Hopf algebra structure of incidence algebras

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Abstract
The incidence algebra of a partially ordered set (poset) carries in a natural way also a coalgebra structure, so that it becomes a weak bialgebra even a weak Hopf algebra. Here weak means that multiplication and comultiplication are not required to be coalgebra- or algebra-morphisms, respectively. A rich theory is obtained in computing modulo an equivalence relation on the set of intervals in the poset.

1 Introduction
Usually, the theory of algebras, coalgebras and bialgebras is formulated without reference to bases of the linear spaces supporting these structures. But in practice, some algebraic structures, especially comultiplication, is given by referring to specific bases.

Incidence algebras of posets are algebras having a canonical base, namely the set of intervals of the poset. So their study can give hints for a general theory of based algebraic objects, i.e. objects furnished with a specific base.

Many interesting incidence algebras are no bialgebras, multiplication and comultiplication are not compatible. Nevertheless they possess an antipode. So we introduce the issues of weak bialgebra and weak Hopf algebra and look what remains valid from the theory of bialgebras and Hopf algebras.

It is supposed that the reader is familiar with tensor products of linear spaces. The elements of coalgebras and bialgebras are repeated in a concise form. In order to ease the access we avoid Sweedler’s notation ([10]) and confine the theory to linear spaces. Most results generalize to modules over a commutative ring. In the next section, Section 2, we fix our notations and present some generalizations and examples needed later on. Section 3 is devoted to the convolution algebra of a coalgebra-algebra pair. The more involved dual problem, to define a coalgebra for an algebra-coalgebra pair is treated in Section 4, except for Example 4.1 Section 4 can be skipped at first reading. Section 5 contains the basics of weak bialgebras and weak Hopf algebras.

The next sections are devoted to incidence algebras and their weak Hopf algebra structure. In Section 6 the interval algebra and coalgebra of a poset are defined
w.r.t. a bialgebra compatible equivalence relation, a strengthening of Schmitt’s issue of order compatibility ([6]). Section 7 gives the usual definition of incidence algebra of a poset and shows that it can be embedded in the convolution algebra of the interval coalgebra-algebra pair. Thus, the Möbius function is the antipode. Section 8 presents classical examples of commutative incidence algebras, which are weak Hopf algebras. Final Section 9 treats morphisms of incidence algebras.

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2 Algebras and Coalgebras

Throughout let $K$ be a field. Applications between $K$-linear spaces are supposed to be linear. They are uniquely defined by their values on a base. We define linear applications mostly by their values on a specific base.

All $K$-algebras are supposed to be unitary. Then a $K$-algebra $A$ is a $K$-linear space $A$ with multiplication $\cdot$ and unit $1 = 1_A$, in the classical notation a triple $A = (A, \cdot, 1)$ or, in the tensor product notation, a triple $(A, \nabla, \eta)$ with linear applications

$$\nabla : A \otimes A \to A, \quad \eta : K \to A,$$

such that associativity and unitarity hold,

$$\nabla \circ (\nabla \otimes \text{id}_A) = \nabla \circ (\text{id}_A \otimes \nabla),$$

$$\nabla \circ (\eta \otimes \text{id}_A) = \text{id}_A = \nabla \circ (\text{id}_A \otimes \eta).$$

Reversing the arrows a coalgebra is defined as a triple $C = (C, \triangle, \epsilon)$ with linear applications called comultiplication and counit

$$\triangle : C \to C \otimes C, \quad \epsilon : C \to K,$$

such that coassociativity (1) and counitarity (2) hold,

$$(\triangle \otimes \text{id}_C) \circ \triangle = (\text{id}_C \otimes \triangle) \circ \triangle,$$

$$(\epsilon \otimes \text{id}_C) \circ \triangle = \text{id}_C = (\text{id}_C \otimes \epsilon) \circ \triangle. \quad (1, 2)$$

By (2) comultiplication is an injective linear mapping. This fact is dual to the fact that $\nabla$ is surjective.

An algebra is commutative iff $\nabla \circ t = \nabla$, where for a linear space $L$ the twist mapping $t$ is defined as

$$t : L \otimes L \to L \otimes L, \quad t(a \otimes b) = b \otimes a.$$  

$t$ is a linear isomorphism. A coalgebra is called cocommutative if $t \circ \triangle = \triangle$. Generally the opposite algebra $A^{\text{op}} := (A, \nabla^{\text{op}}, \eta)$ of the algebra $A$ is defined by $\nabla^{\text{op}} := \nabla \circ t$; similarly the opposite coalgebra $C^{\text{op}} := (C, \triangle^{\text{op}}, \epsilon)$ of the coalgebra $C$ by $\triangle^{\text{op}} := t \circ \triangle$.

A linear application $\alpha : A \to A'$ between algebras is an algebra morphism iff

$$\nabla' \circ (\alpha \otimes \alpha) = \alpha \circ \nabla, \quad \eta' = \alpha \circ \eta.$$  

A linear application $\gamma : C \to C'$ between coalgebras is, by definition, a coalgebra morphism if

$$(\gamma \otimes \gamma) \circ \triangle = \Delta' \circ \gamma, \quad \epsilon = \epsilon' \circ \gamma.$$
The tensor product \((A \otimes A', \nabla, \eta)\) of two algebras \((A, \nabla, \eta)\) and \((A', \nabla', \eta')\) is again an algebra with multiplication
\[
\nabla := (\nabla \otimes \nabla') \circ T
\]
and unit
\[
\nabla := \eta \otimes \eta'
\]
Generally, \(T = T_{L,L'}\) denotes the capital twist mapping for the linear spaces \(L, L'\), defined by
\[
T : (L \otimes L') \rightarrow (L \otimes L'), \quad (a \otimes a') \rightarrow (a \otimes a').
\]
In the sequel we will omit the brackets. Clearly \(T\) is a linear isomorphism and \(T \circ T = \text{id}\).

The tensor product \((C \otimes C', \Delta, \epsilon)\) of two coalgebras \((C, \Delta, \epsilon)\) and \((C', \Delta', \epsilon')\) is again a coalgebra with comultiplication
\[
\Delta := T \circ (\Delta \otimes \Delta')
\]
and counit
\[
\epsilon := \epsilon \otimes \epsilon'
\]
Overlining and underlining is compatible with passing to the opposite,
\[
(\nabla)^{op} = \nabla^{op}, \quad (\Delta)^{op} = \Delta^{op}.
\]
The proof is straightforward using
\[
T \circ t = (t \otimes t) \circ T, \quad t \circ T = T \circ (t \otimes t).
\]

**Proposition 2.1**
1. The tensor product of algebra or coalgebra morphisms is again an algebra or coalgebra morphism, respectively.
2. For an algebra (coalgebra) the twist mapping \(t\) is an algebra (coalgebra) morphism.
3. For algebras (coalgebras) the capital twist mapping \(T\) is an algebra (coalgebra) morphism.

**Proof** We prove the algebra cases. The coalgebra cases prove dually.

1. Let \(\alpha_1 : A_1 \rightarrow A'_1\) be algebra morphisms and \(\alpha := \alpha_1 \otimes \alpha_2 : A_1 \otimes A_2 \rightarrow A'_1 \otimes A'_2\). We know \(\eta'_1 = \alpha_1 \circ \eta_1\) and \(\nabla'_1 \circ (\alpha_1 \otimes \alpha_2) = \alpha_1 \circ \nabla_1\), whence \(\nabla'_1 \otimes \nabla'_2 = (\alpha_1 \otimes \alpha_2) \circ (\eta_1 \otimes \eta_2)\) and \((\nabla'_1 \otimes \nabla'_2) \circ ((\alpha_1 \otimes \alpha_1) \otimes (\alpha_2 \otimes \alpha_2)) = (\alpha_1 \otimes \alpha_2) \circ (\nabla_1 \otimes \nabla_2)\). On the other hand \(T \circ \alpha \otimes \alpha = ((\alpha_1 \otimes \alpha_1) \otimes (\alpha_2 \otimes \alpha_2)) \circ T\) so that \(\nabla' \circ (\alpha \otimes \alpha) = \alpha \circ \nabla\), where \(\nabla := (\nabla_1 \otimes \nabla_2) \circ T\) and \(\nabla' := (\nabla'_1 \otimes \nabla'_2) \circ T\) denote the multiplications of \(A_1 \otimes A_2\) and \(A'_1 \otimes A'_2\), respectively.
2. The proof is straightforward.
3. Let \((A, \triangledown, \eta)\) and \((A', \triangledown', \eta')\) be algebras. We have to show that \(T_{A,A'} : (A \otimes A, (\triangledown \otimes \triangledown) \circ T_{A,A}, \eta \otimes \eta) \rightarrow (A_2 \otimes A'_2, (\triangledown_2 \otimes \triangledown'_2) \circ T_{A_2,A'_2}, \eta_2 \otimes \eta'_2)\) is an algebra morphism. Here we have used the following algebras \((A, \triangledown, \eta) := (A \otimes A', (\triangledown \otimes \triangledown') \circ T_{A,A'}, \eta \otimes \eta')\), \((A_2, \triangledown_2, \eta_2) := (A \otimes A, (\triangledown \otimes \triangledown) \circ T_{A,A}, \eta \otimes \eta)\) and similarly \((A'_2, \triangledown'_2, \eta'_2)\).

Clearly \(T_{A,A'}\) respects the unit. The proof of the other condition
\[
((\triangledown_2 \otimes \triangledown'_2) \circ T_{A_2,A'_2}) \circ (T_{A,A'} \otimes T_{A,A'}) = T_{A,A'} \circ ((\triangledown \otimes \triangledown) \circ T_{A,A})
\]
is straightforward but tedious. It is left to the reader.

The Eckmann-Hilton argument is needed to decide, under which conditions \(\triangledown\) is an algebra morphism.

Theorem 2.2 Let \(A = (A, \triangledown, \eta)\) and \(A' = (A, \triangledown', \eta')\) be \(K\)-algebras on the same linear space \(A\). If \(\triangledown' : (A \otimes A, \triangledown, \eta) \rightarrow (A, \triangledown, \eta)\) is an algebra morphism, then \(\triangledown' = \triangledown, \eta' = \eta\) and \(A\) is commutative.

In fact this is a theorem on monoids ([4] 3.1) since the linear structure of \(A\) is not needed in the proof. Even more, associativity is not needed, too.

**Proof** First we show \(\eta' = \eta\), i.e. \(1_A' = 1_A\) where \(1_A := \eta(1), 1_A' := \eta'(1)\).

\[
1_A = \triangledown(1_A \otimes 1_A) = \triangledown(\triangledown'(1_A' \otimes 1_A) \otimes \triangledown'(1_A \otimes 1_A'))
\]

\[
= \triangledown \circ (\triangledown' \otimes \triangledown')(1_A' \otimes 1_A) \otimes (1_A \otimes 1_A')
\]

\[
= \triangledown' \circ (\triangledown \otimes \triangledown) \circ T((1_A' \otimes 1_A) \otimes (1_A \otimes 1_A'))
\]

\[
= \triangledown' \circ (\triangledown \otimes \triangledown)(1_A' \otimes 1_A) \otimes (1_A \otimes 1_A')
\]

\[
= \triangledown'(1_A' \otimes 1_A') = 1_A'.
\]

In the second equation we used unitarity of \(\triangledown'\) and in the fourth that \(\triangledown'\) is an algebra morphism w.r.t. \(A\). The fifth equation holds since the capital twist \(T\) does not change the argument.

Next we show \(\triangledown' \circ t = \triangledown\) and then \(\triangledown' = \triangledown\), which completes the proof. Within the diagram
the right parallelogram commutes since $\nabla'$ is an algebra morphism, the other cells commute trivially. Furthermore both horizontal compound arrows from $A \otimes A$ to $A \otimes A$ are the identity. So $\nabla' \circ t = \nabla$ is proved. The remaining equation $\nabla' = \nabla$ proves similarly, replacing $t$ with id and interchanging $K$ with $A$ in the left capital twist $T$.

**Corollary 2.3**

1. The multiplication $\nabla : A \otimes A \to A$ of an algebra $A$ is an algebra morphism if and only if $A$ is a commutative algebra.

