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Max-min (σ-)additive representation of monotone measures

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Abstract In non-additive measure and integration (or fuzzy measure and integral) one tries to generalise the issues of product measure and conditional expectation from the additive theory. In the discrete case successful attempts have been made via the max-min additive representation of the monotone measure and the corresponding integrals.
The present paper intends to find, for arbitrary monotone measures, a max-min additive representation and, under certain topological assumptions, a representation with σ-additive measures, thus providing a powerful tool for the theory of non-additive measure and integration.

1 Introduction

The min representation of supermodular set functions and their integrals by means of the additive set functions in the core is well known for a long time [1], especially in cooperative game theory [4]. If one requires σ-additivity in place of (finite) additivity, continuity assuptions have to be made [11], [10].
Totally monotone measures form a subclass of supermodular measures. The representation of an arbitrary monotone measure on a finite set Ω as max of totally monotone measures was first applied in [7] for generalising the Möbius product, which is a good product only for totally monotone measures. The combination of this max representation with the min additive representation mentioned first had then been applied in [8] to define - for the discrete case - conditional expectation w.r.t. monotone set functions, replacing the probability measures.
An important subclass of the totally monotone measures is formed by the lower chain measures (being necessity measures in the discrete case), which are uniquely determined by their values on a certain chain of subsets of Ω.

It turned out that, for the max representation of a monotone measure μ it is sufficient to maximise over all lower chain measures below μ. If one passes to the conjugate set functions one can dualise the max-min additive representation to a min-max additive one.
Looking for a max-min σ-additive representation we build on the results of Parker [10] for the min representation. She has two results, the first (from [11]) with a stronger continuity condition for the supermodular set function, the other with a weaker continuity condition but with the additional structure of a metric space. For the latter result, to which we will refer, she uses Choquets Capacitability Theorem [3], [5]. In order to get a max representation with "continuous" lower chain measures we, of course, need some continuity for the set function μ to be represented, namely continuity from above on the closed sets and inner regularity. Furthermore, for our max-min σ-additive representation we need a compact metric space. The corresponding max-min representation of the Choquet integral is valid for continuous functions, we need the fact that their upper level sets are open. Concluding, we point out further problems to be solved in order to apply our results to products of monotone measures and conditional expectation.

2 Preliminaries

Here we fix the context, giving the basic definitions and notations and preparatory results not available in the literature.
Throughout the paper, Ω denotes a nonempty set and $\mathcal{S} \subset 2^{\Omega}$ a family of subsets containing Ω and the empty set, $\emptyset, \Omega \in \mathcal{S}$. For a set function $\mu : \mathcal{S} \to [0,1]$ we always suppose $\mu(\emptyset) = 0$ and $\mu(\Omega) = 1$. μ is called **monotone**, if $A, B \in \mathcal{S}$, $A \subset B \Rightarrow \mu(A) \le \mu(B)$. The **inner extension** of μ is[1]

$$\mu_*(A) := \bigvee_{\substack{B \in \mathcal{S} \\ B \subset A}} \mu(B)\,, \quad A \in 2^{\Omega}\,.$$

The outer extension μ^* is defined dually. If μ is monotone, so are μ_* and μ^*.

Definition 1 *A monotone set function $\beta : \mathcal{S} \to [0,1]$ is called* **lower chain measure***, if there is a chain w.r.t. set inclusion $\mathcal{K} \subset \mathcal{S}$ with $\emptyset, \Omega \in \mathcal{K}$ such that*

$$\beta = (\beta|\mathcal{K})_* \,|\mathcal{S}$$

Lemma 1 *For a lower chain measure β on $\mathcal{S}$,*

$$\beta\Big(\bigcap_{A \in \mathcal{A}} A\Big) = \bigwedge_{A \in \mathcal{A}} \beta(A) \qquad (1)$$

for all finite set systems $\mathcal{A} \subset \mathcal{S}$ such that $\bigcap_{A \in \mathcal{A}} A \in \mathcal{S}$.

[1] We use $\vee$ to denote the max or sup and $\wedge$ the min or inf.

