Systeme hoher Sicherheit und Qualität
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# Lecture 10:

# Verification Condition Generation

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Frohes Neues Jahr!

# Where are we?

- 01: Concepts of Quality
- 02: Legal Requirements: Norms and Standards
- 03: The Software Development Process
- 04: Hazard Analysis
- 05: High-Level Design with SysML
- 06: Formal Modelling with OCL
- 07: Testing
- 08: Static Program Analysis
- 09: Software Verification with Floyd-Hoare Logic
- 10: Correctness and Verification Condition Generation
- 11-12: Model Checking
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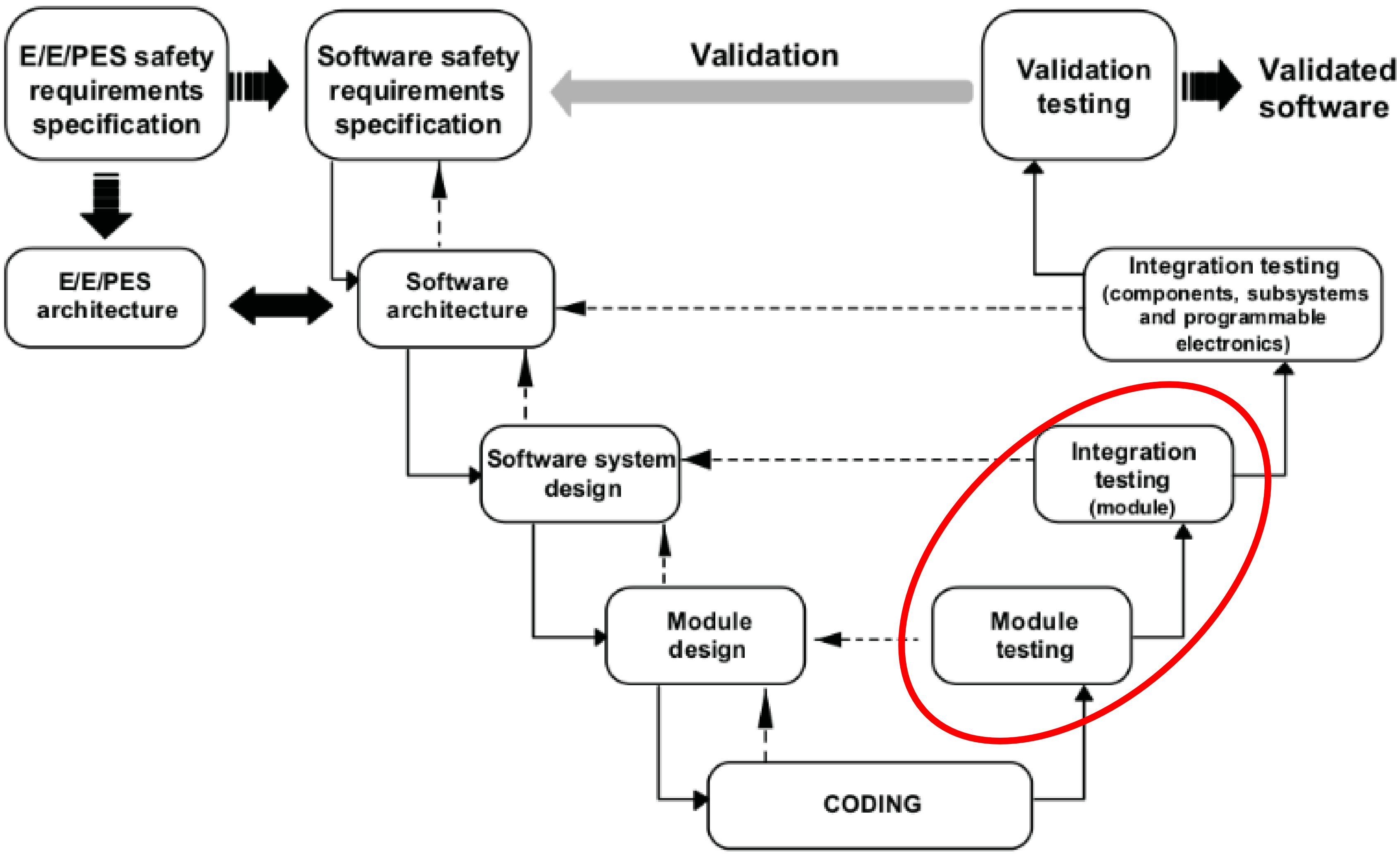
# VCG in the Development Cycle



# Introduction

- In the last lecture, we introduced Hoare triples. They allow us to state and prove correctness assertions about programs, written as $\{P\}\, p\, \{Q\}$
- We introduced two notions, namely:
  - Syntactic derivability, $\vdash \{P\}\, p\, \{Q\}$ (the actual Floyd-Hoare calculus)
  - Semantic satisfaction, $\models \{P\}\, p\, \{Q\}$
- Question: how are the two related?
- The answer to that question also offers help with a practical problem: proofs with the Floyd-Hoare calculus are exceedingly long and tedious. Can we automate them, and how?