2. The comultiplication $\Delta : C \to C \otimes C$ of a coalgebra $C$ is a coalgebra morphism if and only if $C$ is cocommutative.

**Proof** 1. If $\nabla$ is an algebra morphism, apply Theorem 2.2 with $\nabla' = \nabla$ to see that $A$ is commutative. The converse is well known and proves easily. 2. proves with the dual of Theorem 2.2.

Often comultiplication and counit of a coalgebra $C = (C, \Delta, \epsilon)$ are defined by means of a specific base $B$ of the linear space $C$.

**Example 2.1** Let $C$ be a $K$-linear space and $B$ a base of $C$. Define linear applications

$$\Delta_B : C \to C \otimes C, \quad \Delta_B (b) := b \otimes b \quad \text{for } b \in B,$$

$$\epsilon_B : C \to K, \quad \epsilon_B (b) := 1_K \quad \text{for } b \in B.$$

then $(C, \Delta_B, \epsilon_B)$ is a coalgebra.

A converse of Example 2.1 holds.

**Proposition 2.4** Let $(C, \Delta, \epsilon)$ be a coalgebra and $B := \{b \in C \mid \Delta (b) = b \otimes b, \epsilon (b) = 1_K\}$, then $B$ is linearly independent in $C$.


**Example 2.2** Let $C$ be a linear space with a base $B = \{b_n \mid n \in \mathbb{N}_0\}$ which is totally ordered by it’s enumeration. Define for $n \in \mathbb{N}_0$

$$\Delta_B (b_n) := \sum_{k=0}^{n} b_k \otimes b_{n-k},$$

$$\epsilon_B (b_n) := \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{else} \end{cases}.$$

Then $(C, \Delta_B, \epsilon_B)$ is a coalgebra.
Example 2.3 We need characteristic 0, so let $K=\mathbb{C}$. The algebra $(\mathbb{C}[X], \cdot, 1) = (\mathbb{C}[X], \triangledown, \eta)$ of polynomials has two natural bases with the corresponding natural coalgebra structures from Example 2.2.

For more convenient representation of applications we use the natural algebra isomorphism

$$\mathbb{C}[X] \otimes \mathbb{C}[X] \to \mathbb{C}[X, Y], \quad X^m \otimes X^n \mapsto X^m Y^n$$

and identify both spaces. Then the multiplication $\triangledown$ of the algebra of polynomials amounts to setting $Y = X$.

1. Base $B_1 = \{X^n \mid n \in \mathbb{N}_0\}$ generates the coalgebra $(\mathbb{C}[X], \Delta_1, \epsilon_1)$ with

$$\Delta_1 (X^n) = \sum_{k=0}^{n} X^k Y^{n-k}.$$  

2. Base $B_2 = \{X^n \mid n \in \mathbb{N}_0\}$ generates the coalgebra $(\mathbb{C}[X], \Delta_2, \epsilon_2)$. Since

$$\Delta_2 (X^n) = \sum_{k=0}^{n} \binom{n}{k} X^k Y^{n-k} = (X + Y)^n \quad (4)$$

this coalgebra is sometimes called binomial coalgebra. By linearity of $\Delta_2$ we get

$$\Delta_2 (f) (X, Y) = f(X + Y) \quad \text{for } f \in \mathbb{C}[X].$$

Both counits coincide, $\epsilon_1 = \epsilon_2$, (see Example 2.2) and

$$\epsilon_i (f) = f(0), \quad \text{for } f \in \mathbb{C}[X].$$

The above examples being all cocommutative, we finally present a non-cocommutative one.

Example 2.4 Let $M_n$ be the linear space of square $n \times n$ matrices on the field $K$ and $\{B_{i,j} \mid i, j = 1 \ldots n\}$ the standard base. Defining

$$\Delta (B_{i,j}) := \sum_{k=1}^{n} B_{i,k} \otimes B_{k,j},$$

$$\epsilon(B_{i,j}) := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}$$

$(M_n, \Delta, \epsilon)$ becomes a coalgebra, which is not cocommutative. For an arbitrary matrix $A \in M_n$ the counit $\epsilon$ applies $A$ to the trace of $A$. 

7
3 Algebra structures on spaces of linear applications

In this section we study natural algebra structures on spaces $H$ of linear applications. Coalgebra structures will follow in the next section.

**Theorem 3.1** Let $A = (A, \nabla, \eta)$ be an algebra and $C = (C, \Delta, \epsilon)$ a coalgebra on a field $K$ and $H := \text{Hom}_K(C, A)$ the linear space of $K$-linear applications from $C$ to $A$. Define for $f, g \in H$ the product, called convolution,

$$f \ast g := \nabla \circ (f \otimes g) \circ \Delta$$

and the unit

$$u := \eta \circ \epsilon.$$

Then $\mathcal{H}(C, A) := (H, \ast, u)$ is a $K$-algebra, called convolution algebra of $(C, A)$.

If $A$ is commutative or $C$ cocommutative, then the convolution algebra $\mathcal{H}(C, A)$ is commutative, too.

The following commuting diagrams illustrate the convolution and the unit.

**Proof** We first check unitarity regarding the following diagram.

The definition of convolution implies that the upper cell commutes. The inner square commutes by the definition of $u$. The left cell commutes, since $\epsilon$ is a counit, the right one since $\eta$ is a unit. Obviously, the lower cell commutes. So, the outer square commutes as well, i.e. $f \ast u = f$. The equation $u \ast f = f$ proves similarly.

Next we check associativity.
The left cell commutes by coassociativity of $\triangle$, the right one by associativity of $\triangledown$. Associativity of the tensor product implies, that the upper and lower specification of the lower arrow of the central square coincide. Then, by definition of convolution the central square commutes and so does the lower cell. Again the definition of convolution implies that the outer large square commutes and also the upper cell (with the inner specification of the arrows). We showed that the two specifications of the upper arrow coincide, $(f \ast g) \ast h = f \ast (g \ast h)$.

For the last assertion on commutativity we know $\triangledown \circ t = \triangledown$, whence $f \ast g = \triangledown \circ (f \otimes g) \circ \Delta = \triangledown \circ t \circ (f \otimes g) \circ \Delta = \triangledown \circ (g \otimes f) \circ \Delta = g \ast f$. The cocommutative case runs similarly.

**Example 3.1** For a coalgebra $C = (C, \triangle, \epsilon)$ the dual $C^* := (C^*, \ast, \epsilon)$ is an algebra with multiplication $\Delta^* |_{C^* \otimes C^*}$ being the dual of $\Delta$ restricted to $C^* \otimes C^* \hookrightarrow (C \otimes C)^*$ and the counit $\epsilon$ of $C$ being the unit of $C^*$.

**Example 3.2** Applying the preceding example to the coalgebras of polynomials (Example 2.3) we get natural algebra isomorphisms

$$(\mathcal{C}[X], \Delta_1, \epsilon_1) \ast (1, 1) \rightarrow (\mathcal{C}[X], \cdot, 1), \quad f \mapsto \sum_{n=0}^{\infty} f(X^n)X^n,$$

$$(\mathcal{C}[X], \Delta_2, \epsilon_2) \ast (1, 1) \rightarrow (\mathcal{C}[X], \cdot, 1), \quad f \mapsto \sum_{n=0}^{\infty} \frac{f(X^n)}{n!} X^n.$$

Here $(\mathcal{C}[X], \cdot, 1)$ denotes the algebra of formal power series.

**Example 3.3** The dual of the coalgebra $(M_n, \Delta, \epsilon)$ in Example 2.4 is canonically isomorphic to the matrix algebra $(M_n, \cdot, U)$, $U$ denoting the $n \times n$ unit matrix. The isomorphism $(M_n, \cdot, U) \rightarrow (M_n, \Delta, \epsilon)^*$ maps $A = (a_{i,j})_{i,j=1,\ldots,n}$ to the linear form given by $(B_{i,j} \mapsto a_{i,j})_{i,j=1,\ldots,n}$, especially $U \mapsto \epsilon$. 
Passing from pairs \((C, A)\) to the convolution algebra \(\mathcal{H}(C, A)\) is compatible with morphisms, i.e. this is a bifunctor, contravariant in the coalgebras and covariant in the algebras.

**Proposition 3.2** Let \(A, A'\) be algebras, \(C, C'\) coalgebras and \(\alpha : A \to A'\) an algebra morphism, \(\gamma : C' \to C\) a coalgebra morphism, then

\[
h : \text{Hom}_K(C, A) \to \text{Hom}_K(C', A') , \quad f \mapsto \alpha \circ f \circ \gamma
\]

is an algebra morphism from the convolution algebra \(\mathcal{H}(C, A)\) to the convolution algebra \(\mathcal{H}(C', A')\).

**Proof** We have to show

\[
h(\varphi \ast \psi) = h(\varphi) \ast h(\psi) , \quad h(u) = u'.
\]

First the unit is applied to the unit: We know \(\alpha \circ \eta = \eta'\) since \(\alpha\) is an algebra morphism, and \(\epsilon \circ \gamma = \epsilon'\) since \(\gamma\) is a coalgebra morphism and \(u = \eta \circ \epsilon\). Then \(h(u) = \alpha \circ \eta \circ \epsilon \circ \gamma = \eta' \circ \epsilon' = u'\).

Regard the following diagram.

The definition of convolution implies that the inner and outer squares commute. The upper and lower cells commute by the definition of \(h\) and functoriality of the tensor product. The left cell commutes, since \(\gamma\) is a coalgebra morphism, the right since \(\alpha\) is an algebra morphism. So, the upper arrow equals \(h(\varphi \ast \psi)\) (upper cell) and equals also \(h(\varphi) \ast h(\psi)\) (outer square). We are done. \(\square\)

The opposite of the convolution algebra is the convolution algebra of the opposites.
**Proposition 3.3** Let $A$ be an algebra, $C$ a coalgebra. Then

$$\mathcal{H}^{\text{op}}(C, A) = \mathcal{H}(C^{\text{op}}, A^{\text{op}}).$$

**Proof** Denote with $\ast^{\text{op}}$ the convolution $f \ast^{\text{op}} g = \nabla^{\text{op}} \circ (f \otimes g) \circ \Delta^{\text{op}}$ of the convolution algebra $\mathcal{H}(C^{\text{op}}, A^{\text{op}})$. We have to show

$$f \ast^{\text{op}} g = g \ast f, \quad f, g \in \text{Hom}_K(C, A).$$

But this is plain since the diagram

$$\begin{array}{ccc}
L & \xrightarrow{f \ast^{\text{op}} g} & L \\
\downarrow{\Delta} & & \downarrow{\nabla} \\
L \otimes L & \xrightarrow{g \otimes f} & L \otimes L \\
\downarrow{t} & & \downarrow{t} \\
L \otimes L & \xleftarrow{f \otimes g} & L \otimes L
\end{array}$$

commutes. \(\square\)

Convolution is compatible with the tensor product.

**Proposition 3.4** Let $A, A'$ be algebras, $C, C'$ coalgebras and let $f, g \in \mathcal{H} := (\text{Hom}_K(C, A), \ast, u)$ and $f', g' \in \mathcal{H}' := (\text{Hom}_K(C', A'), \ast, u')$ be applications in the convolution algebras. Then, in the convolution algebra $\mathcal{H} := (\text{Hom}_K(C \otimes C', A \otimes A'), \ast, u \otimes u')$, one has

$$(f \otimes f') \ast (g \otimes g') = (f \ast g) \otimes (f' \ast g').$$

If $f, g$ are invertible for convolution, then their tensor product is invertible too, and

$$(f \otimes f')^{*-1} = f^{*-1} \otimes f'^{*-1}.$$  

**Proof** Regard the following diagram.

\(^1\)We use the same symbol $\ast$ for the convolution in the three convolution algebras.
The defining diagrams for \( f \ast g \) and \( f' \ast g' \) together imply that the upper cell with the inner applications commutes. The outer square with \( (f \otimes f') \ast (g \otimes g') \) on top commutes by definition of convolution in \( \mathcal{H} \). The other three cells commute trivially. So both applications on the top horizontal arrow coincide.

Setting \( g = f^{*-1} \) and \( g' = f'^{*-1} \) the last assertion derives directly from the first. \( \square \)
4 Coalgebra structures on spaces of linear applications

In this section we study coalgebra structures on subspaces of $H = \text{Hom}_K(L, L')$. Compared to the algebra structure in the last section, the coalgebra structure requires a preassigned base but needs only very few structure on $L, L'$.