If property (1) holds for arbitrary $\mathcal{A}$ and $\mathcal{S}$ is closed under arbitrary intersection, then β is called a **necessity measure**. Any necessity measure is a lower chain measure, a defining chain consisting of the sets $K_x := \bigcap\{B \mid B \in \mathcal{S},\, \beta(B) \geq x\}$, $x \in \mathbb{R}$. But the converse does not hold since a necessity measure has a very strong continuity property. Therefore, necessity measures are not used any more in this article. But, of course, the class of necessity measures coincides with the class of lower chain measures if Ω is finite.

Proof For simplicity we suppose $\mathcal{A} = \{A_1, A_2\}$. Let $\mathcal{K}$ be a chain defining β. Since $K_1 \subset K_2$ or $K_1 \supset K_2$ if $K_1,\ K_2 \in \mathcal{K}$, we get $\beta(K_1) \wedge \beta(K_2) = \beta(K_1 \cap K_2)$ and, using Definition 1,

$$\begin{aligned}\beta(A_1) \wedge \beta(A_2) &= \bigvee_{\substack{K_1 \in \mathcal{K} \\ K_1 \subset A_1}} \beta(K_1) \wedge \bigvee_{\substack{K_2 \in \mathcal{K} \\ K_2 \subset A_2}} \beta(K_2) \\ &= \bigvee_{\substack{K_1, K_2 \in \mathcal{K} \\ K_1 \subset A_1, K_2 \subset A_2}} \beta(K_1) \wedge \beta(K_2) \\ &= \bigvee_{\substack{K_1, K_2 \in \mathcal{K} \\ K_1 \subset A_1, K_2 \subset A_2}} \beta(K_1 \cap K_2) \\ &= \bigvee_{\substack{K \in \mathcal{K} \\ K \subset A_1 \cap A_2}} \beta(K) \\ &= \beta(A_1 \cap A_2)\,,\end{aligned}$$

which proves (1) □

Definition 2 *Let $\mathcal{S} \subset 2^{\Omega}$ be closed under finite union and intersection and $\beta : \mathcal{S} \to [0,1]$ a set function. β is called k-***monotone***, $k \in \mathbb{N}$, $k \geq 2$, if it is monotone and, for any system $A_1, A_2, \ldots, A_k \in \mathcal{S}$ of k sets,*

$$\beta\left(\bigcup_{i=1}^{k} A_i\right) + \sum_{\substack{I \subset \{1,\ldots,k\} \\ I \neq \emptyset}} (-1)^{|I|}\ \beta\left(\bigcap_{i \in I} A_i\right) \geq 0\,.$$

2-monotonicity is also called **supermodularity**. **Submodularity** *is the same condition with the inequality reversed. β is called* **totally monotone** *or a* **belief measure** *(often denoted belief function), if β is k-monotone for all $k \geq 2$.*

For a given set function $\mu : \mathcal{S} \to [0,1]$ we define the **lower chain core** as

$$\bar{C}_{lc}\,(\mu) := \{\beta : \mathcal{S} \to [0,1] \mid \beta \text{ lower chain measure},\ \beta \leq \mu\}$$

If $\mathcal{S}$, in addition, is closed under finite union and intersection, we define the **totally monotene core** as

$$\bar{C}_{tm}\,(\mu) := \{\beta : \mathcal{S} \to [0,1] \mid \beta \text{ monotone and totally monotone},\ \beta \leq \mu\}\,.$$

The **additive core** for a set function $\beta : \mathcal{S} \to [0,1]$ is defined as

$$C(\beta) := \{\alpha : \mathcal{S} \to [0,1] \mid \alpha \text{ monotone and additive, } \alpha \geq \beta\}.$$

Notice, that here monotonicity of α is guaranteed if $\mathcal{S}$ is closed under complementation of sets. Be aware, that all our set functions are normalised, so we omitted the condition $\alpha(\Omega) = \beta(\Omega)$ which usually appears in the definition of the core. The name core for $C(\beta)$ is well established in the literature whereas $\bar{C}_{lc}(\mu)$ and $\bar{C}_{tm}(\mu)$ are introduced in this article by analogy. The bar is added to indicate that $\geq$ in the definition of the additive core has been changed to $\leq$ (see also end of Section 3).