# Correctness and Completeness

- In general, given a syntactic calculus with a semantic meaning, **correctness** means the syntactic calculus implies the semantic meaning, and **completeness** means all semantic statements can be derived syntactically.
  - Cf. also Static Program Analysis

- **Correctness** should be a basic property of verification calculi.
- **Completeness** is elusive due to Gödel‘s first incompleteness theorem:
  - Any logics which is strong enough to encode the natural numbers and primitive recursion* is incomplete.**

* Or any other notion of computation.

** Or inconsistent, which is even worse.

# Correctness of the Floyd-Hoare calculus

**Theorem** (Correctness of the Floyd-Hoare calculus)
If $\vdash \{P\}\, p\, \{Q\}$, then $\models \{P\}\, p\, \{Q\}$.

- Proof: by induction on the derivation of $\vdash \{P\}\, p\, \{Q\}$.
- More precisely, for each rule we show that:
  - If the conclusion is $\vdash \{P\}\, p\, \{Q\}$, we can show $\models \{P\}\, p\, \{Q\}$
  - For the premisses, this can be assumed.
- Example: for the assignment rule, we show that

# Completeness of the Floyd-Hoare calculus

- Predicate calculus is incomplete, so we cannot hope F/H is complete. But we get the following:

> **Theorem** (Relative completeness)
> If $\vDash \{P\}\, p\, \{Q\}$, then $\vdash \{P\}\, p\, \{Q\}$ *except* for the proofs occuring in the weakenings.

- To show this, we construct the **weakest precondition**.

> **Weakest precondition**
> Given a program c and an assertion P, the weakest precondition $wp(c, P)$ is an assertion W such that
> 1. $W$ is a valid precondition $\vDash \{W\}\, c\, \{P\}$
> 2. And it is the weakest such: for any other $Q$ such that $\vDash \{Q\}\, c\, \{P\}, W \to Q$

# Constructing the weakest precondition

- Consider a simple program and its verification:

$$\{\, x = X \wedge y = Y \}$$
$$\leftrightarrow$$
$$\{\, y = Y \wedge x = X \}$$
```
z := y;
```
$$\{\, z = Y \wedge x = X \}$$
```
y := x;
```
$$\{\, z = Y \wedge y = X \}$$
```
x := z;
```
$$\{\, x = Y \wedge y = X \}$$

- Note how proof is **constructed backwards systematically**.
- The idea is to construct the weakest precondition inductively.
- This also gives us a methodology to automate proofs in the calculus.

# Constructing the weakest precondition

- There are four straightforward cases:

  *(1)* $wp(\mathbf{skip}, P) = P$

  *(2)* $wp(X := e, P) = P\,[e\,/\,X]$

  *(3)* $wp(c_0; c_1, P) = wp(c_0, wp(c_1, P))$

  *(4)* $wp(\mathbf{if}\ b\ \{c_0\}\ \mathbf{else}\ \{c_1\}, P) = (b \wedge wp(c_0, p)) \vee (\neg\, b \wedge wp(c_1, P))$

- The complicated one is iteration (unsurprisingly, since it is the source of the computational power and Turing-completeness of the language). It can be given recursively:

  *(5)* $wp(\mathbf{while}\ b\ \{c\}, P) = (\neg\, b \wedge P) \vee wp\big(c, wp\ (\mathbf{while}\ b\ \{c\}, P)\big)$

- A closed formula can be given, but it can be infinite and is not practical. It shows the relative completeness, but does not give us an effective way to automate proofs.
- Hence, $wp(c, P)$ is not effective for proof automation, but it shows the right way: we just need something for iterations.

# Verification Conditions: Annotations

- The idea is that we have to give the invariants manually by annotating them.
- We need a language for this:
  - Arithmetic expressions and boolean expressions stays as they are.
  - Statements are augmented to **annotated statements**:

    S ::= x := a | **skip** | S1; S2 | **if** (b) S1 **else** S2
    | **assert** P | **while** (b) **inv** P S

  - Each while loop needs to its invariant annotated.
    - This is for partial correctness, total correctness also needs a variant: an expression which is strictly decreasing in a well-founded order such as $(<, \mathbb{N})$ after the loop body.
  - The assert statement allows us to force a weakening.

# Preconditions and Verification Conditions

- We are given an annotated statement $c$, a precondition P and a postcondition Q.
  - We want to know: when does $\vDash \{P\}\ c\ \{Q\}$ hold?