**Proposition 4.1** Let $L$ be a $K$-linear space endowed with a unit $\eta : K \to L$, $1_L := \eta(1_K)$, and let $L'$ be a space with unit $\eta' : K \to L'$ and counit $\epsilon' : L' \to K$, and suppose $\epsilon' \circ \eta' = \text{id}_K$. Let $B \subseteq H := \text{Hom}_K(L, L')$ be a linear independent set with $b \circ \eta = \eta'$ for all $b \in B$. Let $H_B \subseteq H$ be the span of $B$, then the coalgebra $(H_B, \triangle_B, \epsilon_B)$ of Example 2.1 has the property

$$\triangle_B(f) = \sum_i x_i b_i \otimes b_i , \quad f = \sum_i x_i b_i \in H_B ,$$

$$\epsilon_B(f) = \epsilon' \circ f \circ \eta(1_K) = \epsilon' \circ f(1_L) , \quad f \in H_B .$$

**Proof** The first equation follows by linearity from the definition of $\triangle_B$ in Example 2.1. For $b \in B$ our assumptions imply $\epsilon_B(b) = 1_K = \epsilon' \circ \eta'(1_K) = \epsilon' \circ b \circ \eta(1_K) = \epsilon' \circ b(1_L)$. These equations extend by linearity to $f \in H_B$. □

Let $L, L'$ be linear spaces with unit $\eta$ respectively $\eta'$. A base of a subspace of the morphism space $\text{Hom}_K(L, L')$ with the property $b \circ \eta = \eta'$ for all base elements $b$, will be called a unitary base.

**Example 4.1** Let $\mathcal{A} = (A, \vee, \eta)$ be an algebra structure on $L = A$ and set $L' = K$, then $\text{Hom}_K(\mathcal{A}, K) = A^*$ is the dual space of $A$. With the notations of Proposition 4.1 set $1_A = \eta(1_K)$, $\eta' = \text{id}_K$, $\epsilon' = \text{id}_K$. We show that there exists a unitary base $B$ of $A^*$, i.e. $b(1_A) = 1_K$, $b \in B$.

It is sufficient to show, that $H_1 := \{ f \in A^* \mid f(1_A) = 1_K \}$ spans $H$. Let $f \in H$. First, in case $f(1_A) \neq 0$ we get $h := f/f(1_A) \in H_1$ and $f$ is a multiple of $h$. In the remaining case $f(1_A) = 0$ select a function $g \in H_1$ and set $h := f + g$.

Then $h(1_A) = f(1_A) + g(1_A) = 1_K$, whence $h \in H_1$, too. So $f$ equals the difference $h - g$ of functions in $H_1$.

The assumptions of Proposition 4.1 are verified with $H_B = A^*$. So $(A^*, \triangle_B, \epsilon_B)$ is a coalgebra and $\epsilon_B(f) = f(1_A)$ is the evaluation of $f \in A^*$ at the point $1_A$.

Well known is the case $\dim_K A < \infty$. Let $B$ be the dual base of a base $\{a_1, \ldots, a_n\}$ of $A$ such that $1_A = \sum_{i=1}^n a_i$. The last condition implies $b(1_A) = 1_K$, $b \in B$, our assumption in Proposition 4.1. Then $\triangle_B = g^{-1} \circ \nabla^* : A^* \to A^* \otimes A^*$ (Proposition 4.3 below with $A = A'$, $C = C' = K$), where $\nabla^* : A^* \to (A \otimes A)^*$ is the dual of the multiplication of $A$ and $g : A^* \otimes A^* \to (A \otimes A)^*$ the natural isomorphism (see Lemma 4.2). We have the remarkable result, that in finite dimensions the coalgebra structure on $A^*$ does not depend on the specific base $B$.

In Section 3 we have seen that $\text{Hom}_K(C, A)$ has a natural algebra structure. Here we ask: Is $\text{Hom}_K(A, C)$ a coalgebra? As we know from Example 4.1 this
problem has no canonical solution, but it solves canonically w.r.t. a base. We start
with some preparation. The following well known lemma proves easily by means
of the universal property of tensor products.

**Lemma 4.2** For $K$-linear spaces $A$, $A'$, $C$ and $C'$ let

$$H := \text{Hom}_K(A, C), \quad H' := \text{Hom}_K(A', C'), \quad H := \text{Hom}_K(A \otimes A', C \otimes C')$$

Define the linear application

$$\varrho : H \otimes H' \to H, \quad f \otimes f' \mapsto f \otimes f',$$

where $f \otimes f'$ denotes the tensor product of the linear applications $f$, $f'$, whereas $f \otimes f'$ denotes the tensor product of elements of the spaces $H$ and $H'$. Then $\varrho$ is
injective and an isomorphism if the spaces have finite dimensions.

**Proof** By the universal property of the tensor product we get injections

$$H \times H' \hookrightarrow \text{Bilin}_K(A \times A', C \otimes C') \hookrightarrow H.$$

Now, if $\varrho(\sum f_i \otimes f'_i) = 0$, i.e. $\sum f_i \otimes f'_i = 0$ then $\sum f_i \times f'_i = 0$. It’s image
$\sum f_i \otimes f'_i$ under the canonical bilinear application $H \times H' \to H \otimes H'$ equals zero, too.

In the finite dimensional case one easily proves $\dim_K H \otimes H' = \dim_K H < \infty$, whence $\varrho$ is an isomorphism. □

Next we give formulas for calculating the inverse of $\varrho$. They had already been
applied in Example 4.1.

**Proposition 4.3** Let $A = (A, \cdot, \eta)$ and $A' = (A', \cdot', \eta')$ be algebras and $C = (C, \Delta, \epsilon)$ and $C' = (C', \Delta', \epsilon')$ coalgebras. With the notations from Lemma 4.2
define the (non-linear) split mapping

$$s : H \to H \otimes H', \quad F \mapsto F_1 \otimes F_2,$$

where the linear applications $F_1 : A \to C$, $F_2 : A' \to C'$ are defined by

$$F_1(x) := (\text{id}_C \otimes \epsilon') \circ F(x \otimes 1_{A'}), \quad \text{for } x \in A,
F_2(x') := (\epsilon \otimes \text{id}_{C'}) \circ F(1_A \otimes x'), \quad \text{for } x' \in A'.$$

Then $s$ is quadratic, $s(kF) = k^2 s(F)$, $k \in K$, and

$$s \circ \varrho (f \otimes f') = \epsilon \circ f(1_A) \cdot \epsilon' \circ f'(1_{A'}), \quad f \otimes f', \quad f \in H, f' \in H'. \quad (5)$$
Proof We set $F := g(f \otimes f') = f \otimes f'$. For $x \in A$, $x' \in A'$ we then have $F(x \otimes x') = f(x) \otimes f'(x')$. Since $g$ is injective it is sufficient to prove, that both sides of equation (5) have equal images under $g$.

$$g \circ s \circ g(f \otimes f')(x \otimes x') = F_1 \otimes F_2 (x \otimes x')$$

$$= (id_C \otimes \epsilon') \circ F(x \otimes 1_{A'}) \otimes (\epsilon \otimes id_{C'}) \circ F(1_A \otimes x')$$

$$= (f(x) \otimes \epsilon'(f'(1_{A'}))) \otimes (\epsilon(f(1_A) \otimes f'(x'))$$

$$= \epsilon'(f'(1_{A'})) \cdot (\epsilon(f(1_A) \otimes f'(x') \otimes x)$$

$$= g \circ (\epsilon'(f'(1_{A'})) \cdot (\epsilon(f(1_A) \otimes f'(x') \otimes x')$$

For the fourth equality we used the canonical isomorphisms $C \otimes K \simeq C$, $K \otimes C \simeq C$. Since these equations hold for all $x \otimes x' \in A \otimes A'$, we are done.

In (5) the constant factor on the right hand side can be zero. We are looking for conditions, such that this factor becomes 1.

Corollary 4.4 Under the assumptions of Proposition 4.3 suppose that the underlying spaces of $A$ and $C$ are identical, $A = C$, and that counit and unit are compatible, $\epsilon \circ \eta = id_K, \quad \epsilon' \circ \eta' = id_K$.

Regard the affine spaces

$$H_1 := \{ f \in H \mid f(1_A) = 1_A \}, \quad H'_1 := \{ f' \in H' \mid f'(1_{A'}) = 1_{A'} \}$$

$$H_1 := \{ F \in H \mid F(1_{A \otimes A'}) = 1_{A \otimes A'} \}$$

of linear mappings respecting the unit. Then

$$s(H_1) \subseteq H_1 \otimes H'_1$$

$$s \circ g(f \otimes f') = s(f \otimes f') = f \otimes f', \quad f \in H_1, f' \in H'_1.$$

Proof If $F \in H_1$ then, using compatibility of counit and unit, $F_2(1_{A'}) = (\epsilon \otimes id_{C'}) \circ F(1_A \otimes 1_{A'}) = (\epsilon \otimes id_{C'}) (1_A \otimes 1_{A'}) = 1_{A'}$ that is $F_2 \in H_2$ and similarly $F_1 \in H_1$. We get $s(F) = F_1 \otimes F_2 \in H_1 \otimes H'_1$. The last assertion is just (5), where the coefficients on the right hand side equal 1 by our assumptions.

Our question "Is $\text{Hom}_K(A, C)$ a coalgebra?" can now be answered affirmatively, but only on suitable subspaces.

Proposition 4.5 Let $L$ be a $K$-linear space, which supports an algebra $A = (L, \cdot, \eta)$ and a coalgebra $C = (L, \vartriangle, \epsilon)$. Suppose that the structures of $A$ and $C$ are compatible in the weak sense, that the unit $\eta$ is a coalgebra morphism, i.e.

$$\epsilon \circ \eta = id_K, \quad \vartriangle \circ \eta = \eta \otimes \eta,$$

(6)
Like above set

\[ H := \text{Hom}_K(L, L), \quad H := \text{Hom}_K(L \otimes L, L \otimes L) \]
\[ H_1 := \{ f \in H \mid f(1_A) = 1_A \}, \quad H_1 := \{ F \in H \mid F(1_{A \otimes A}) = 1_{A \otimes A} \} \]

Then \( \Delta \circ f \circ \nabla \in H_1 \) if \( f \in H_1 \) and (recall that \( s \) is not linear)

\[ s(\Delta \circ f \circ \nabla) = f \otimes f \quad \text{for } f \in H_1. \]

Restricting this application \( f \mapsto f \otimes f \) to the base \( B \subseteq H_1 \) of its span \( H_B \subseteq H \) generates a coproduct. Namely defining \( \Delta_B \) and \( \epsilon_B \) like in Proposition 4.1 then \( (H_B, \Delta_B, \epsilon_B) \) is a coalgebra.

**Proof** Let \( f \in H_1 \) then (6) immediately implies \( \Delta \circ f \circ \nabla \in H_1 \). That \( (H_B, \Delta_B, \epsilon_B) \) is a coalgebra, follows from Proposition 4.1. \( \square \)
5 Bialgebras and Hopf algebras

Here we collect the definitions and basic properties of bialgebras and Hopf algebras, weakening the compatibility condition between multiplication and comultiplication in order to access a greater domain of applications. Some of them, weak incidence Hopf algebras, will be treated in the remaining sections.