Proposition 1 *Let $\mathcal{S} \subset 2^{\Omega}$ be closed under finite union and intersection and $\beta : \mathcal{S} \to [0,1]$ monotone and k-monotone, then β_* is k-monotone, too.*

Proof Let $A_1, \ldots, A_k \subset 2^{\Omega}$. For $I \subset \{1, \ldots, k\}$, $I \neq \emptyset$ we use the abbreviation $A_I := \bigcap_{i \in I} A_i$. We have to show

$$\beta_*\left(\bigcup_{i=1}^{k} A_i\right) + \sum_{\substack{I \subset \{1,\ldots,k\} \\ I \neq \emptyset}} (-1)^{|I|} \ \beta_*(A_I) \geq 0. \tag{2}$$

According to the definition of the inner extension there is, for any $\epsilon > 0$ and $I \subset \{1, \ldots, k\}$, $I \neq \emptyset$ a set $B_I \in \mathcal{S}$, $B_I \subset A_I$, such that

$$\beta_*(A_I) - \epsilon \leq \beta(B_I) \leq \beta_*(A_I). \tag{3}$$

We define new sets

$$C_i := \bigcup_{\substack{J \subset \{1,\ldots,k\} \\ i \in J}} B_J, \quad i = 1, \ldots, k,$$

and like for the A_i

$$C_I := \bigcap_{i \in I} C_i, \quad I \subset \{1, \ldots, k\}, \ I \neq \emptyset.$$

By construction $B_I \subset C_I \in \mathcal{S}$ and C_I approximates A_I from the interior,

$$\beta_*(A_I) - \epsilon \leq \beta(C_I) \leq \beta_*(A_I) \tag{4}$$

For proving (2) we split the sum in two sums with $|I|$ even and odd respectively,

$$\beta_*\left(\bigcup_{i=1}^{k} A_i\right) + \sum_{\substack{I\subset\{1,\ldots,k\}\\ |I|\ \text{even}\\ I\neq\emptyset}} \beta_*\left(A_I\right)$$

$$\geq \beta\left(\bigcup_{i=1}^{k} C_i\right) + \sum_{\substack{I\subset\{1\ldots,k\}\\ |I|\ \text{even}\\ I\neq\emptyset}} \beta\left(C_I\right) \qquad \text{since } C_i \subset A_i,\ \beta \text{ monotone, } C_i \in \mathcal{S}$$

$$\geq \sum_{\substack{I\subset\{1,\ldots,k\}\\ |I|\ \text{odd}}} \beta\left(C_I\right) \qquad \text{since } \beta \text{ k-monotone}$$

$$\geq \sum_{\substack{I\subset\{1,\ldots,k\}\\ |I|\ \text{odd}}} \left(\beta_*\left(A_I\right) - \epsilon\right) \qquad \text{by (4)}$$

$$= \left(\sum_{\substack{I\subset\{1,\ldots,k\}\\ |I|\ \text{odd}}} \beta_*\left(A_I\right)\right) - 2^{k-1}\epsilon\,.$$

Since $\epsilon > 0$ was arbitrary, we get

$$\beta_*\left(\bigcup_{i=1}^{k} A_i\right) + \sum_{\substack{I\subset\{1,\ldots,k\}\\ |I|\ \text{even}\\ I\neq\emptyset}} \beta_*\left(A_I\right) \geq \sum_{\substack{I\subset\{1,\ldots,k\}\\ |I|\ \text{odd}}} \beta_*\left(A_I\right)\,.$$

which proves (2) □

Corollary 1 *Let $\mathcal{S}$ be closed under finite union and intersection. Then a lower chain measure β on $\mathcal{S}$ is totally monotone.*

