Systeme hoher Sicherheit und Qualität
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Lecture 09: Software Verification with Floyd-Hoare Logic

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Where are we?

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- 02: Legal Requirements: Norms and Standards
- 03: The Software Development Process
- 04: Hazard Analysis
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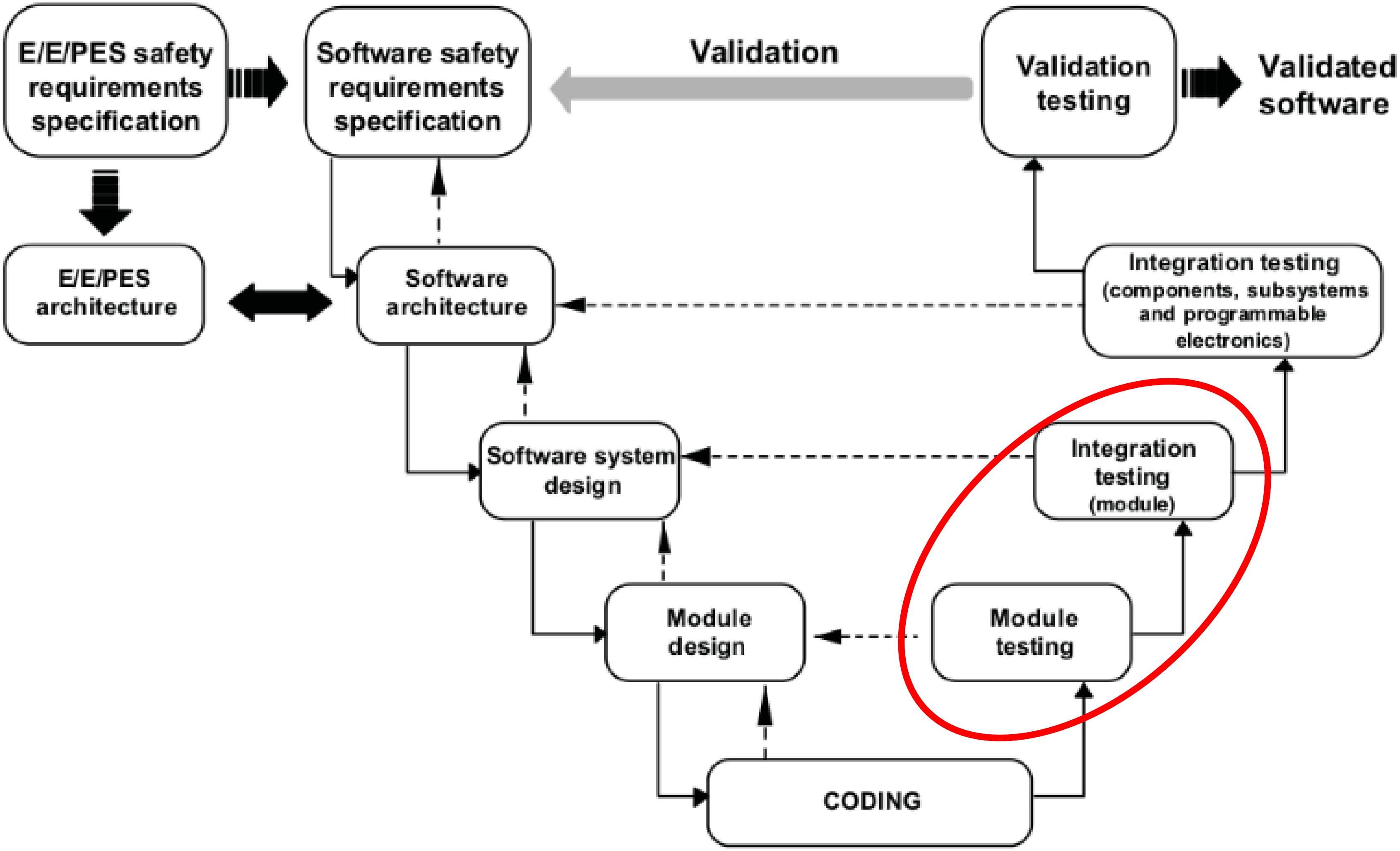
Software Verification in the Development Cycle



