
- It is far more powerful than static analysis — and hence, far more complex to use (it requires user interaction, and is not automatic).

Idea

- What does this compute?

```
P := 1;
C := 1;
while (C ≤ N) {
  P := P * C;
  C := C + 1
};
```

Idea

- What does this compute? $P = N!$
- How can we prove this?

```
P := 1;
C := 1;
while (C ≤ N) {
  P := P * C;
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};
```

Idea

- What does this compute? $P = N!$
- How can we prove this?
- Inuitively, we argue about which value variables have at certain points in the program.
- Thus, to prove properties of imperative programs like this, we need a formalism where we can formalise assertions of the program properties at certain points in the exection, and which tells us how these assertions change with program execution.

$\{1 \leq N\}$
```
P := 1;
C := 1;
while (C ≤ N) {
  P := P * C;
  C := C + 1
};
```
$\{P = N!\}$

Floyd-Hoare-Logic

- Floyd-Hoare-Logic consists of a set of rules to derive valid assertions about programs. The assertions are denoted in the form of Floyd-Hoare-Triples $\{P\}\, p\, \{Q\}$, with P the precondition, p a program and Q the postcondition.
- The logical language has both logical variables (which do not change), and program variables (the value of which changes with program execution).
- Floyd-Hoare-Logic has one basic principle and one basic trick.
- The principle is to abstract from the program state into the logical language; in particular, assignment is mapped to substitution.
- The trick is dealing with iteration: iteration corresponds to induction in the logic, and thus is handled with an inductive proof. The trick here is that in most cases we need to strengthen our assertion to obtain an invariant.

Recall Our Small Language

- Arithmetic Expressions (**AExp**)

$$a ::= \mathbf{N} \mid \mathbf{Loc} \mid a_1 + a_2 \mid a_1 - a_2 \mid a_1 \times a_2$$

  with variables **Loc**, numerals **N**

- Boolean Expressions (**BExp**)

$$b ::= \mathbf{true} \mid \mathbf{false} \mid a_1 = a_2 \mid a_1 < a_2 \mid \neg b \mid b_1 \wedge b2 \mid b_1 \vee b_2$$

- Statements (**Com**)

$$\begin{aligned} c ::= {} & \mathbf{skip} \mid \mathbf{Loc} := \mathbf{AExp} \mid \mathbf{skip} \mid c_1; c_2 \\ & \mid \mathbf{if}\ \ b\ \{c_1\}\ \ \mathbf{else}\ \{c_2\} \mid \mathbf{while}\ \ b\ \{c\} \end{aligned}$$

Semantics of our Small Language

- The semantics of an imperative language is state transition: the program has an ambient state, and changes it by assigning values to certain locations
- Concrete example: execution starting with $N = 3$

P	?		P	1		P	1		P	1		P	6
C	?	$\leadsto$	C	?	$\leadsto$	C	1	$\leadsto$	C	1	$\leadsto \ldots$	C	4
N	3		N	3		N	3		N	3		N	3

Semantics in a nutshell

- Expressions evaluate to values **Val**(in our case, integers)
- A program state maps locations to values: $\Sigma = \mathbf{Loc} \rightharpoonup \mathbf{Val}$
- A programs maps an initial state to possibly a final state (if it terminates)
- Assertions are predicates over program states.

Floyd-Hoare-Triples

Partial Correctness ($\models \{P\}\, c\, \{Q\}$)

c is partial correct with precondition P and postcondition Q if:
for all states σ which satisfy P
if the execution of c on σ terminates in σ'
then σ' satisfies Q

Total Correctness ($\models [P]\, c\, [Q]$)

c is total correct with precondition P and postcondition Q if:
for all states σ which satisfy P
the execution of c on σ terminates in σ'
and σ' satisfies Q

- $\models \{\textbf{true}\}\ \textbf{while}\ \ \textbf{true}\ \{\textbf{skip}\}\ \{\textbf{true}\}$ holds
- $\models [\textbf{true}]\ \textbf{while}\ \ \textbf{true}\ \{\textbf{skip}\}\ [\textbf{true}]$ does not hold

Assertion Language

- Extension of **AExp** and **BExp** by
  - logical variables **Var** $v := n, m, p, q, k, l, u, v, x, y, z$
  - defined functions and predicates on **Aexp** $n!, \sum_{i=1}^{n}, \ldots$
  - implication, quantification $b_1 \Rightarrow b_2, \forall v.\, b, \exists v.\, b$
- **Aexpv**

$$a ::= \mathbf{N} \mid \mathbf{Loc} \mid a_1 + a_2 \mid a_1 - a_2 \mid a_1 \times a_2 \mid \mathbf{Var} \mid f(e_1, \ldots, e_n)$$