Let $L$ be a $K$-linear space, which supports an algebra $A = (L, \triangleleft, \eta)$ and a coalgebra $C = (L, \triangleright, \epsilon)$. We regard the following compatibility conditions.

\[
(\triangleleft \otimes \triangleleft) \circ T \circ (\triangleright \otimes \triangleright) = \triangleright \circ \triangleleft : L \otimes L \rightarrow L \otimes L \quad (7)
\]
\[
\triangleright \circ \triangleright = \eta \otimes \eta : K \simeq K \otimes K \rightarrow L \otimes L \quad (8)
\]
\[
\epsilon \circ \triangleright = \epsilon \otimes \epsilon : L \otimes L \rightarrow K \simeq K \otimes K \quad (9)
\]
\[
\epsilon \circ \eta = \text{id}_K : K \rightarrow K \quad (10)
\]

Recall that $\triangleright = (\triangleleft \otimes \triangleleft) \circ T$ is the multiplication of the algebra $A \otimes A$ and $\triangleright = T \circ (\triangleright \otimes \triangleright)$ the comultiplication of the coalgebra $C \otimes C$. Condition (7) is represented by the commuting diagram

and it is obvious that

- $\triangleright$ is an algebra morphism $\iff$ (7), (8) hold
- $\epsilon$ is an algebra morphism $\iff$ (9), (10) hold
- $\triangleright$ is a coalgebra morphism $\iff$ (7), (9) hold
- $\eta$ is a coalgebra morphism $\iff$ (8), (10) hold

The quintuple $(L, \triangleright, \eta; \triangleright, \epsilon)$ is called a weak bialgebra if $(L, \triangleright, \eta)$ is an algebra and $(L, \triangleright, \epsilon)$ a coalgebra and (8), (9), (10) hold. A weak bialgebra is called a (strong) bialgebra if equation (7) holds, too.
Example 5.1 The coalgebra \((M_n, \Delta, \epsilon)\) of \(n \times n\) matrices (Example 2.4) and its dual matrix algebra \((M_n^*, \cdot, U)\) (Example 3.3) forms a bialgebra. The proof is straightforward using the property \(B_{i,j} \cdot B_{k,\ell} = \delta_{j,k} B_{i,\ell}\) of the standard base \(\{B_{i,j} \mid i,j = 1, \ldots, n\}\) of \(M_n^*\).

Be aware, that in a bialgebra the multiplication \(\triangledown\) is a coalgebra morphism, but it is an algebra morphism only in the commutative case (Corollary 2.3), similarly with the comultiplication. Does there exist a more symmetric issue between weak and strong bialgebra?

A bialgebra morphism is an application between weak bialgebras which is simultaneously an algebra morphism and a coalgebra morphism.

The linear space \(H = \text{Hom}_K(L, L)\) supports two algebra structures. First the convolution algebra \(\mathcal{H}(\mathcal{C}, A) := (H, \ast, u)\) (cf. Theorem 3.1). The other one is simpler with composition of applications as multiplication and the identity as unit, \((H, \circ, \text{id}_L)\). We call it the composition algebra of the linear space \(L\).

Proposition 5.1 Let \(L\) be a \(K\)-linear space endowed with linear applications \(\eta : K \to L\) and \(\epsilon : L \to K\), which are compatible, i.e. \((10)\) holds. Let \(H_B\) be a subalgebra of the composition algebra \((\text{Hom}_K(L, L), \circ, \text{id}_L)\), which has a unitary base \(B\), i.e. \(b \circ \eta = \eta\) for all \(b \in B\). Then \(H_B\) becomes a bialgebra,

\[
\mathcal{H}_B := (H_B, \circ, \text{id}_L; \Delta_B, \epsilon_B),
\]

where \((H_B, \Delta_B, \epsilon_B)\) is the coalgebra defined in Proposition 4.1,

\[
\Delta_B (b) := b \otimes b, \quad \epsilon_B(b) := 1_K, \quad b \in B.
\]

Examples for such bases and subspaces will be given in the context of incidence algebras, see Proposition 7.6.

**Proof** We denote multiplication and unit of \(\mathcal{H}_B\) with \(\triangledown\) and \(\eta_0\), respectively. It is sufficient to verify \((7)\) through \((10)\) for base vectors. \(\{b \otimes b' \mid b, b' \in B\}\) is a base of \(H_B \otimes H_B\).

\[
(\triangledown \otimes \triangledown) \circ T \circ (\Delta_B \otimes \Delta_B) (b \otimes b') = (b \circ b') \otimes (b \circ b') = \Delta_B \circ \triangledown (b \otimes b')
\]

\[
\Delta_B \circ \eta_0(1_K) = \text{id}_L \otimes \text{id}_L = \eta_0 \otimes \eta_0 (1_K \otimes 1_K)
\]

\[
\epsilon_B \circ \triangledown (b \otimes b') = 1_K \cdot 1_K = \epsilon_B \otimes \epsilon_B (b \otimes b')
\]

\[
\epsilon_B \circ \eta_0(1_K) = 1_K
\]

The proof is complete. 

In analogy to Proposition 5.1 we ask: Does there exist a natural coalgebra structure on \(H = \text{Hom}_K(L, L)\), such that, together with the convolution algebra \((H, \ast, u)\), we get a bialgebra? The examples in Section 8 will give an affirmative answer, but like in Proposition 5.1 only on a subspace of \(H\) and the coalgebra structure depending on a preassigned base.
By definition a (weak) Hopf algebra is a (weak) bialgebra \((L, \nabla, \eta; \delta, \epsilon)\) for which the unit \(\text{id}_L\) of the composition algebra \((H, \circ, \text{id}_L)\) has a multiplicative inverse in the convolution algebra \(\mathcal{H} = (H, \ast, u)\). This inverse \(S := \text{id}_L^{-1}\) is called the antipode of the (weak) Hopf algebra. Recall that the antipode \(S\), if it exists, is uniquely determined by the two equations

\[
\text{id}_L \ast S = u, \quad S \ast \text{id}_L = u. \tag{11}
\]

The bialgebra \(\mathcal{H}_B\) in Proposition 5.1 has no antipode, whence it is no Hopf algebra. The same holds for the matrix bialgebra in Example 5.1. But restricting to upper triangular matrices, one gets weak Hopf algebras as we shall see in the next sections.

**Example 5.2** With the notations of Example 2.3 \((\mathbb{C}[X], \cdot, 1; \delta_2, \epsilon)\) is a bialgebra, whereas \((\mathbb{C}[X], \cdot, 1; \delta_1, \epsilon)\) is only a weak bialgebra since equation (7) fails.

\[
\delta_1 \text{ is an algebra morphism: } \delta_1 (X^m X^n) = \sum_{k=0}^{m+n} X^k \otimes X^{m+n-k} \text{ whereas } \delta_1 (X^m) \delta_1 (X^n) = \sum_{i=0}^{m} \sum_{j=0}^{n} X^{i+j} \otimes X^{m+n-(i+j)} = \sum_{k=0}^{m+n} a_k X^k \otimes X^{m+n-k}, \text{ with } a_k := |\{(i, j) \in \mathbb{N}_0 \times \mathbb{N}_0 \mid i \leq m, j \leq n, i + j = k\}|.
\]

Both weak bialgebras have an antipode, namely

\[
S_1(X^n) = \begin{cases} 1 & \text{if } n = 0 \\ -X & \text{if } n = 1 \\ 0 & \text{else} \end{cases}, \quad S_2(X^n) = (-X)^n = S_2(X)^n.
\]

\(S_2\) is a bialgebra morphism. \(S_1\) is neither an algebra nor a coalgebra morphism.

**Example 5.3** Hamilton’s skew field \(\mathbb{H}\) of quaternions can be made a weak Hopf algebra, if one takes the coalgebra structure from Example 2.1 w.r.t. the natural base \(B = \{1, i, j, k\}\) of \(\mathbb{H}\) over the field \(\mathbb{R}\) of reals. It is no Hopf algebra since (7) fails, \((\nabla \otimes \nabla) \circ T \circ (\Delta \otimes \Delta)(i \otimes i) = (-1) \otimes (-1) \neq -1 \otimes 1 = \Delta \circ \nabla(i \otimes i)\). The antipode \(S\) is conjugation, \(S(a + ib + jc + kd) = a - ib - jc - kd, \quad a, b, c, d \in \mathbb{R}\). One easily checks that \(S(x \cdot y) = S(y) \cdot S(x)\), i.e., \(S : \mathbb{H} \to \mathbb{H}\) is an antimorphism of \(\mathbb{R}\)-algebras.

The sub-\(\mathbb{R}\)-algebra \(\mathbb{C} \subset \mathbb{H}\) of complex numbers is a commutative weak Hopf algebra.

The antipode is compatible with many operations and constructions as we will show next. First, the opposite bialgebra has the same antipode as the original one.

**Proposition 5.2** The opposite \((L, \nabla^{op}, \eta; \Delta^{op}, \epsilon)\) of a (weak) bialgebra \((L, \nabla, \eta; \Delta, \epsilon)\) is a (weak) bialgebra. The opposite of a weak Hopf algebra with antipode \(S\) is again a weak Hopf algebra with the same antipode \(S\).

**Proof** Since the field \(K\) is commutative, the first assertion is trivial for weak bialgebras. To prove condition (7) for \((L, \nabla^{op}, \eta; \Delta^{op}, \epsilon)\) we get from the diagram (7) for \((L, \nabla, \eta; \Delta, \epsilon)\) and from \(\nabla \circ t = \nabla \circ t\) (see (3)), that \(\Delta\) is an algebra morphism.
w.r.t. $\nabla^{op}$. That is, $(L, \nabla^{op}, \eta; \Delta, \epsilon)$ is a bialgebra. Dually, $(L, \nabla, \eta; \Delta^{op}, \epsilon)$ is a bialgebra, too. Starting again with the last bialgebra we see that $\Delta^{op}$ is an algebra morphism w.r.t. $\nabla^{op}$, i.e. $(L, \nabla^{op}, \eta; \Delta^{op}, \epsilon)$ is a bialgebra.

Using Proposition 3.3 and (11) we get

\[
\text{id} \star^{op} S = S \star \text{id}, \quad S \star^{op} \text{id} = \text{id} \star S = u
\]

which shows that $S$ is the $\star^{op}$-inverse of $\text{id}$, too. \hfill\qed

The tensor product preserves bialgebras and the antipode.

**Proposition 5.3** If $B, B'$ are (weak) bialgebras then their tensor product is a (weak) bialgebra. If $B, B'$ are (weak) Hopf algebras with antipodes $S$ and $S'$, respectively, then $B \otimes B'$ is a (weak) Hopf algebra with antipode $S \otimes S'$.

Also, the tensor product of bialgebra morphisms is a bialgebra morphism.

**Proof** The tensor product of algebras is an algebra and the same holds with coalgebras (see Section 2). Since the counits $\epsilon, \epsilon'$ are algebra morphisms, so is $\epsilon \otimes \epsilon'$ by Proposition 2.1. Similarly the unit $\eta \otimes \eta'$ is a coalgebra morphism. That is, conditions (8), (9), (10) are inherited by the tensor product. So the tensor product of weak bialgebras is a weak bialgebra. In case of strong bialgebras it remains to prove that (7) is inherited, too. Again Proposition 2.1 implies that the comultiplication $\Delta = T \circ (\Delta \otimes \Delta')$ of $B \otimes B'$ is an algebra morphism.

Furthermore, if antipodes exist, Proposition 3.4 implies $\text{id}^{*_{B \otimes B'}}^{-1} = (\text{id}_B \otimes \text{id}_{B'})^{*_{B \otimes B'}}^{-1} = \text{id}_B^{*_{B}}^{-1} \otimes \text{id}_{B'}^{*_{B'}}^{-1} = S \otimes S'$.

Finally, Proposition 2.1 implies the assertion on morphisms. \hfill\qed

The operations composition and convolution obey certain distributive laws. Especially the convolution inverse of (co)algebra morphisms can be computed by composition with the antipode.

**Proposition 5.4** 1. Let $\alpha \in \text{Hom}_K(B, A)$ be an algebra morphism from a weak bialgebra $B$ in an algebra $A$, then

\[
\alpha \circ (f \star g) = (\alpha \circ f) \star (\alpha \circ g), \quad f, g \in (\text{Hom}_K(B, B), \star, u).
\]

Especially $\alpha \circ S$ is the convolution inverse of $\alpha$, if $B$ is a weak Hopf algebra with antipode $S$.