Software Verification

- Software Verification **proves** properties of programs. That is, given the basic problem of program P satisfying a property p we want to show that for **all possible inputs and runs** of P , the property p holds.
- Software verification is far **more powerful** than static analysis. For the same reasons, it cannot be fully automatic and thus requires user interaction. Hence, it is **complex to use**.
- Software verification does not have false negatives, only failed proof attempts. If we can prove a property, it holds.
- Software verification is used in **highly critical systems**.

The Basic Idea

- What does this program compute?
  - The index of the maximal element of the array a if it is non-empty.

- How to prove it?
  (1) We need a language in which to **formalise** such **assertions**.
  (2) We need a notion of meaning (**semantics**) for the program.
  (3) We need to way to **deduce valid assertions**.

- Floyd-Hoare logic provides us with (1) and (3).

```
i := 0;
x := 0;
while (i < n) {
  if (a[i] ≥ a[x]) {
    x := i;
    }
  i := i + 1;
  }
```

Formalizing correctness:

$$\mathrm{array}(a, n) \wedge n > 0 \Longrightarrow a[x] = \max(a, n)$$

$$\forall i.\, 0 \leq i < n \Longrightarrow a[i] \leq \max(a, n)$$

$$\exists j.\, 0 \leq j < n \Longrightarrow a[j] = \max(a, n)$$

Recall our simple programming language

- **Arithmetic** expressions:

$$a \ ::= x \mid n \mid a_1[a_2] \mid a_1 \ op_a \ a_2$$

  - Arithmetic operators: $op_a \in \{+, -, *, /\}$

- **Boolean** expressions:

$$b \ \coloneqq \text{true} \mid \text{false} \mid \text{not } b \mid b_1 op_b \ b_2 \mid a_1 op_r \ a_2$$

  - Boolean operators: $op_b \in \{and, or\}$
  - Relational operators: $op_r \in \{=, <, \leq, >, \geq, \neq\}$

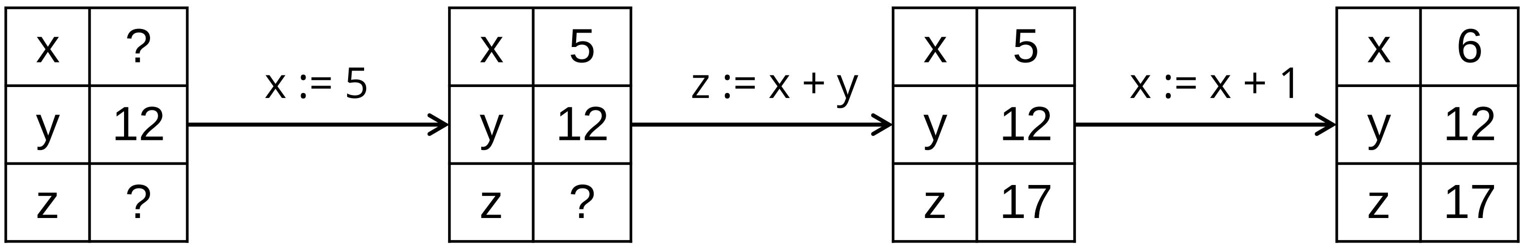
- **Statements**:

$$\mathrm{S} ::= \mathrm{x} := \mathrm{a} \mid \textbf{skip} \mid \mathrm{S1};\ \mathrm{S2} \mid \textbf{if}\ (\mathrm{b})\ \ \mathrm{S1}\ \textbf{else}\ \mathrm{S2} \mid \textbf{while}\ (\mathrm{b})\ \mathrm{S}$$

  - Labels from basic blocks omitted, only used in static analysis to derive cfg.
  - Note this abstract syntax, operator precedence and grouping statements is not covered.