- For this, we calculate a **precondition** $pre(c, Q)$ and a **set** of **verification conditions** $vc(c, Q)$.
  - The idea is that if all the verification conditions hold, then the precondition holds:

$$\bigwedge_{R \in vc(c,Q)} R \Rightarrow \vDash \{pre(c,Q)\} c\ \{Q\}$$

  - For the precondition $P$, we get the additional weaking $P \Rightarrow pre(c, Q)$.

# Calculation Verification Conditions

- Intuitively, we calculate the verification conditions by stepping through the program backwards, starting with the postcondition $Q$.
- For each of the four simple cases (assignment, sequencing, case distinction and $\boldsymbol{skip}$), we calculate new current postcondition $Q$
- At each iteration, we calculate the precondition $R$ of the loop body working backwards from the invariant $I$, and get two verification conditions:
  - The invariant $I$ and negated loop condition implies $Q$.
  - The invariant $I$ and loop condition implies $R$.
- Asserting $R$ generates the verification condition $R \Rightarrow Q$.

- Let‘s try this.

# Example: deriving VCs for the factorial.

```
{ 0 <= n }
{ 1 == (1-1)! && (1- 1) <= n }
p := 1;
{ p == (1-1)! && (1- 1) <= n }
c := 1;
{ p == (c-1)! && (c- 1) <= n }
while (c <= n)
  inv (p == (c-1)! && c-1 <= n) {
  { p*c == ((c+1)-1)! && ((c+1)- 1) <= n }
  p := p* c;
  { p == ((c+1)-1)! && ((c+1)- 1) <= n }
  c := c+1;
  { p == (c-1)! && (c- 1) <= n }
  }
{ p == (c-1)! && (c- 1) <= n && ! (c <= n) }
{ p = n! }
```

VCs (unedited):

1. p == (c-1)! && (c- 1) <= n && ! (c <= n)
   ==> p= n!
2. p == (c-1)! && c-1 <= n && c<= n
   ==> p* c= ((c+1)-1)! && ((c+1)-1) <= n
3. 0 <= n ==> 1= (1-1)! && 1-1 <= n

VCs (simplified):

1. p == (c-1)! && (c- 1) <= n && c> n
   ==> p= n!
2. p == (c-1)! && c-1 <= n && c<= n
   ==> p* c= c!
2. p == (c-1)! && c-1 <= n && c<= n
   ==> c <= n
3. 0 <= n ==> 1= 0! && 0 <= n

# Formal Definition

- Calculating the precondition:

$$pre(\textbf{skip}, Q) = Q$$
$$pre(X := e, Q) = Q\,[e\,/\,X]$$
$$pre(c_0; c_1, Q = pre(c_0, pre(c_1, Q))$$
$$pre(\textbf{if}\ (b)\ c_0\ \textbf{else}\ c_1, Q) = \big(b \wedge pre(c_0, Q)\big) \vee \big(\neg\, b \wedge pre(c_1, Q)\big)$$
$$pre(\textbf{assert}\ R, Q) = R$$
$$pre\ (\textbf{while}\ (b) \textbf{inv}\ I\ c, Q) = I$$

- Calculating the verification conditions:

$$vc(skip, Q) = \emptyset$$
$$vc(X := e, Q) = \emptyset$$
$$vc(c_0; c_1, Q) = vc(c_0, pre(c_1, Q)) \cup vc(c_1, Q)$$
$$vc(\textbf{if}\ (b)\ c_0\ \textbf{else}\ c_1, Q) = vc(c_0, Q) \cup vc(c_1, Q)$$
$$vc(\textbf{while}\ (b)\ \textbf{inv}\ I\ c, Q) = vc(c, I) \cup \{I \wedge b \Rightarrow pre(c, I), I \wedge \neg b \Rightarrow Q\}$$
$$vc(\textbf{assert}\ R, Q) = \{R \Rightarrow Q\}$$

- The main definition:

$$vcg(\{P\}\ c\ \{Q\}) = \{P \Rightarrow pre(c, Q)\} \cup vc(c, Q)$$

# Correctness of VC

- The correctness calculus is correct: if we can prove all the verifcation conditons, the program is correct w.r.t to given pre- and postconditions.

- Formally:

**Theorem** (Correctness of the VCG calculus)
Given assertions $P$ and $Q$ (with $P$ the precondition and $Q$ the postcondition), and an annotated program, then

$$\bigwedge_{R \in vcg(c, Q)} R \Rightarrow \models \{P\}\, c\, \{Q\}$$

- Proof: by induction on $c$.