2. Let $\gamma \in \text{Hom}_K(C, B)$ be a coalgebra morphism from a coalgebra $C$ to a weak bialgebra $B$, then

\[
(f \star g) \circ \gamma = (f \circ \gamma) \star (g \circ \gamma), \quad f, g \in (\text{Hom}_K(B, B), \star, u).
\]

Especially $S \circ \gamma$ is the convolution inverse of $\gamma$, if $B$ is a weak Hopf algebra with antipode $S$.
Proof 1. Applying Proposition 3.2 with $\gamma = \text{id}_B$ we get an algebra morphism
\[ h : \text{Hom}_K(B, B) \to \text{Hom}_K(B, A), \quad f \mapsto \alpha \circ f, \]
whence $\alpha \circ (f \ast g) = h(f \ast g) = h(f) \ast h(g) = ((\alpha \circ f) \ast (\alpha \circ g))$. Denoting with $u_A$ the unit of the right hand side convolution algebra, we get with (11) $u_A = h(u) = h(\text{id}_B \ast S) = S \ast (\alpha \circ S)$ and $u_A = h(u) = h(S \ast \text{id}_B) = (\alpha \circ S) \ast \alpha$.

2. proves similarly.

Bialgebra morphisms respect the antipode.

Corollary 5.5 Let $\beta : B \to B'$ be a bialgebra morphism between weak Hopf algebras with antipodes $S$ and $S'$, then
\[ \beta \circ S = S' \circ \beta. \]

Proof Proposition 5.4 implies $\beta \circ S = \beta^{-1} = S' \circ \beta$. 

According to their importance in Proposition 5.4 we regard the algebra morphisms among the linear morphisms in the morphism space $H$. Applying the unit to the unit enforces that they form an affine subspace, but no linear subspace (cf. Corollary 4.4). We show that in the abelian case this affine space forms a group with convolution as group operation (cf. [2] 2.5).

Proposition 5.6 Let $(B, \triangleright, \eta; \triangle, \epsilon)$ be a bialgebra, $(A', \triangleright', \eta')$ an abelian algebra and
\[ H_{\text{alg}} := \text{Hom}_{K-\text{alg}}(B, A') \subseteq H = \text{Hom}_K(B, A') \]
the affine subspace of the $K$-algebra morphisms $\varphi : B \to A'$. Then
\[ \varphi, \psi \in H_{\text{alg}} \Rightarrow \varphi \ast \psi \in H_{\text{alg}} \]
and $(H_{\text{alg}}, \ast, u)$ is an abelian monoid.
\[ (H_{\text{alg}}, \ast, u) \text{ is an abelian group if } (B, \triangleright, \eta; \triangle, \epsilon) \text{ is a Hopf algebra.} \]

Proof $\varphi \otimes \psi$ is an algebra morphism by Proposition 2.1, $\Delta$ is an algebra morphism by the bialgebra property (7) and $\triangleright'$ is one since the algebra $A'$ is abelian (Proposition 2.3), whence $\varphi \ast \psi = \Delta' \circ (\varphi \otimes \psi) \circ \Delta$ is an algebra morphism as well. Since $A'$ is abelian, the monoid $(H_{\text{alg}}, \ast, u)$ is abelian, too (Theorem 3.1).

Under the final assumption, $B$ has an antipode and Proposition 5.4 implies that any $\varphi \in (H_{\text{alg}}, \ast, u)$ has an inverse. 

Corollary 5.7 Let $B$ be an abelian Hopf algebra with antipode $S$, then $S$ is an algebra isomorphism and
\[ S \circ S = \text{id}_B. \]
Example 5.2 shows that the corollary fails for weak Hopf algebras.

**Proof** Clearly $\text{id}_B \in \mathcal{H}_{\text{alg}}$, so Proposition 5.6 implies $S = \text{id}^{-1} \in \mathcal{H}_{\text{alg}}$. Applying now Proposition 5.4 with $\varphi = S$ we get $S \circ S = S^{-1} = \text{id}_B$. 

In the non abelian case one only gets (cf. Example 5.3)

**Theorem 5.8** Let $B$ be a Hopf algebra with antipode $S$, then $S$ is an antomorphism of algebras and coalgebras, i.e.

\[ S \circ \nabla = \nabla^{\text{op}} \circ (S \otimes S), \quad \Delta^{\text{op}} \circ S = (S \otimes S) \circ \Delta. \]


Finally an example to illustrate Proposition 5.6.

**Example 5.4** We resume the Hopf algebra $(C[X], \cdot, 1; \Delta_2, \epsilon)$ of Example 5.2. Any polynomial $g \in C[X]$ induces via $X \mapsto g$ an algebra endomorphism

\[ \varphi_g : C[X] \to C[X], \quad f \mapsto f \circ g \]

and any algebra endomorphism of $C[X]$ is of this form. Using (4), one easily checks $\varphi_g \ast \varphi_h(f) = f \circ (g + h)$, that is

\[ \varphi_g \ast \varphi_h = \varphi_{g+h}. \]

So we have a group isomorphism

\[(C[X], +, 0) \to (H_{C-\text{alg}}(C[X], C[X]), \ast, u), \quad g \mapsto \varphi_g,\]

especially $\varphi_X = \text{id}_{C[X]}$, $\varphi_{-1} = \varphi_{-g}$, $S_2 = \varphi_{-X}$. Notice that $\varphi_g$ is a coalgebra morphism only if $g(X) = aX$ with $a \in C$. Hence $S_2$ is one.
6 Posets and their interval spaces

The linear space, spanned by the intervals of a locally finite ordered set, supports canonically a multiplication and a comultiplication. Under a unitarian condition it becomes a weak bialgebra. This holds, too, if one passes to the quotient space w.r.t. a suitable equivalence relation on intervals. Many important examples are addressed.

Let \( P = (P, \preceq) \) be a countable or finite partially ordered set (poset for short). We always suppose that it is locally finite, i.e. all intervals \( [a, b] := \{ x \in P \mid a \preceq x \preceq b \} \), \( a, b \in P \), are finite. We write \( [a, b] \preceq \) for an interval, if it is necessary to indicate the ordering. \( \text{Intv}(P) \) denotes the set of all intervals \( \neq \emptyset \).

Concatenation and decomposition of intervals lead to the following natural definitions. We denote with \( L = L(P) \) the \( K \)-linear space with base \( \text{Intv}(P) \). A multiplication \( \nabla : L \otimes L \to L \) is defined by the following values on the base \( \nabla([a, b] \otimes [c, d]) := \begin{cases} [a, d] & \text{if } b = c \\ 0 & \text{else} \end{cases} \).

Similarly, a comultiplication \( \Delta : L \to L \otimes L \) is defined by \( \Delta ([a, b]) := \sum_{a \preceq x \preceq b} [a, x] \otimes [x, b] \), where local finiteness is essential. One easily checks that these operations are associative and coassociative, respectively.

Notice that \( \nabla \circ \Delta : L \to L \) is an important combinatorial function, essentially it counts the elements of an interval,

\[
\nabla \circ \Delta ([a, b]) = |[a, b]| [a, b]. \tag{12}
\]

**Example 6.1** Let \( P \) be finite. Then the partial order \( \preceq \) can be extended to a total order on \( P \) (\cite{8} 3.5), i.e. we get an enumeration \( P = \{a_1, a_2, \ldots, a_n\} \) of \( P \) such that \( a_i \preceq a_j \Rightarrow i \leq j \). Applying an interval \( [a_i, a_j] \) to the matrix \( B_{i,j} \in M_n \) (Example 3.3) with entry 1 at \( (i, j) \) and 0 elsewhere, we see that \( L \) can be perceived as subspace of \( M_n \), where concatenation \( \nabla \) corresponds to matrix multiplication and \( \sum_{k=1}^n [a_k, a_k] \in L \) to the unit matrix \( U \). The elements of \( L \) correspond to upper triangular matrices.

But the just defined comultiplication \( \Delta_L \) on \( L \) does not coincide with the comultiplication \( \Delta_{M_n} \) of the matrix bialgebra \( M_n \) (Example 5.1), we rather get

\[ \Delta_L = (\pi \otimes \pi) \circ \Delta_{M_n}|_L, \]

where \( \pi : M_n \to L \) denotes the natural projection, applying \( B_{i,j} \) to \([a_i, a_j]\) if \( a_i \preceq a_j \) and to 0 else. Thus the compatibility condition (7) for a strong bialgebra is lost. But this loss will be compensated by existence of an antipode as we shall see in Corollary 7.5.
Before considering the weak bialgebra structure on $L$ we are looking for equivalence relations on the base $\text{Intv} (\mathcal{P})$, which allow essentially to maintain the above definitions when replacing intervals by their equivalence classes. An equivalence relation $\sim$ on $\text{Intv} (\mathcal{P})$ is called **bialgebra compatible**, if it is

1. **$\triangledown$-compatible**, i.e.
   $$[a, b] \sim [a', b'], \ [b, c] \sim [b', c'] \implies [a, c] \sim [a', c'];$$

2. **$\triangle$-compatible**, i.e. for $[a, b] \sim [a', b']$ there exists a bijection $f : [a, b] \to [a', b']$ such that for all $x \in [a, b]$
   $$[a, x] \sim [a', f(x)], \ [x, b] \sim [f(x), b'];$$

3. **unitary**, i.e. all one point intervals are equivalent,
   $$[a, a] \sim [b, b] \text{ for all } a, b \in \mathcal{P}.$$

In the literature (e.g. [6]) a $\triangle$-compatible equivalence relation is called **order compatible**. Notice that $\triangle$-compatibility and unitarity of intervals depends only on the intervals themselves, whereas $\triangledown$-compatibility depends on that part of the poset surrounding the intervals.

**Example 6.2** On a locally finite poset define $[a, b] \sim [a', b']$ iff the two intervals $([a, b], \preceq), ([a', b'], \preceq)$ with the induced ordering are isomorphic posets. This equivalence relation is only $\triangle$-compatible and unitary but generally not $\triangledown$-compatible as the poset with the following Hasse diagram shows.

$$\begin{array}{c}
\begin{array}{c}
\overset{c}{\searrow}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\overset{c'}{\searrow}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\overset{b'}{\searrow}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\overset{b}{\searrow}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\overset{a'}{\searrow}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\overset{a}{\searrow}
\end{array}
\end{array}
\end{array}$$

Given an equivalence relation $\sim$ on $\text{Intv} (\mathcal{P})$, we denote the equivalence class of an interval $[a, b]$ with $[[a, b]]$ or $[[a, b]]_{\sim}$. The set $\text{Intv} (\mathcal{P}, \sim) := \text{Intv} (\mathcal{P})/\sim$ of equivalence classes of intervals is now taken as base for the $K$-linear space $L = L(\mathcal{P}, \sim)$.

\^\footnote{For some results it would be sufficient to require that the one point intervals are partitioned in finitely many equivalence classes $I_k$, so that $1_L := \sum_k I_k$ exists. But our main results need a single class.}
Proposition 6.1 Let \( \mathcal{P} \) be a locally finite poset with a bialgebra compatible equivalence relation \( \sim \) on its intervals. Then the linear space \( L = L(\mathcal{P}, \sim) \) becomes a weak bialgebra \( (L, \nabla, \eta; \triangle, \epsilon) \), if \( \nabla, \eta \) and \( \triangle, \epsilon \) are defined by\(^3\)

\[
\nabla(I \otimes J) := \sum_{x \in [a,b]} [[a,b]], \quad 1_L := [[a,a]], \quad \eta : K \hookrightarrow L, \quad \eta(r) = r1_L; \\
\triangle([[a,b]]) := \sum_{x \in [a,b]} [[a,x]] \otimes [[x,b]], \\
\epsilon([[a,b]]) := \begin{cases} 1 & \text{if } a = b \\ 0 & \text{else} \end{cases}, \quad \epsilon : L \rightarrow K.
\]

If the equivalence relation is only \( \nabla \)-compatible and unitary, then \( (L, \nabla, \eta) \) is an algebra. If the equivalence relation is only \( \triangle \)-compatible, then \( (L, \triangle, \epsilon) \) is a coalgebra.

We will call \( \mathcal{A}(\mathcal{P}, \sim) := (L, \nabla, \eta) \) the interval algebra, \( \mathcal{C}(\mathcal{P}, \sim) := (L, \triangle, \epsilon) \) the interval coalgebra and \( (L, \nabla, \eta; \triangle, \epsilon) \) the (weak) interval bialgebra.

We remark here, that \( \nabla(I \otimes J) = [[a,b]] \) if there exists a single \( x \in [a,b] \) such that \( I = [[a,x]], J = [[x,b]] \). This will be the case in most of our examples. The prominent exception is Example 6.6, where several such \( x \) exist.