Semantics of our simple language

- The semantics of an **imperative** language is state transition: the program has an ambient state, which is changed by assigning values to certain locations.
- Example:



σ $\quad\quad$ $\sigma_1 = \sigma[x/5]$ $\quad\quad$ $\sigma_2 = \sigma_1[z/17] = \sigma[x/5, z/17]$ $\quad\quad$ $\sigma_3 = \sigma_2[x/6] = \sigma[x/6, z/17]$

- Semantics in a nutshell:

Expressions evaluate to values Val (for our language integers).
Locations Loc are variable names.
A **program state** maps locations to values: $\Sigma = Loc \rightharpoonup Val$
A program maps an initial state to a final state, **if it terminates**.
Assertions are predicates over program states.

Semantics in a nutshell

- There are three major ways to denote semantics.

(1) As a relation between program states, described by an abstract machine (**operational semantics**).

(2) As a function between program states, defined for each statement of the programming langauge (**denotational semantics**).

(3) As the set of all assertions which hold for a program (**axiomatic semantics**).

- Floyd-Hoare logic covers the third aspect, but it is important that all three semantics agree.
  - We will not cover semantics in detail here, but will concentrate on how to **use** Floyd-Hoare logic to prove correctness.

Extending our simple language

- We introduce a set Var of **logical variables.**
- **Assertions** are boolean expressions, which may not be executable, and arithmetic expressions containing logical variables.

- Arithmetic assertions

$$ae \ ::= x \mid X \mid n \mid ae_1[ae_2] \mid ae_1 \ op_a \ ae_2 \mid \ f(ae_1, \dots, ae_n)$$

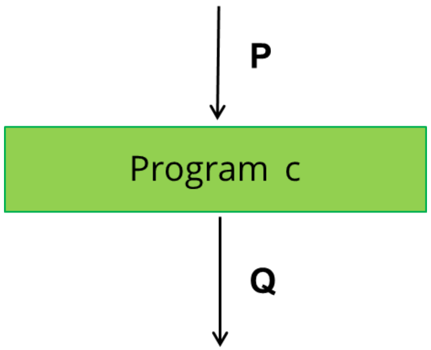
  - where $x \in Loc, X \in Var, op_a \in \{+, -, *, /\}$

- Boolean assertions:

$$be \ \coloneqq \text{true} \mid \text{false} \mid \text{not } be \mid be_1 op_b \ be_2 \mid ae_1 op_r \ ae_2$$
$$\mid p(ae_1, \dots, \ ae_n) \mid \forall X. be \mid \exists X. be$$

  - Boolean operators: $op_b \in \{\wedge, \vee, \Longrightarrow\}$
  - Relational operators: $op_r \in \{=, <, \leq, >, \geq, \neq\}$

Floyd-Hoare Triples



The basic build blocks of Floyd-Hoare logic are Hoare triples of the form $\{P\}c\,\{Q\}$.

- P, Q are assertions using variables in Loc and Var
  - e.g. x < 5 + y, Odd(x), …

- A state σ satisfies P (written $\sigma \vDash P$) iff $P[^{\sigma(x)}/_x]$ is true for all $x \in Loc$ and all possible values for $X \in Var$:
  - e.g. let

σ =

x	5
y	12
z	17

then σ satisfies x < 5 + y, Odd(x)

- A formula P describes a set of states, i.e. all states that satisfy the formula P.

Partial and Total Correctness

- **Partial correctness**: $\models \{P\}c\{Q\}$
  - c is partial correct with precondition P and postcondition Q iff, for all states σ which satisfy P and for which the execution of c terminates in some state σ' then it holds that σ' satisfies Q.
    $$\forall \sigma.\, \sigma \models P \land \exists \sigma'.\, \langle \sigma, c \rangle \to \sigma' \implies \sigma' \models Q$$
- **Total correctness**: $\models [P]c[Q]$
  - c is total correct with precondition P and postcondition Q iff, for all states σ which satisfy P the execution of c terminates in some state σ' which satisfies Q.
    i.e $\forall \sigma.\, \sigma \models P \implies \exists \sigma'.\, \langle \sigma,\ c \rangle \to \sigma' \land \sigma' \models Q$
- Examples: $\models \{true\} while(true)\ skip\ \{true\}$,
  $\not\models [true]\ while(true) skip\ \ [true]$