Proof There is a chain $\mathcal{K} \subset \mathcal{S}$ such that $\beta = \left(\beta|\mathcal{K}\right)_* |\mathcal{S}$. Now $\beta|\mathcal{K}$ is modular, i.e. $\beta(A_1 \cup A_2) + \beta(A_1 \cap A_2) - \beta(A_1) - \beta(A_2) = 0$ for $A_1, A_2 \in \mathcal{K}$. This is the first step of an induction which shows the inclusion exclusion equation

$$\beta\left(\bigcup_{i=1}^{k} A_i\right) + \sum_{\substack{I\subset\{1,\ldots,k\}\\ I\neq\emptyset}} (-1)^{|I|}\ \beta\left(\bigcap_{i\in I} A_i\right) = 0$$

for sets $A_1, A_2, \ldots, A_k \in \mathcal{K}$ and any $k \geq 2$. Hence $\beta|\mathcal{K}$ is totally monotone. By Proposition 1 $(\beta|\mathcal{K})_*$ is totally monotone, too, and so is β. □

An alternative method to prove Corollary 1 without using Proposition 1 had been communicated to us by Michèle Cohen: Since k-monotonicity is a condition for a finite number of sets, one can restrict the lower chain measure to the algebra generated by these sets (we suppose that $\mathcal{S}$ is an algebra). Becoming a discrete necessity measure by Lemma 1, it is totally monotone since it's Möbius transform is non-negative (see e.g. [9] Example 6.3).

Finally we recall the definition of the **Choquet integral** of a $\mathcal{S}$-measurable function $X : \Omega \to \overline{\mathbb{R}}$ w.r.t. a monotone set function $\mu : \mathcal{S} \to [0,1]$,

$$\int X \, \mathrm{d}\mu := \int_0^\infty \mu\,(X > x)\;\mathrm{d}x \; - \int_{-\infty}^0 1 - \mu\,(X > x)\;\mathrm{d}x\,. \qquad (5)$$

Here X is $\mathcal{S}$-**measurable** if the upper level sets $\{X > x\} := \{\omega \in \Omega \mid X(\omega) > x\}$, $x \in \overline{\mathbb{R}}$, are contained in $\mathcal{S}$ (a weaker definition of measurability in this context can be found in [6], Chapter 4). The integral (5) may not exist, namely if both (Riemann) integrals on the right hand side equal ∞. The integral always exists if the function X is bounded below or above. The integral is finite if X is bounded.

3 The max-min representation with finitely additive measures

The min representation of a totally monotone measure with additive measures is well known. We prove here the max representation of a monotone measure with lower chain measures, which are totally monotone. The resulting max-min additive representation will also be dualised to the min-max additive representation.

Proposition 2 *Let $\mathcal{S}$ be closed under finite union and intersection, $\mu : \mathcal{S} \to [0,1]$ monotone and $X : \Omega \to \overline{\mathbb{R}}$ a $\mathcal{S}$-measurable function. Then there exist a lower chain measure $\beta \in \bar{C}_{lc}\,(\mu)$ and an additive measure $\alpha \in C\,(\beta)$ such that*

$$\int X \, \mathrm{d}\mu = \int X \, \mathrm{d}\beta = \int X \, \mathrm{d}\alpha\,.$$

Proof Denote with $\mathcal{K}_X$ the family of the upper level sets $\{X > x\}$, $x \in \overline{\mathbb{R}}$, and Ω added if not of the form $\{X > -\infty\}$. Then $\mathcal{K}_X$ is a chain with $\emptyset, \Omega \in \mathcal{K}_X$ and is contained in $\mathcal{S}$ since X is $\mathcal{S}$-measurable. Define $\beta := (\mu|\mathcal{K}_X)_* \,|\mathcal{S}$. By construction β is a lower chain measure and $\beta|\mathcal{K}_X = \mu|\mathcal{K}_X$. So we see from the definition of the Choquet integral that $\int X \, \mathrm{d}\mu = \int X \, \mathrm{d}\beta$. Since β is the smallest monotonic extension of $\mu|\mathcal{K}_X$ we have $\beta \leq \mu$, whence $\beta \in \bar{C}_{lc}\,(\mu)$.
Finally the existence of $\alpha \geq \beta$ with the desired properties is well known (see e.g. [6], Proposition 10.1). □

Since the integral behaves monotone in the set functions (Proposition 5.2 (iii) in [6]),

$$\mu \leq \nu\,,\ \mu\,(\Omega) = \nu\,(\Omega)\ \text{ implies } \int X \, \mathrm{d}\mu \leq \int X \, \mathrm{d}\nu\,,$$

we immediately get the following representation of the Choquet integral.