**Proof** The proof is straightforward, since the properties of the equivalence relation had been defined such that \( \nabla, \eta \) and \( \triangle, \epsilon \) are well defined, i.e. not depending on the intervals representing a class. \( \square \)

**Example 6.3** On a locally finite poset the equivalence relation

\[
[a, b] \sim_0 [a', b'] \iff a = b, \ a' = b'
\]

is bialgebra compatible.

All unitary, whence all bialgebra compatible equivalence relations are refinements of the relation in Example 6.3.

**Example 6.4** Let \( \hat{\mathcal{P}} = P \cup \hat{0} \) be an antichain \( P \) with adjoined minimal element \( \hat{0} \). With the bialgebra compatible equivalence relation \( \sim_1 \) from Example 6.3 and, with the notations of Proposition 6.1, the comultiplication becomes

\[
\triangle([[a,b]]) = \begin{cases} 1_L \otimes 1_L & \text{if } a = b \\ 1_L \otimes [[\hat{0},b]] + [[\hat{0},b]] \otimes 1_L & \text{if } a = \hat{0} \neq b \end{cases}.
\]

\(^3\)We use the convention \( \sum_{\emptyset} = 0 \).
Example 6.5 Let $\Omega$ be an arbitrary set and $\mathcal{P}_f(\Omega) := \{ A \subseteq \Omega \mid A \text{ finite} \}$, then $\mathcal{P} = (\mathcal{P}_f(\Omega), \subseteq)$ is a locally finite poset. There is a natural equivalence relation on the set of intervals,

$$[A, B] \sim_1 [A', B'] \iff B \setminus A = B' \setminus A'.$$

It is bialgebra compatible and $(L(\mathcal{P}, \sim_1), \nabla, \eta; \triangle, \epsilon)$ is a weak bialgebra. Via $[[A, B]] \mapsto B \setminus A$ we identify the base $\text{Intv}(\mathcal{P}, \sim)$ of $L(\mathcal{P}, \sim_1)$ with $\mathcal{P}_f(\Omega)$. Then, for base elements $A, B \in \mathcal{P}_f(\Omega)$, multiplication is disjoint union, $\nabla(A \otimes B) = A \cup B$ if $A \cap B = \emptyset$ and $= 0$ else.

In case of finite $\Omega$, say $\Omega = \{1, \ldots, n\}$, we get an algebra isomorphism

$$A(\mathcal{P}, \sim_1) \rightarrow K[X_1, \ldots, X_n]/(X_1^2, \ldots, X_n^2), \ A \mapsto \prod_{a \in A} X_a$$

of the interval algebra with the algebra of polynomials in $n$ variables modulo the ideal, generated by the squared variables.

Example 6.6 Regarding the same poset $\mathcal{P} = (\mathcal{P}_f(\Omega), \subseteq)$ as in Example 6.5 there is another equivalence relation on the set of intervals,

$$[A, B] \sim_2 [A', B'] \iff |B \setminus A| = |B' \setminus A'|.$$

It is again bialgebra compatible and $(L(\mathcal{P}, \sim_2), \nabla, \eta; \triangle, \epsilon)$ is a weak bialgebra. Denote with $I_k$ the class of intervals $[A, B]$, with $|B \setminus A| = k$, then

$$\nabla(I_k \otimes I_l) = \begin{cases} (k+l)I_{k+l} & \text{if } k + l \leq |\Omega| \\ 0 & \text{else} \end{cases}, \quad \triangle I_n = \sum_{k=0}^n \binom{n}{k} I_k \otimes I_{n-k}.$$

We get an algebra epimorphism

$$K[X] \rightarrow A(\mathcal{P}, \sim_2), \quad X^k \mapsto \begin{cases} k!I_k & \text{if } k \leq |\Omega| \\ 0 & \text{else} \end{cases}.$$

For infinite $\Omega$ it is an isomorphism and if $\Omega$ is finite, the kernel is the ideal $(X^{\lceil |\Omega| \rceil + 1})$.

Example 6.7 $(\mathbb{N}_0, \leq)$ is the poset of natural numbers including 0 with the usual total ordering. It is locally finite. Here we have the natural relation

$$[a, b] \sim_n [a', b'] \iff b - a = b' - a'.$$

It is bialgebra compatible and $(L((\mathbb{N}_0, \leq), \sim_n), \nabla, \eta; \triangle, \epsilon)$ is a weak bialgebra. Denoting with $I_n$ the class of intervals $[a, b]$, with $b - a = n$, then

$$\nabla(I_m \otimes I_n) = I_{m+n}, \quad \triangle I_n = \sum_{k=0}^n I_k \otimes I_{n-k}.$$
Example 6.8 \((\mathbb{N}, |)\) is the set of natural numbers with divisibility as partial order. It is locally finite and the smallest element is 1. Here again we have a natural relation
\[
[a, b] \sim_n [a', b'] \iff \frac{b}{a} = \frac{b'}{a'}.
\]
It is bialgebra compatible and \((L((\mathbb{N}, |), \sim_n), \triangledown, \eta; \Delta, \epsilon)\) is a weak bialgebra. Denoting with \(I_n\) the class of intervals \([a, b]\), with \(\frac{b}{a} = n\), then
\[
\triangledown(I_m \otimes I_n) = I_{mn}, \quad \Delta I_n = \sum_{d|n} I_d \otimes I_{\frac{n}{d}}.
\]
Let \(\mathcal{P}\) be a locally finite poset with a bialgebra compatible equivalence relation \(\sim\). We denote the convolution algebra \(\mathcal{H}(\mathcal{C}, \mathcal{A})\) of the coalgebra \(\mathcal{C} = (L(\mathcal{P}, \sim), \Delta, \epsilon)\) and the algebra \(\mathcal{A} = (L(\mathcal{P}, \sim), \triangledown, \eta)\) with
\[
\mathcal{H}(\mathcal{P}, \sim) := (H, \star, u), \quad H = \text{Hom}_K(L, L), \quad L = L(\mathcal{P}, \sim),
\]
and call it the convolution algebra of \((\mathcal{P}, \sim)\).

For later use let us suppose that our poset \(\mathcal{P}\) has a minimal element \(\hat{0}\). This had been the case in Examples 6.4 through 6.8. An atom of \(\mathcal{P}\) is an element \(a\) such that the interval \([\hat{0}, a]\) has exactly two elements. Then for \(\varphi, \psi \in H\) preserving the unit
\[
\varphi \star \psi([\hat{0}, a]) = \varphi([\hat{0}, a]) + \psi([\hat{0}, a]) \quad \text{if} a \text{ is an atom} \quad (13)
\]
7 The incidence algebra

The incidence algebra of a locally finite poset is the dual of the interval coalgebra. Under a unitarian assumption it comprises the interval algebra as subalgebra and for finite posets both coincide. The incidence algebra is naturally embedded in the convolution algebra of the poset (Theorem 7.4) and thus becomes a weak Hopf algebra. In this section \( \mathcal{P} = (P, \preceq) \) is a locally finite poset and \( \sim \) a \( \triangle \)-compatible equivalence relation on \( \text{Intv} (\mathcal{P}) \).

In combinatorics the incidence algebra \( \mathcal{I} (\mathcal{P}, \sim) = (K^{\text{Intv} (\mathcal{P}, \sim)}, *, U) \) is defined as the linear space \( K^{\text{Intv} (\mathcal{P}, \sim)} = \{ \Phi : \text{Intv} (\mathcal{P}) \rightarrow K \} \) furnished with

\[
\Phi \ast \Psi ([a, b]) := \sum_{a \leq x \leq b} \Phi([a, x]) \cdot \Psi([x, b]),
\]

\[
U([a, b]) := \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{else} \end{cases}.
\]

We write \( \mathcal{I} (\mathcal{P}) := \mathcal{I} (\mathcal{P}, =) \) if \( \sim \) is the trivial equivalence relation =. The name incidence algebra is justified in

**Proposition 7.1** Let \( \mathcal{P} = (P, \preceq) \) be a locally finite poset and \( \sim \) a \( \triangle \)-compatible equivalence relation. Then \( \mathcal{I} (\mathcal{P}, \sim) \) is an algebra with multiplication \( \ast \) and unit \( U \). An incidence function \( \Phi \in \mathcal{I} (\mathcal{P}, \sim) \) is right invertible iff it is left invertible and this holds iff \( \Phi([a, a]) \neq 0 \) for all \( a \in P \).

**Proof** \( \mathcal{I} (\mathcal{P}, \sim) \) can be perceived as subspace of \( \mathcal{I} (\mathcal{P}) \), where an application \( \Phi : \text{Intv} (\mathcal{P}) \rightarrow K \) belongs to \( \mathcal{I} (\mathcal{P}, \sim) \) iff it is constant on the equivalence classes of \( \sim \). Therefore it is sufficient to prove the case \( \mathcal{I} (\mathcal{P}) \), what is done in e.g. [8] 3.6.2.

We expand the domain of the incidence functions \( \Phi \in \mathcal{I} (\mathcal{P}, \sim) \) to the linear space \( L(\mathcal{P}, \sim) \) with base \( \text{Intv} (\mathcal{P}, \sim) \).

\[
\Phi \mapsto \overline{\Phi} : L(\mathcal{P}, \sim) \rightarrow K \quad \text{with} \quad \overline{\Phi}|_{\text{Intv}(\mathcal{P}, \sim)} = \Phi,
\]

i.e. \( \overline{\Phi} \) denotes the linear form, coinciding on the base \( \text{Intv} (\mathcal{P}, \sim) \) with \( \Phi \).

**Proposition 7.2** The incidence algebra \( \mathcal{I} (\mathcal{P}, \sim) \) is canonically isomorphic to the dual \( C^* \) of the interval coalgebra \( C := (L(\mathcal{P}, \sim), \triangle, \epsilon) \),

\[
\mathcal{I} (\mathcal{P}, \sim) \rightarrow C^*, \quad \Phi \mapsto \overline{\Phi}.
\]

**Proof** Clearly \( \overline{U} = \epsilon \), the unit of \( C \). For the multiplication the following commuting diagram proves the desired equation.
Here $L$ abbreviates $L(P, \sim)$ as usual and $\text{Intv}(P, \sim) \hookrightarrow L(P, \sim)$ is the embedding of the base in its span.

In the sequel we identify $I(P, \sim)$ with $C^\ast$ and perceive $\Phi \in I(P, \sim)$ as application $L \to K$.

**Proposition 7.3** Let $P = (P, \preceq)$ be a locally finite poset with a bialgebra compatible equivalence relation $\sim$ on its intervals. Then the interval algebra $A(P, \sim) = (L(P, \sim), \triangledown, \eta)$ is a subalgebra of the incidence algebra $I(P, \sim)$.

Furthermore, $A(P, \sim) = I(P, \sim)$ iff $P$ is finite.

**Proof** Clearly $L(P, \sim)$ is linearly embedded in $K^{\text{Intv}(P, \sim)}$ by means of

$$\alpha := \sum_{I \in \text{Intv}(P, \sim)} \alpha_I I \mapsto (I \mapsto \alpha_I)$$

Denote with $\Phi_\alpha$ the image of $\alpha$ under this embedding, then $\Phi_I(J) = \delta_{I,J}$. We have to show that

$$\Phi_I \ast \Phi_J = \Phi_{\triangledown(I \otimes J)}, \quad \Phi_{[a,a]} = U. \quad (14)$$

The second equation being obvious, we check the first.

$$\Phi_I \ast \Phi_J([a, b]) = \triangledown \circ (\Phi_I \otimes \Phi_J) \circ \triangle ([a, b])$$

$$= \sum_{x \in [a,b]} \Phi_I([a,x]) \Phi_J([x,b])$$

$$= |\{x \in [a,b] \mid [a,x] = I, [x,b] = J\}|$$

$$= \Phi_{\triangledown(I \otimes J)}([a,b]).$$

Since any $\alpha \in L(P)$ is a linear combination of finitely many intervals the last assertion is clear.

Proposition 7.3 holds, too, with finite $P$ and the trivial equivalence relation $=$, if one modifies the second equation in (14) to $\Phi_{\sum a \in P [a,a]} = U$.

29
The incidence algebra can be embedded in the convolution algebra \( \mathcal{H}(\mathcal{P}, \sim) \). Recall from end of Section 6 that the latter is defined only for bialgebra compatible equivalence relations.