- **Bexpv**

$$b ::= \mathbf{true} \mid \mathbf{false} \mid a_1 = a_2 \mid a_1 \leq a_2 \mid \neg b \mid b_1 \wedge b2 \mid b_1 \vee b_2 \mid b_1 \Rightarrow b_2 \mid p(e_1, \ldots, e_n) \mid \forall v.\, b \mid \exists v.\, b$$

Rules of Floyd-Hoare-Logic

- The Floyd-Hoare logic allows us to derive assertions of the form $\vdash \{P\}\, c\, \{Q\}$
- The calculus of Floyd-Hoare logic consists of six rules of the form

$$\frac{\vdash \{P_1\}\, c_1\, \{Q_1\} \ldots \vdash \{P_n\}\, c_n\, \{Q_n\}}{\vdash \{P\}\, c\, \{Q\}}$$

- This means we can derive $\vdash \{P\}\, c\, \{Q\}$ if we can derive $\vdash \{P_i\}\, c_i\, \{Q_i\}$
- There is one rule for each construction of the language.

Rules of Floyd-Hoare Logic: Assignment

$$\overline{\vdash \{B[e/X]\}\, X := e\, \{B\}}$$

- An assigment X:=e changes the state such that at location X we now have the value of expression e. Thus, in the state before the assignment, instead of X we must refer to e.
- It is quite natural to think that this rule should be the other way around.
- Examples:

```
X := 10;
```
$\{0 < 10 \longleftrightarrow (X < 10)[X/0]\}$
```
X := 0
```
$\{X < 10\}$

$\{X < 9 \longleftrightarrow X + 1 < 10\}$
```
X := X+ 1
```
$\{X < 10\}$

Rules of Floyd-Hoare Logic: Conditional and Sequencing

$$\frac{\vdash \{A \land b\}\, c_0\, \{B\} \qquad \vdash \{A \land \neg b\}\, c_1\, \{B\}}{\vdash \{A\}\ \textbf{if}\ \ b\ \{c_0\}\ \ \textbf{else}\ \{c_1\}\, \{B\}}$$

- In the precondition of the positive branch, the condition b holds, whereas in the negative branch the negation $\neg b$ holds.
- Both branches must end in the same postcondition.

$$\frac{\vdash \{A\}\, c_0\, \{B\} \qquad \vdash \{B\}\, c_1\, \{C\}}{\vdash \{A\}\, c_0; c_1\, \{C\}}$$

- We need an intermediate state predicate B.

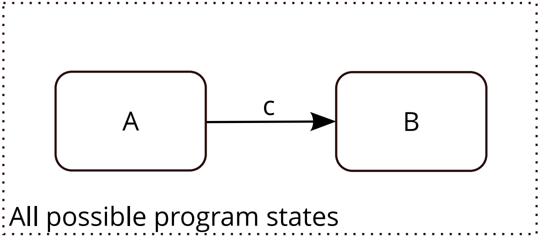
Rules of Floyd-Hoare Logic: Iteration

$$\frac{\vdash \{A \land b\}\, c\, \{A\}}{\vdash \{A\}\ \textbf{while}\ \ b\ \{c\}\, \{A \land \neg b\}}$$

- Iteration corresponds to induction. Recall that in (natural) induction we have to show the same property P holds for 0, and continues to hold: if it holds for n, then it also holds for $n + 1$.
- Analogously, here we need an invariant A which has to hold both before and after the body (but not necessarily in between).
- In the precondition of the body, we can assume the loop condition holds.
- The precondition of the iteration is simply the invariant A, and the postcondition of the iteration is A and the negation of the loop condition.

Rules of Floyd-Hoare Logic: Weakening

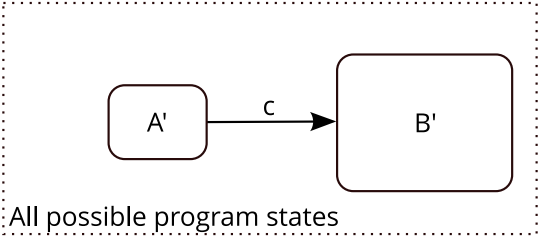
$$\frac{A' \longrightarrow A \qquad \vdash \{A\}\, c\, \{B\} \qquad B \longrightarrow B'}{\vdash \{A'\}\, c\, \{B'\}}$$



- $\models \{A\}\, c\, \{B\}$ means that whenever we start in a state where A holds, c ends (if it does) in state where B holds.
- Further, for two sets of states, $P \subseteq Q$ iff $P \longrightarrow Q$.