# Using VCG in Real Life

- We have just a toy language, but VCG can be used in real life. What features are missing?
- **Modularity**: the language must have modularity concepts, e.g. functions (as in C), or classes (as in Java), and we must be able to verify them separately.
- **Framing**: in our simple calculus, we need to specify which variables stay the same (e.g. when entering a loop). This becomes tedious when there are a lot of variables involved; it is more practical to specify which variables may change.
- **References**: languages such as C and Java use references, which allow aliasing. This has to be modelled semantically; specifically, the assignment rule has to be adapted.
- **Machine arithmetic**: programs work with machine words and floating point representations, not integers and real numbers. This can be the cause of insidious errors.

# VCG Tools

- Often use an intermediate language for VCG and front-ends for concrete programming languages.

- The Why3 toolset (http://why3.lri.fr)
  - A verification condition generator
  - Front-ends for different languages:
    C (Frama-C), Java (defunct?)

- Boogie (Microsoft Research)
  - Frontends for programming languages such C, C#, Java.
- VCC – a verifying C compiler built on top of Boogie
  - Interactive demo:
    https://www.rise4fun.com/Vcc/

# VCC Example: Binary Search

- A correct (?) binary search implementation:

```
#include <limits.h>

unsigned int bin_search(unsigned int a [], unsigned int a_len, unsigned int key)
{
  unsigned int lo= 0;
  unsigned int hi= a_len;
  unsigned int mid;

  while (lo <= hi)
     {
      mid= (lo+ hi)/2;
      if (a[mid] < key) lo= mid+1;
      else hi= mid;
    }

  if (!(lo < a_len && a[lo] == key)) lo= UINT_MAX;

  return lo;
}
```

# VCC: Correctness Conditions?

- We need to annotate the program.
- Precondition:
  - `a` is an array of length `a_len`;
  - The array `a` is sorted.
- Postcondition:
  - Let `r` be the result, then:
  - if `r` is `UINT_MAX`, all elements of `a` are unequal to `key`;
  - if `r` is not `UINT_MAX`, then `a[r] == key`.
- Loop invariants:
  - `hi` is less-equal to `a_len`;
  - everything „left“ of `lo` is less then `key`;
  - everything „right“ of `hi` is larger-equal to `key`.

# VCC Example: Binary Search

- Source code as annotated for VCC:

```
#include <limits.h>
#include <vcc.h>
unsigned int bin_search(unsigned int a [], unsigned int a_len, unsigned int key)
  _(requires \thread_local_array(a, a_len))
  _(requires \forall unsigned int i, j; i < j && j < a_len ==> a[i] <= a[j])
  _(ensures \result != UINT_MAX ==> a[\result] == key)
  _(ensures \result == UINT_MAX ==> \forall unsigned int i; i < a_len ==> a[i] !=  key)
{
  unsigned int lo= 0;
  unsigned int hi= a_len;
  unsigned int mid;

  while (lo <= hi)
    _(invariant hi <= a_len)
    _(invariant \forall unsigned int i; i < lo ==> a[i] <  key)
    _(invariant \forall unsigned int i; hi <= i && i < a_len ==>a[i] >= key)
    {
      mid= (lo+ hi)/2;
      if (a[mid] < key) lo= mid+1;
      else hi= mid;
    }
  if (!(lo < a_len && a[lo] == key)) lo= UINT_MAX;
  return lo;
}
```

# Binary Search: the Corrected Program

- Corrected source code:

```
#include <limits.h>
#include <vcc.h>
unsigned int bin_search(unsigned int a [], unsigned int a_len, unsigned int key)
  _(requires \thread_local_array(a, a_len))
  _(requires \forall unsigned int i, j; i < j && j < a_len ==> a[i] <= a[j])
  _(ensures \result != UINT_MAX ==> a[\result] == key)
  _(ensures \result == UINT_MAX ==> \forall unsigned int i; i < a_len ==> a[i] !=  key)
{
  unsigned int lo= 0;
  unsigned int hi= a_len;
  unsigned int mid;

  while (lo < hi)
    _(invariant hi <= a_len)
    _(invariant \forall unsigned int i; i < lo ==> a[i] <  key)
    _(invariant \forall unsigned int i; hi <= i && i < a_len ==>a[i] >= key)
    {
     mid= (hi-lo)/2+ lo;
     if (a[mid] < key) lo= mid+1;
     else hi= mid;
    }
  if (!(lo < a_len && a[lo] == key)) lo= UINT_MAX;
  return lo;
}
```

# Summary

- Starting from the relative completeness of the Floyd-Hoare calculus, we devised a verification condition generation (vcg) calculus which makes program verification viable.
- Verification condition generation reduces the question whether the given pre/postconditions hold for a program to the validity of a set of logical properties.
  - We do need to annotate the while loops with invariants.
  - Most of these logical properties can be discharged with automated theorem provers.
- To scale to real-world programs, we need to deal with framing, modularity (each function/method needs to be verified independently), and machine arithmetic (integer word arithmetic and floating-points).