Corollary 2 *Let $\mathcal{S}$ be closed under finite union and intersection, $\mu : \mathcal{S} \to [0,1]$ monotone and X a $\mathcal{S}$-measurable function $X : \Omega \to \overline{\mathbb{R}}$ with finite integral, $\int X \,\mathrm{d}\mu \in \mathbb{R}$. Then*

$$\int X \,\mathrm{d}\mu = \bigvee_{\beta \in \bar{C}_{lc}(\mu)} \int X \,\mathrm{d}\beta \,, \tag{6}$$

where all integrals on the right hand side exist. Furthermore, for any $\beta \in \bar{C}_{lc}(\mu)$

$$\int X \,\mathrm{d}\beta = \bigwedge_{\alpha \in C(\beta)} \int X \,\mathrm{d}\alpha \,, \tag{7}$$

where the inf on the right hand side is understood to be taken for all $\alpha \in C(\beta)$ for which the integral $\int X \,\mathrm{d}\alpha$ exists (there is at least one). Altogether we get

$$\int X \,\mathrm{d}\mu = \bigvee_{\beta \in \bar{C}_{lc}(\mu)} \bigwedge_{\alpha \in C(\beta)} \int X \,\mathrm{d}\alpha \,. \tag{8}$$

Equation (8) is called the **max-min additive representation** of the Choquet integral. Especially, taking indicator functions for X, we get the max-min additive representation of a monotone set function $\mu : \mathcal{S} \to [0,1]$,

$$\mu = \bigvee_{\beta \in \bar{C}_{lc}(\mu)} \bigwedge_{\alpha \in C(\beta)} \alpha \,. \tag{9}$$

Since, by Corollary 1, $\bar{C}_{lc} \subset \bar{C}_{tm}$ we can also write

$$\int X \,\mathrm{d}\mu = \bigvee_{\beta \in \bar{C}_{tm}(\mu)} \bigwedge_{\alpha \in C(\beta)} \int X \,\mathrm{d}\alpha \,. \tag{10}$$

Now we dualise the max-min additive representation. First we need some definitions. For $\mathcal{S} \subset 2^{\Omega}$ we set $\mathcal{S}_c := \{A^c \mid A \in \mathcal{S}\}$, A^c denoting the complement $\Omega \setminus A$ of A. The **conjugate** or **dual** of a set function $\mu : \mathcal{S} \to [0,1]$ (with $\mu(\Omega) = 1$ as always supposed) is the set function $\overline{\mu} : \mathcal{S}_c \to [0,1]$,

$$\overline{\mu}(A) := 1 - \mu(A^c) \,, \quad A \in \mathcal{S}_c \,.$$

Definition 3 *Let $\mu : \mathcal{S} \to [0,1]$ be monotone. μ is called an* **upper chain measure** *if $\overline{\mu}$ is a lower chain measure. Let, in addition, $\mathcal{S}$ be closed under finite intersection and union, then μ is called a* **plausibility measure** *if $\overline{\mu}$ is a belief measure, i.e. totally monotone.*

For a given set function $\mu : \mathcal{S} \to [0,1]$ we define the **upper chain core** as

$$C_{uc}(\mu) := \{\nu : \mathcal{S} \to [0,1] \mid \nu \text{ upper chain measure, } \nu \geq \mu\} \,.$$

If $\mathcal{S}$, in addition, is closed under finite union and intersection, we define the **plausibility core** as

$$C_{pl}(\mu) := \{\nu : \mathcal{S} \to [0,1] \mid \nu \text{ plausibility measure, } \nu \geq \mu\}.$$

Since $\mu \leq \nu$ iff $\overline{\mu} \geq \overline{\nu}$ we get

$$C_{uc}(\overline{\mu}) = \bar{C}_{lc}(\mu)\ , \quad C_{pl}(\overline{\mu}) = \bar{C}_{tm}(\mu)\ .$$

Similarly, for the additive core we get

$$C(\overline{\mu}) = \{\alpha : \mathcal{S} \to [0,1] \mid \alpha \text{ monotone and additive, } \alpha \leq \mu\},$$

which could also be denoted $\bar{C}(\mu)$.
Now one easily derives from (8) the **min-max additive representations** of the Choquet integral

$$\int X \,\mathrm{d}\mu = \bigwedge_{\nu \in C_{uc}(\mu)} \bigvee_{\alpha \in C(\overline{\nu})} \int X \,\mathrm{d}\alpha\,, \tag{11}$$

which holds under the (dual) conditions of Corollary 2. Again, on the right hand side, $C_{uc}(\mu)$ can be replaced by $C_{pl}(\mu)$ (cf. (8) and (10)).