**Theorem 7.4** Let \( \mathcal{P} = (\mathcal{P}, \preceq) \) be a locally finite poset and \( \sim \) a bialgebra compatible equivalence relation on \( \text{Intv}(\mathcal{P}) \). For \( \Phi \in \mathcal{I}(\mathcal{P}, \sim) \) let \( \hat{\Phi} \in \mathcal{H}(\mathcal{P}, \sim) \) be the linear application given on the base \( \text{Intv}(\mathcal{P}, \sim) \) by

\[
\hat{\Phi}(I) := \Phi(I)I, \quad I \in \text{Intv}(\mathcal{P}, \sim).
\]

Then the application

\[ \mathcal{I}(\mathcal{P}, \sim) \hookrightarrow \mathcal{H}(\mathcal{P}, \sim), \quad \Phi \mapsto \hat{\Phi} \]

is a monomorphism of algebras. Then the image \( \hat{\mathcal{I}}(\mathcal{P}, \sim) \) of \( \mathcal{I}(\mathcal{P}, \sim) \) is a subalgebra of \( \mathcal{H}(\mathcal{P}, \sim) \).

Furthermore \( f \in \hat{\mathcal{I}}(\mathcal{P}, \sim) \) respects the unit, \( f \circ \eta = \eta \), iff it respects the counit, \( \epsilon \circ f = \epsilon \). And this holds iff \( f \) has a convolution inverse \( f^{-1} \in \hat{\mathcal{I}}(\mathcal{P}, \sim) \).

**Proof** The application \( \hat{\cdot} \) clearly is linear. It preserves the unit, \( \hat{U} = \eta \circ \epsilon = \eta \) since \( \sim \) is unitary. Finally \( \hat{\cdot} \) is compatible with convolution. It is sufficient to check this on the base elements \( [a, b] \in \text{Intv}(\mathcal{P}, \sim) \). Using the definitions of convolution in Theorem 3.1 and of \( \triangle \) in Proposition 6.1 we get

\[
\hat{\Phi} \ast \hat{\Psi}([[a, b]]) = \nabla \circ (\hat{\Phi} \otimes \hat{\Psi}) \circ \triangle ([[a, b]])
\]

\[ = \sum_{x \in [a, b]} \nabla(\hat{\Phi}([[a, x]]) \otimes \hat{\Psi}([[x, b]])) \]

\[ = \sum_{x \in [a, b]} \Phi(a, x)\Psi(x, b) \nabla([[a, x]] \otimes [[x, b]]) \]

\[ = \hat{\Phi} \ast \hat{\Psi}(a, b) [[a, b]] \]

\[ = \hat{\Phi} \ast \hat{\Psi}([[a, b]]). \]

For the fourth equation we used the definition of \( \ast \) in \( \mathcal{I}(\mathcal{P}) \) and of \( \nabla \). The reader should draw a diagram like in the proof of Proposition 3.2.

The constant function 1 on \( \text{Intv}(\mathcal{P}) \) is called the **zeta function** of the poset \( \mathcal{P} \) and denoted \( Z \). It is invertible (Proposition 7.1) and it’s inverse \( \mathcal{M} := Z^{-1} \) is called the **Möbius function**. These names had been given (see [5]) in analogy to the classical case \( \mathcal{P} = (\mathbb{N}, |) \) (Example 8.3). The image \( \hat{\mathcal{M}} \) of the Möbius function in the convolution algebra is the antipode. The application \( \hat{\cdot} \) transforms \( Z \) in \( \text{id}_L \), \( L = \text{L}(\mathcal{P}, \sim) \), and using (12) we see that the application

\[
\hat{Z} \ast \hat{Z} = \nabla \circ (\text{id}_L \otimes \text{id}_L) \circ \triangle = \nabla \circ \triangle
\]

is a linear isomorphism. (15)
Corollary 7.5 Under the assumptions of Theorem 7.4 the weak bialgebra \((L(P, \sim), \nabla, \eta; \Delta, \epsilon)\) (see Proposition 6.1) is a weak Hopf algebra. The antipode is \(S = \hat{M}\).

**Proof** \(M = Z^*^{-1}\) and \(\hat{Z} = \text{id}_L\) imply \(\hat{M} = Z^*^{-1} = \hat{Z}^*^{-1} = \text{id}_L^*^{-1}\) and this is the antipode \(S\). □

\(S\) is a bialgebra morphism iff

\[ S \circ \nabla = \nabla \circ (S \otimes S) \rho \Delta = (S \otimes S) \circ \Delta, \]

i.e. iff the following diagram commutes.

In general this is not the case but all paths from north west to south east give the same application,

\[ S \circ \nabla \circ \Delta = \nabla \circ (S \otimes S) \circ \Delta = S \star S = \nabla \circ \Delta \circ S. \]

The linear space \(K^{\text{Intv}(P, \sim)}\) which underlies the incidence algebra \(I(P, \sim) = (K^{\text{Intv}(P, \sim)}, \star, U)\) supports a second algebra \(\hat{A}_0(P, \sim) := (K^{\text{Intv}(P, \sim)}, \cdot, Z)\) with \(\cdot\) denoting usual multiplication of functions and the constant function \(Z \equiv 1\) being the unit. Since the application \(\sim\) transforms \(Z\) in \(\text{id}_L\) and the product to composition,

\[ \Phi : \Psi = \hat{\Phi} \circ \hat{\Psi}, \]

the image\(^4\) \(\hat{A}_0(P, \sim) \subseteq H = \text{Hom}_K(L, L)\), \(L = L(P, \sim)\), of this algebra is a subalgebra of the composition algebra \((H, \circ, \text{id}_L)\). Even more it becomes a bialgebra, but not canonically.

**Proposition 7.6** Let \(P = (P, \preceq)\) be a locally finite poset and \(\sim\) a bialgebra compatible equivalence relation on \(\text{Intv}(P)\). There exists a unitary base \(B\) of \(\hat{A}_0(P, \sim)\) and for any such base \((\hat{A}_0(P, \sim), \circ, \text{id}_L; \Delta_B, \epsilon_B)\) is a bialgebra, where

\[
\Delta_B(\hat{b}) = \hat{b} \otimes \hat{b}, \quad \hat{b} \in B, \\
\epsilon_B(\Phi) = \Phi([a, a]), \quad \Phi \in K^{\text{Intv}(P, \sim)}.
\]

In the finite dimensional case, coproduct and counit don’t depend on the base.

\(^4\)the underlying linear spaces of \(\hat{A}_0(P, \sim)\) and \(\hat{I}(P, \sim)\) are identical.
Proof From Example 4.1 we know that there exists a unitary base \( B_0 \) of \( L^* \), that is \( b \circ \eta(1_K) = 1_K \) for all \( b \in B_0 \). Then \( B := \{ \hat{b} \mid b \in B_0 \} \) is a base of \( \hat{A}_\circ(P, \sim) \) and \( \hat{b}([[a,a]]) = \hat{b}([[a,a]]) \cdot [[a,a]] = [[a,a]] \) for all \( \hat{b} \in B \). The assumptions of Proposition 5.1 are verified and we are done.

In this context the following result is remarkable.

Proposition 7.7 Let \( P = (P, \leq) \) be a finite poset and \( A \subseteq I(P) \) a subalgebra of the incidence algebra. Then \( \hat{A} \subseteq \hat{A}_\circ(P, \sim) \) is a subalgebra of the composition algebra iff there exists a \( \Delta \)-compatible equivalence relation \( \sim \) on \( \text{Intv}(P) \) such that \( A = I(P, \sim) \).

Proof See e.g. [7] Proposition 1.3.9.
8 Examples and incidence bialgebra

Proposition 7.6 shows, that the dual of the algebra \((L, \nabla, \eta)\) is not the appropriate coalgebra (see Example 4.1) in order to make the incidence algebra a bialgebra. Rather, like in the case of finite dimensions (Proposition 7.3), one has to use the comultiplication \(\Delta\) on \(L\) to define the suitable comultiplication on the incidence algebra. The next examples will pave the way. Notice that all examples presented here are commutative and cocommutative.

Example 8.1 Let \(K = \mathbb{C}\). We resume Example 6.6 of the poset \(\mathcal{P} = (\mathcal{P}_f(\Omega), \subseteq)\) with countable \(\Omega\) and the relation \(\sim_2\). We denoted with \(I_n\) the interval class \([A,B]\), with \(|B\setminus A| = n\) and have seen, that \((L(\mathcal{P}, \sim_2), \nabla, \Delta, \epsilon) \to (\mathbb{C}[X], \cdot, 1; \Delta_2, \epsilon_2)\)

\[ I_n \mapsto \frac{1}{n!} X^n \]

is an algebra isomorphism. One easily checks that it is a coalgebra morphism, too, whence a bialgebra morphism. So the interval bialgebra is a Hopf algebra with antipode \(S(I_n) = (-1)^n I_n\) (Example 5.2). Then the Möbius function is \(M(I_n) = (-1)^n\) (Corollary 7.5).

This isomorphism, Proposition 7.2 and Example 3.2 give us the algebra isomorphism

\[ \mathcal{I}(\mathcal{P}, \sim_2) \to ((\mathbb{C}[X], \Delta_2, \epsilon_2)^*, \ast_2, 1) \simeq (\mathbb{C}[[X]], \cdot, 1) \]

\[ \Phi \mapsto \tilde{\Phi} := \sum_{n=0}^{\infty} \frac{\Phi(I_n)}{n!} X^n \]

Especially \(\tilde{Z} = e^X\) is the exponential function and \(\tilde{M} = e^{-X}\) it’s inverse function.

We regard the incidence function \(\Phi(I_n) = \frac{1}{n!+1}\) from [1]. It’s image \(\tilde{\Phi} \in \mathcal{H}(\mathcal{P}, \sim_2)\) is neither an algebra morphism nor a coalgebra morphism. It is invertible and one easily computes, that \(\tilde{\Phi}^{-1}\) is the exponential generating function \(e^{X-1} = \sum_{n=0}^{\infty} \frac{\beta_n}{n!} X^n\) of the Bernoulli numbers \((\beta_n)_{n \in \mathbb{N}_0}\), whence \(\Phi^{-1}(I_n) = \)

\[ \beta_n. \]

The comultiplication \(\Delta_2\) and the counit \(\epsilon_2\) of the bialgebra of polynomials can be extended to the algebra of formal power series,

\[ \Delta_2 (f)(X,Y) = f(X+Y), \quad \epsilon_2(f) = f(0) \quad \text{for } f \in \mathbb{C}[[X]], \]

but the question is, if the power series \(\Delta_2 (f)\) in two variables is contained in \(\mathbb{C}[[X]] \otimes \mathbb{C}[[X]],\) which is only a subalgebra of \(\mathbb{C}[[X,Y]]\). The answer is affirmative:

The linear space \(\mathbb{C}[[X]]\) is the directed colimit\(^5\) of the directed poset (with set inclusion as ordering) of the linear spaces \(D_n, \ n \in \mathbb{N}_0,\) of polynomials of

\(^5\)Directed colimits are often called direct limit or inductive limit.
degree \leq n. Since the functor $M \mapsto M \otimes M$ on the category of $K$-linear spaces preserves directed colimits ([3] Proposition 8), $\mathbb{C}[[X]] \otimes \mathbb{C}[[X]]$ is the colimit of $(D_n \otimes D_n)_{n \in \mathbb{N}_0}$. The universal property of directed colimits implies that the linear applications $\Delta_2: D_n \to D_n \otimes D_n$, $n \in \mathbb{N}_0$, have a directed colimit. Use the $(X)$-adic and $(X,Y)$-adic topologies to show that this application coincides with $\Delta_2: \mathbb{C}[[X]] \to \mathbb{C}[[X,Y]] \supset \mathbb{C}[[X]] \otimes \mathbb{C}[[X]]$ as defined above.

(\mathbb{C}[[X]], \cdot, 1; \Delta_2, \epsilon_2) is a bialgebra. especially $\Delta_2(\tilde{Z})(X,Y) = e^X e^Y$.