Reasoning with Floyd-Hoare Triples

- How do we know that $\models \{P\}c\{Q\}$ in practice ?
- Calculus to derive triples, written as $\vdash \{P\}c\{Q\}$
  - Rules operate along the constructs of the programming language (cf. operational semantics)
  - Only one rule is applicable for each construct (!)
  - Rules are of the form

$$\frac{\vdash \{P_1\}c_1\{Q_1\}, \dots, \vdash \{P_n\}c_n\{Q_n\}}{\vdash \{P\}c\ \{Q\}}$$

meaning we can derive $\vdash \{P\}c\{Q\}$ if all $\vdash \{P_i\}c_i\{Q_i\}$ are derivable.

Floyd-Hoare Rules: Assignment

- Assignment rule:

$$\frac{}{\vdash \{P[^{e}/_{x}]\}\ x := \ e\ \{P\}}$$

- $P[^{e}/_{x}]$ replaces all occurrences of the program variable x by the arithmetic expression e.

- Examples:
  - $\vdash\ \{0\ <\ 10\}\ x := \ 0\ \{x\ <\ 10\}$
  - $\vdash \{\underbrace{x - 1\ <\ 10}_{x < 11}\}\ x := \ x\ - 1\ \{x\ <\ 10\}$
  - $\vdash \{\underbrace{x\ +\ 1\ +\ x\ +\ 1\ <\ 10}_{x + x < 8}\}\ x := \ x\ +\ 1\ \{x\ +\ x\ <\ 10\}$

Rules: Sequencing and Conditional

- Sequence:

$$\frac{\vdash \{P\}\, c_1\, \{Q\} \quad \vdash \{Q\}\, c_2\, \{R\}}{\vdash \{P\}\, c_1; c_2\, \{R\}}$$

  - Needs an intermediate state predicate Q.

- Conditional:

$$\frac{\vdash \{P \land b\}\, c_1\, \{Q\} \quad \vdash \{P \land \neg b\}\, c_2\, \{Q\}}{\vdash \{P\}\, \mathbf{if}(\mathrm{b})\ c_1\, \mathbf{else}\ c_2\, \{Q\}}$$

  - Two preconditions capture both cases of b and $\neg\, b$.
  - Both branches end in the same postcondition Q.

Rules: Iteration and Skip

$$\frac{\vdash \{P \wedge b\}\ c\ \{P\}}{\vdash \{P\}\ \textbf{while}\ (b)\ c\ \{P \wedge \neg\, b\}}$$

- P is called the **loop invariant**. It has to hold both before and after the loop (but not necessarily in the whole body).
- Before the loop, we can assume the loop condition b holds.
- After the loop, we know the loop condition b does not hold.
- In practice, the loop invariant has to be **given**– this is the creative and difficult part of working with the Floyd-Hoare calculus.

$$\frac{}{\vdash \{P\}\ \textbf{skip}\ \{P\}}$$

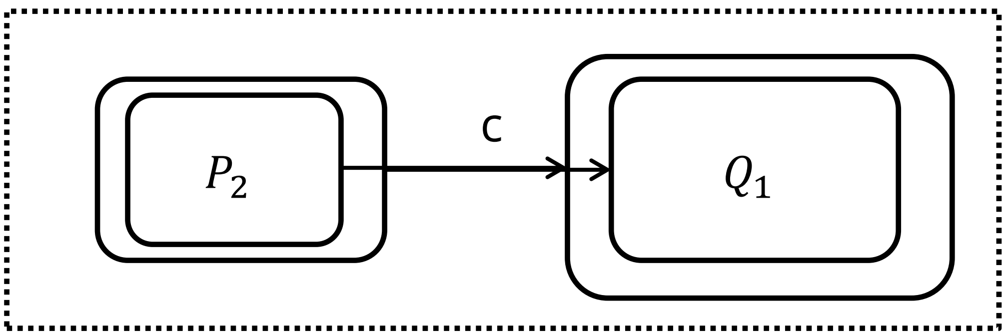
- **skip** has no effect: pre- and postcondition are the same.