4 Min-max representations with σ-additive measures

In this section we impose an additional topological structure on the set Ω and require some types of continuity and regularity for our monotone measures so that Choquets Capacitability Theorem can be employed to get a min-max σ-additive representation. The min is taken over upper chain measures which are - in the sense of Choquet - capacities w.r.t. the closed sets. The max is taken over the σ-core.
Here we will restrict ourselves to σ-additive measures, so we start defining the **σ-core** of a given set function $\mu : \mathcal{S} \to [0,1]$ as

$$C^{\sigma}(\mu) := \{\alpha : \mathcal{S} \to [0,1] \mid \alpha \text{ monotone and } \sigma\text{-additive, } \alpha \geq \mu\}.$$

For the sake of completeness we quote a theorem of Schmeidler [11] Theorem 3.2 which is proved, too, as Theorem 1 in [10]. In our context we don't formulate it in the most general form, actually it holds for exact β.

Proposition 3 *Let $\beta : \mathcal{A} \to [0,1]$ be monotone and supermodular on an algebra $\mathcal{A} \subset 2^{\Omega}$. Then $C(\beta) = C^{\sigma}(\beta)$ iff β is σ-continuous from below.*

Here, a set function $\mu : \mathcal{S} \to [0,1]$ is called **σ-continuous from below** if for any sequence A_n with $A_n \subset A_{n+1}$, $n \in \mathbb{N}$, and $A := \bigcup_n A_n \in \mathcal{S}$ one has $\mu(A) = \lim_{n\to\infty} \mu(A_n)$. Similarly, **$\sigma$-continuity from above** is defined. Under the joint assumptions of Proposition 3 and Corollary 2 the min representation (7) remains valid with the σ-core. But, as we will see below, these assumptions are too restrictive for our purposes.

Since, for our max-min or min-max additive representations, we want to restrict ourselves to σ-additive measures, we of course have to impose continuity conditions on the lower or upper chain measures in the representations (8) or (11). We only treat the min-max representation since it fits better in the existing literature [3], [5], [10]. The other case can then be derived by dualisation.
The continuity conditions mentioned above require topological assumptions. Hence let Ω be a topological space with $\mathcal{G}(\Omega)$ denoting the family of open sets and $\mathcal{F}(\Omega)$ the family of closed sets in Ω. As usual, $\mathcal{B}(\Omega)$ is the σ-Algebra of the Borel sets of Ω which contains $\mathcal{F}(\Omega)$ and $\mathcal{G}(\Omega)$.

Definition 4 *A set function $\nu : \mathcal{B}(\Omega) \to [0,1]$ is called a* **(Choquet) capacity** *(w.r.t. $\mathcal{F}(\Omega)$), if ν is monotone, σ-continuous from below and $\nu|\mathcal{F}(\Omega)$ is σ-continuous from above.*

These asymmetric continuity conditions arise quite naturally as the following example shows.

Example 1 Let Ω be a metric space, $K \subset \Omega$ compact and

$$u_K(A) := \begin{cases} 1 & \text{if } A \supset K \\ 0 & \text{else} \end{cases} , \quad A \in \mathcal{B}(\Omega) ,$$

the unanimity game for K. Clearly u_K is σ-continuous from above and one can show with the open covering argument that $u_K|\mathcal{G}(\Omega)$ is σ-continuous from below. But u_K itself is not σ-continuous from below in general. Take e.g. $\Omega = K = \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\} \subset \mathbb{R}$. Then the sequence of sets $A_n := \{0, 1, \frac{1}{2}, \dots \frac{1}{n}\}$ approximates K from below but $\lim_{n\to\infty} u_K(A_n) = 0 \neq 1 = u_K(K)$.
u_K has just the dual properties of a capacity, so the conjugate $\overline{u_K}$ of u_K is a capacity. Furthermore u_K is a lower chain measure and $\overline{u_K}$ an upper chain measure. □

We give a construction principle for capacities which generalises the previous example in case $\Omega = K$. The **closure from below** $\overline{\mathcal{S}}$ of a set system $\mathcal{S} \subset 2^{\Omega}$ consists of all sets in Ω which are the union of an increasing sequence of sets in $\mathcal{S}$.