Rules of Floyd-Hoare Logic: Weakening

$$\frac{A' \longrightarrow A \qquad \vdash \{A\}\, c\, \{B\} \qquad B \longrightarrow B'}{\vdash \{A'\}\, c\, \{B'\}}$$



- $\models \{A\}\, c\, \{B\}$ means that whenever we start in a state where A holds, c ends (if it does) in state where B holds.
- Further, for two sets of states, $P \subseteq Q$ iff $P \longrightarrow Q$.
- We can restrict the set A to A' ($A' \subseteq A$ or $A' \longrightarrow A$) and we can enlarge the set B to B' ($B \subseteq B'$ or $B \longrightarrow B'$), and obtain $\models \{A'\}\, c\, \{B'\}$.

Overview: Rules of Floyd-Hoare-Logic

$$\frac{}{\vdash \{A\}\ \textbf{skip}\ \{A\}} \qquad \frac{}{\vdash \{B[e/X]\}\ X := e\ \{B\}}$$

$$\frac{\vdash \{A \land b\}\ c_0\ \{B\} \qquad \vdash \{A \land \neg b\}\ c_1\ \{B\}}{\vdash \{A\}\ \textbf{if}\ \ b\ \{c_0\}\ \ \textbf{else}\ \{c_1\}\ \{B\}}$$

$$\frac{\vdash \{A \land b\}\ c\ \{A\}}{\vdash \{A\}\ \textbf{while}\ \ b\ \{c\}\ \{A \land \neg b\}} \qquad \frac{\vdash \{A\}\ c_0\ \{B\} \qquad \vdash \{B\}\ c_1\ \{C\}}{\vdash \{A\}\ c_0; c_1\ \{C\}}$$

$$\frac{A' \longrightarrow A \qquad \vdash \{A\}\ c\ \{B\} \qquad B \longrightarrow B'}{\vdash \{A'\}\ c\ \{B'\}}$$

Properties of Hoare-Logic

Soundness

If $\vdash \{P\}\, c\, \{Q\}$, then $\models \{P\}\, c\, \{Q\}$

- If we derive a correctness assertion, it holds.
- This is shown by defining a formal semantics for the programming language, and showing that all rules are correct wrt. to that semantics.

Relative Completeness

If $\models \{P\}\, c\, \{Q\}$, then $\vdash \{P\}\, c\, \{Q\}$ except for the weakening conditions.

- Failure to derive a correctness assertion is always due to a failure to prove some logical statements (in the weakening).
- First-order logic itself is incomplete, so this result is as good as we can get.

The Need for Verification

Consider the following variations of the faculty example.
Which are correct?

$\{1 \leq N\}$
```
P := 1;
C := 1;
while (C≤N) {
  C := C+1;
  P := P*C
}
```
$\{P = N!\}$

$\{1 \leq N\}$
```
P := 1;
C := 1;
while (C<N) {
  C := C+1;
  P := P*C
}
```
$\{P = N!\}$

$\{1 \leq N \land n = N\}$
```
P := 1;
while (70<N) {
  P := P*N;
  N := N-1
}
```
$\{P = n!\}$

A Hatful of Examples

$\{i = Y\}$

```
X := 1;
while (¬ (Y = 0)) {
 Y := Y−1;
 X := 2∗X
}
```

$\{X = 2^i\}$

$\{A \geq 0 \wedge B \geq 0\}$

```
Q := 0;
R := A−(B∗Q);
while (B ≤ R) {
 Q := Q+1;
 R := A−(B∗Q)
}
```

$\{A = B * Q + R \wedge R < B\}$

$\{0 < A\}$

```
T:= 1;
S:= 1;
I:= 0;
while (S ≤ A) {
  T := T+ 2;
  S := S+ T;
  I := I+ 1
  }
```

$\{I * I <= A \wedge A < (I + 1) * (I + 1)\}$

A Hatful of Examples

$\{i = Y \wedge Y \geq 0\}$

```
X := 1;
while (¬ (Y = 0)) {
 Y := Y−1;
 X := 2∗X
}
```

$\{X = 2^i\}$

$\{A \geq 0 \wedge B \geq 0\}$

```
Q := 0;
R := A−(B∗Q);
while (B ≤ R) {
 Q := Q+1;
 R := A−(B∗Q)
}
```

$\{A = B * Q + R \wedge R < B\}$

$\{0 < A\}$

```
T:= 1;
S:= 1;
I:= 0;
while (S ≤ A) {
  T := T+ 2;
  S := S+ T;
  I := I+ 1
  }
```

$\{I * I <= A \wedge A < (I + 1) * (I + 1)\}$

Summary

- Floyd-Hoare logic in a nutshell:
  - The logic abstracts over the concrete program state by program assertions
  - Program assertions are boolean expressions, enriched by logical variables (and more)
  - We can prove partial correctness assertions of the form $\models \{P\}\, c\, \{Q\}$ (or total $\models [P]\, c\, [Q]$).
- Validity (correctness wrt a real programming language) depends very much on capturing the exact semantics formally.
- Floyd-Hoare logic itself is rarely used directly in practice, verification condition generation is — see next lecture.