**Example 8.2** We resume Example 6.7 of the poset $\mathcal{P} = (\mathbb{N}_0, \leq)$ with the relation $\sim_n$. Denote with $I_n$ the interval class $[[a,b]]$, with $|b-a| = n$. Then one easily checks that

$$(L((\mathbb{N}, \leq), \sim_n), \nabla, \eta; \Delta, \epsilon) \to (\mathbb{C}[X], \cdot, 1; \Delta_1, \epsilon_1)$$

$I_n \to X^n$

is a bialgebra isomorphism. So the interval bialgebra is a weak Hopf algebra with antipode $S = \hat{M}$ (Example 5.2) and Möbius function $M(I_0) = 1$, $M(I_1) = -1$ and $M(I_n) = 0$ for $n > 1$ (Corollary 7.5).

This isomorphism induces the algebra isomorphism

$$\mathcal{I}((\mathbb{N}, \leq), \sim_n) \to ((\mathbb{C}[X], \Delta_1, \epsilon_1)^*, *, 1) \simeq (\mathbb{C}[[X]], \cdot, 1)$$

$$\Phi \mapsto \tilde{\Phi} := \sum_{n=0}^{\infty} \Phi(I_n) X^n$$

Especially $\tilde{Z} = \sum_{n=0}^{\infty} X^n$ is the geometric series and $\tilde{M} = 1 - X$ it’s inverse.

Like in the last example, $\Delta_1$ can be extended from polynomials to power series and we get a weak bialgebra $(\mathbb{C}[[X]], \cdot, 1; \Delta_1, \epsilon_1)$.

**Example 8.3** We resume Example 6.8 of the weak bialgebra $L((\mathbb{N}, |), \sim_n), \nabla, \eta; \Delta, \epsilon)$.

Here we regard the $\mathbb{C}$-linear space $\mathbb{C}(s)$ spanned by the linearly independent functions $(b_n)_{n \in \mathbb{N}_0}$.

$$b_n: \mathbb{C} \to \mathbb{C}, \quad b_n(s) := n^{-s}, s \in \mathbb{C}.$$  

Notice that $b_m \cdot b_n = b_{mn}$, i.e. $\nabla(b_m \otimes b_n) = b_{mn}$.  

$\mathbb{C}(s)$ is a subalgebra of the algebra $\mathbb{C}(\langle s \rangle)$ of formal Dirichlet series $f = \sum_{n=1}^{\infty} \alpha_n b_n$, $\alpha_n \in \mathbb{C}$. We call the functions in $\mathbb{C}(s)$ Dirichlet polynomials. Like for power series there is an algebra monomorphism of $\mathbb{C}(\langle s \rangle) \otimes \mathbb{C}(\langle s \rangle)$ to the algebra $\mathbb{C}(\langle s, t \rangle)$ of formal Dirichlet series in two variables,

$$\mathbb{C}(\langle s \rangle) \otimes \mathbb{C}(\langle s \rangle) \to \mathbb{C}(\langle s, t \rangle), \quad m^{-s} \otimes n^{-s} \mapsto m^{-s} n^{-t}.$$  

(16)
Now we get algebra isomorphisms
\[
(L((\mathbb{N}, |), \sim_n), \triangledown, \eta) \to (\mathbb{C}(s), \cdot, 1), \quad [[1, n]] \mapsto b_n,
\]
\[
(I((\mathbb{N}, |), \sim_n) \to (\mathbb{C}(\langle s \rangle), \cdot, 1), \Phi \mapsto \widetilde{\Phi} := \sum_{n=1}^{\infty} \Phi([[1, n]]) n^{-s}.
\]

The image of\( Z \in I((\mathbb{N}, |), \sim_n)\) is the Riemann zeta function\( \zeta(s) = \sum_{n=1}^{\infty} n^{-s} \) and the image of\( M = Z^{-1} \) is the Dirichlet series\( \frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \mu_n n^{-s} \), where the coefficients are the values of the classical Möbius function,
\[
\mu_n = \begin{cases} (-1)^k & \text{if } n = p_1 p_2 \cdots p_k \text{ with primes } p_1 < p_2 < \cdots < p_k \\ 0 & \text{else} \end{cases}
\]

Like in the preceding examples the coalgebra structure
\[
\Delta(b_n) = \sum_{d|n} b_d \otimes b_{n/d}, \quad \epsilon(b_n) := \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{else} \end{cases}
\]
on\( L((\mathbb{N}, |), \sim_n)\) transports to \( \mathbb{C}(s) \) and extends \( \mathbb{C}(\langle s \rangle) \) to \( \mathbb{C}(\langle \langle s \rangle \rangle) \). Using (16), \( \Delta \) can be written as
\[
\Delta(b_n)(s, t) = \sum_{d|n} d^{-s} \left( \frac{n}{d} \right)^{-t} = n^{-t} \sum_{d|n} d^{-(s-t)}.
\]

Especially \( \Delta(\zeta)(s, t) = \zeta(s) \cdot \zeta(t) \) and \( \Delta(\zeta^{-1})(s, t) = \zeta^{-1}(s) \cdot \zeta^{-1}(t) \). This holds despite the fact, that we have only a weak bialgebra. \( \Delta(f) \) reproduces \( f \in \mathbb{C}(\langle s \rangle) \),
\[
\Delta(f)(s, t) \to_{t \to +\infty} f(s), \quad \Delta(f)(s, t) \to_{s \to +\infty} f(t).
\]

Setting \( t = s \) we get \( \Delta(n^{-s})(s, s) = d_n n^{-s} \) with \( d_n \) denoting the number of divisors of \( n \) (cf. (12)).

In all these examples the incidence algebra is a function space and the comultiplicaton, introduced on it, transforms a function of one variable in a function of two variables. We have just seen, that inserting specific values for the new covariable can reveal interesting combinatorial functions of the respective poset. Here is another example.

**Example 8.4** Inserting in Example 8.1 \( Y = -a \in \mathbb{C} \) in the coproduct of \( f \in \mathbb{C}[[X]] \) gives the power series expansion \( \Delta_2(f)(X, -a) = f(X - a) \) of the incidence function \( f \) at the point \( a \).

Setting \( Y = X \) gives \( \Delta_2(X^n)(X, X) = 2^n X^n \). Here \( 2^n \) is the number of elements in the intervals of the class \( I_n \) (cf. (12)).
How to construct generally a comultiplication on the incidence algebra? The examples suggest to extend the finite dimensional case via directed colimits. Let \( \mathcal{P} = (P, \preceq) \) be a locally finite poset and \( \sim \) a \( \Delta \)-compatible equivalence relation on \( \text{Intv}(\mathcal{P}) \) and suppose additionally that \( \mathcal{P} \) has a minimal element \( 0 \) and is directed, i.e. any finite subset of \( \mathcal{P} \) has an upper bound. Then

\[
\begin{align*}
\dim_K D_p &< \infty \quad \text{for} \quad D_p := \text{span}\{[[a, b]] \mid [a, b] \subseteq [0, p]\}, \ p \in \mathcal{P}, \\
D_p &\hookrightarrow D_q \quad \text{for} \quad p \leq q.
\end{align*}
\]
9 Morphisms of incidence algebras

Given two locally finite posets $P_i = (P_i, \preceq_i)$ with $\triangle$-compatible equivalence relations $\sim_i$ on $\text{Intv}(P_i)$, $i = 1, 2$. A linear application

$$\gamma : L_1 \to L_2$$

between the corresponding interval spaces $L_i := L(P_i, \sim_i)$ induces by dualisation a linear morphism between the incidence algebras $I_i := I(P_i, \sim_i)$,

$$\gamma^* : I_2 \to I_1, \quad \Phi_2 \mapsto \Phi_2 \circ \gamma.$$

If $\gamma$ is injective (surjective), then $\gamma^*$ is surjective (injective). The question is, if $\gamma^*$ is an algebra morphism.

**Proposition 9.1** Given two locally finite posets $P_i = (P_i, \preceq_i)$ with $\triangle$-compatible equivalence relations $\sim_i$ on $\text{Intv}(P_i)$, $i = 1, 2$, and a coalgebra morphism $\gamma : C_1 \to C_2$ between the interval coalgebras $C_i = (L_i, \triangle_i, \epsilon_i)$, then $\gamma^*$ is an algebra morphism.

**Proof** Applying Proposition 3.2 with $\alpha = \text{id}_K$ we get an algebra morphism $C^*_2 \to C^*_1$. By Proposition 7.2 we are done. $\square$

**Example 9.1** Let $\mathcal{P}$ be a locally finite poset with two bialgebra compatible equivalence relations $\sim_1$ and $\sim_2$, the latter being a refinement of the first. The natural projection $\text{Intv}(\mathcal{P}, \sim_1) \to \text{Intv}(\mathcal{P}, \sim_2)$ induces a linear application

$$\gamma : L(\mathcal{P}, \sim_1) \to L(\mathcal{P}, \sim_2), \quad [[a, b]]_{\sim_1} \mapsto [[a, b]]_{\sim_2}.$$

This is a coalgebra morphism so that by Proposition 9.1

$$\gamma^* : I(\mathcal{P}, \sim_2) \to I(\mathcal{P}, \sim_1), \quad \Phi \mapsto \Phi \circ \gamma$$

is an algebra morphism. It is a monomorphism since $\gamma$ is surjective.

**Example 9.2** Regard Example 9.1 with the poset $\mathcal{P}(\{1, \ldots, n\}, \subseteq)$ and the equivalence relations $\sim_1$ (Example 6.5) and $\sim_2$ (Example 6.6). Then we get

$$\gamma^* : \mathbb{C}[X]/(X_{n+1}) \hookrightarrow \mathbb{C}[X_1, \ldots, X_n]/(X_1^2, \ldots, X_n^2), \quad X \mapsto X_1 + \cdots + X_n$$

**Example 9.3** We denote with $\mathbb{P}$ the set of prime numbers. Let $\mathcal{P}_1$ be the poset $(\mathcal{P}_f(\mathbb{P}), \subseteq)$ with the relation $\sim_1$ (Example 6.5) and $\mathcal{P}_2 = (\mathbb{N}, |)$ with $\sim_n$ (Example 8.3). They are closely related through the poset embedding

$$T : (\mathcal{P}_f(\mathbb{P}), \subseteq) \hookrightarrow (\mathbb{N}, |), \quad P \mapsto \prod_{p \in P} p, \quad \emptyset \mapsto 1, \quad (17)$$
assigning to any finite set \( P \subset \mathbb{P} \) of primes their product. The image of this application is

\[
Q := \{ n \in \mathbb{N} \mid p^2 \nmid n \text{ for all } p \in \mathbb{P} \} = \{1, 2, 3, 5, 6, 7, 10, 11, 13, \ldots \},
\]

the set of squarefree numbers. \((\mathcal{P}_f(\mathbb{P}), \subseteq)\) and \((Q, |)\) are canonically isomorphic posets. Furthermore \((Q, |)\) is a subposet of \((\mathbb{N}, |)\) with the property that any interval \([x, y]\) in \(Q\) is also an interval in \(\mathbb{N}\) since a divisor of a squarefree number is itself squarefree. Also the equivalence relations \(\sim_1\) on \((\mathcal{P}_f(\mathbb{P}), \subseteq)\) (Example 6.5) and \(\sim_n\) on \((Q, |)\) (Example 6.8) are the same modulo the isomorphism. Then the injection

\[
\text{Intv} ((\mathcal{P}_f(\mathbb{P}), \subseteq), \sim_1) \hookrightarrow \text{Intv} ((\mathbb{N}, |), \sim_n).
\]

induces a linear application between the spans of these interval sets,

\[
\gamma : L((\mathcal{P}_f(\mathbb{P}), \subseteq), \sim_1) \hookrightarrow L((\mathbb{N}, |), \sim_n).
\]

Clearly it is a coalgebra morphism and by Proposition 9.1 we get an algebra morphism

\[
\gamma^* : \mathcal{I}((\mathbb{N}, |), \sim_n) \rightarrow \mathcal{I}((\mathcal{P}_f(\mathbb{P}), \subseteq), \sim_1) \simeq \mathcal{I}((Q, |), \sim_n), \quad \Phi \mapsto \Phi|_Q.
\]

It is an epimorphism since \(\gamma\) is injective.
References


http://www.math.uni-bremen.de/~porst/download/VorlesungSS2010.pdf