Lemma 2 *Let Ω be a compact topological space and $\mu : \mathcal{G}(\Omega) \to [0,1]$ monotone and σ-continuous from below. Furthermore let $\mathcal{K} \subset \mathcal{G}(\Omega)$ be a chain with $\emptyset, \Omega \in \mathcal{K}$. Then $\overline{\mathcal{K}} \subset \mathcal{G}(\Omega)$ and $\nu := \left(\mu|\overline{\mathcal{K}}\right)^* |\mathcal{B}(\Omega)$ is a capacity.*

Proof First $\overline{\mathcal{K}} \subset \mathcal{G}(\Omega)$ since any union of open sets is again open. $\overline{\mathcal{K}}$ is a chain, hence $\mu|\overline{\mathcal{K}}$ is modular and, by Proposition 2.4 (iii) in [6], σ-continuity from below is inherited by $\nu := \left(\mu|\overline{\mathcal{K}}\right)^* |\mathcal{B}(\Omega)$.
So it remains to prove that $\nu|\mathcal{F}(\Omega)$ is σ-continuous from above. Given a sequence $(F_n)_{n\in\mathbb{N}}$ with $F_n \in \mathcal{F}(\Omega)$, $F_n \supset F_{n+1}$, $\forall n \in \mathbb{N}$, $F := \bigcap_{n\in\mathbb{N}} F_n$ we have to show

$$\nu(F) = \lim_{n\to\infty} \nu(F_n) . \tag{12}$$

Since $\nu(F)$ is defined as $\bigwedge\left\{\mu(D) \mid D \in \overline{\mathcal{K}},\ D \supset F\right\}$ we consider a set $D \in \overline{\mathcal{K}}$ with $D \supset F$. Then the sets $F_n \setminus D$, $n \in \mathbb{N}$, are compact and have intersection $F \setminus D = \emptyset$. So compactness of Ω implies, by the finite intersection property, that already a finite intersection of certain $F_n \setminus D$ is empty. Since the $F_n \setminus D$ form a decreasing sequence we get $F_n \setminus D = \emptyset$ for all n sufficiently large. To summarise, for any $D \in \overline{\mathcal{K}}$ with $D \supset F$ there exists a natural number m such that $F_n \subset D$ for all $n \geq m$. This fact implies

$$\left\{D \mid D \in \overline{\mathcal{K}},\ D \supset F\right\} = \bigcup_{n \in \mathbb{N}} \left\{D \mid D \in \overline{\mathcal{K}},\ D \supset F_n\right\},$$
$$\bigwedge\left\{\mu(D) \mid D \in \overline{\mathcal{K}},\ D \supset F\right\} = \lim_{n \to \infty} \bigwedge\left\{\mu(D) \mid D \in \overline{\mathcal{K}},\ D \supset F_n\right\},$$

which is equation (12) to be proved. □

The capacity ν in Lemma 2 is an upper chain measure since its conjugate $\overline{\nu}$ is a lower chain measure because of the general identity $\overline{(\mu^*)} = (\overline{\mu})_*$ which holds for a monotone set function μ on a set system $\mathcal{S} \subset 2^{\Omega}$. Before formulating our main result we define, for a monotone set function $\mu : \mathcal{B}(\Omega) \to [0,1]$,

$$C^{\sigma}_{uc}(\mu) := \{\nu \mid \nu \text{ is a capacity and } \nu \in C_{uc}(\mu)\},$$

and for use in Section 5,

$$C^{\sigma}_{pl}(\mu) := \{\nu \mid \nu \text{ is a capacity and } \nu \in C_{pl}(\mu)\},$$
$$\bar{C}^{\sigma}_{tm}(\mu) := \{\nu \mid \overline{\nu} \text{ is a capacity and } \nu \in \bar{C}_{tm}(\mu)\}.$$