Final Rule: Weakening

- Weakening is crucial, because it allows us to change pre- or postconditions by applying rules of logic.

$$\frac{P_2 \Longrightarrow P_1 \qquad \vdash \{P_1\}\ c\ \{Q_1\} \qquad Q_1 \Longrightarrow Q_2}{\vdash \{P_2\}\ c\ \{Q_2\}}$$

- We can **weaken** the precondition and **strengthen** the postcondition:
  - $\models \{P\}c\{Q\}$ means whenever c starts in a state in which P holds, it ends in a state in which Q holds. So, we can reduce the starting set, and enlarge the target set.



How to derive and denote proofs

```
// {P}
// {P_1}
x:= e;
// {P_2}
// {P_3}
while (x< n) {
    // {P_3 ∧ x < n}
    // {P_4}
    z := a
    // {P_3}
  }
// {P_3 ∧ ¬(x < n)}
// {Q}
```

- The example shows $\vdash \{P\}c\{Q\}$
- We annotate the program with valid assertions: the precondition in the preceding line, the postcondition in the following line.
- The sequencing rule is applied implicitly.
- Consecutive assertions imply weaking, which has to be proven separately.
  - In the example:

    $P \Longrightarrow P_1,$
    $P_2 \Longrightarrow P_3,$
    $P_3 \land x < n \Longrightarrow P_4,$
    $P_3 \land \neg(x < n) \Longrightarrow Q$

More Examples

P ==

$p := 1;$
$c := 1;$
$\text{while } (c \leq n) \ \{$
$\quad p := p * c;$
$\quad c := c + 1$
$\quad \}$

Q ==

$p := 1;$
$\text{while } (0 \leq n) \ \{$
$\quad p := p * n;$
$\quad n := n - 1$
$\quad \}$

R ==

$r := a;$
$q := 0;$
$\text{while } (b \leq r)\{$
$\quad r := r - b;$
$\quad q := q + 1$
$\quad \}$

Specification:

$$\vdash \{ 1 \leq n \} \ P \ \{ p = n! \}$$

Specification:

$$\vdash \{ 1 \leq n \land n = N \} \ Q \ \{ p = N! \}$$

Specification:

$$\vdash \{ a \geq 0 \land b \geq 0 \} \ R \ \{ a = b * q + r \land 0 \leq r \land r < b \}$$

Invariant:

$$p = (c - 1)!$$

Invariant:

$$p = \prod_{i=n+1}^{N} i$$

Invariant:

$$a = b * q + r \land 0 \leq r$$

How to find invariants

- Going backwards: try to split/weaken postcondition Q into negated loop-condition and „something else“ which becomes the invariant.
- Many while-loops are in fact for-loops, i.e. they count uniformly:

$$
\begin{array}{l}
\mathrm{i} := 0; \\
\textbf{while}\ (i < n)\ \{ \\
\quad \ldots; \\
\quad i := i + 1 \\
\quad \}
\end{array}
$$

- In this case:
  - If post-condition is $P(n)$, invariant is $P(i) \wedge i \leq n$.
  - If post-condition is $\forall j.\, 0 \leq j < n.\, P(j)$ (uses indexing, typically with arrays), invariant is $\forall j.\, j \leq 0 < i.\, i \leq n \wedge\ P(j)$.

Summary

- Floyd-Hoare-Logic allows us to **prove** properties of programs.
- The proofs cover all possible inputs, all possible runs.
- There is **partial** and **total correctness**:
  - Total correctness = partial correctness + termination.
- There is one rule for each construct of the programming language.
- Proofs can in part be constructed automatically, but iteration needs an **invariant** (which cannot be derived mechanically).
- Next lecture: correctness and completeness of the rules.