Theorem 1 *Let Ω be a compact metric space, $\mu : \mathcal{B}(\Omega) \to [0,1]$ monotone, $\mu|\mathcal{G}(\Omega)$ σ-continuous from below and $X : \Omega \to \mathbb{R}$ a continuous function. Then there exist an upper chain measure $\nu \in C^{\sigma}_{uc}(\mu)$ and a σ-additive measure $\alpha \in C^{\sigma}(\overline{\nu})$ such that*

$$\int X \,\mathrm{d}\mu = \int X \,\mathrm{d}\nu = \int X \,\mathrm{d}\alpha.$$

If, in addition, $\int X \,\mathrm{d}\mu \in \mathbb{R}$ then, with the understanding (cf. Corollary 2) that the sup has to be extended only over those integrals which exist,

$$\int X \,\mathrm{d}\mu = \bigwedge_{\nu \in C^{\sigma}_{uc}(\mu)} \bigvee_{\alpha \in C^{\sigma}(\overline{\nu})} \int X \,\mathrm{d}\alpha.$$

Proof The chain $\mathcal{K}$ of the upper level sets $\{X > x\}$, $x \in \overline{\mathbb{R}}$, is contained in $\mathcal{G}$ since X is continuous. Furthermore $\mathcal{K} = \overline{\mathcal{K}}$ because we get $\bigcup_{n \in \mathbb{N}} \{X > x_n\} = \{X > x\}$ for any sequence $(x_n)_{n \in \mathbb{N}}$ with $x_n > x_{n+1}\ \forall n \in \mathbb{N},\ \inf_{n \in \mathbb{N}} x_n = x$. Finally a metric space is normal and all assumptions of Lemma 2 are veryfied. Hence $\nu := (\mu|\mathcal{K})^* |\mathcal{B}(\Omega)$ is a capacity. ν being the largest extension of $\mu|\mathcal{K}$ and an upper chain measure we get $\nu \in C^{\sigma}_{uc}(\mu)$. By construction $\mu|\mathcal{K} = \nu|\mathcal{K}$ such that

$$\int X \, \mathrm{d}\mu = \int X \, \mathrm{d}\nu \, .$$

The existence of an $\alpha \in C^{\sigma}(\overline{\nu})$ with the desired properties derives from Theorem 5 in [10], which requires a metric space. First we recall that the definition (5) of the Choquet integral with $\mathcal{S} = \mathcal{B}(\Omega)$ could have equivalently been given with the distribution function $G_{\geq}(x) := \mu(X \geq x)$ in place of $G_{>}(x) := \mu(X > x)$ since both functions can differ only in points of discontinuity of $G_{>}$ and the latter are countable for a decreasing function on the reals. Again by continuity of X we know that the chain $\mathcal{K}_{\geq}$ of the upper level sets $\{X \geq x\}$, $x \in \overline{\mathbb{R}}$, is contained in $\mathcal{F}$. In this context the theorem of Parker mentioned above guarantees that there is already an $\alpha \in C^{\sigma}(\overline{\nu})$ with $\alpha|\mathcal{K}_{\geq} = \nu|\mathcal{K}_{\geq}$ if there is an $\alpha_0 \in C(\overline{\nu})$ with $\alpha_0|\mathcal{K}_{\geq} = \nu|\mathcal{K}_{\geq}$. Since the last assertion is well known (see e.g. [6] Proposition 10.1) we get the desired α with

$$\int X \, \mathrm{d}\nu = \int X \, \mathrm{d}\alpha \, .$$

Finally the min-max representation derives like in Corollary 2. □

We remark that Choquets Capacitability Theorem [5] III.28 was used by Parker to prove her theorem which, in turn, entered our proof.
Since indicator functions are not continuous in general, we cannot apply Theorem 1 to get a min-max additive representation for monotone set functions. An additional regularity condition is needed. $\mu : \mathcal{B}(\Omega) \to [0,1]$ is called **outer regular** if

$$\mu(A) = \bigwedge_{\substack{B \in \mathcal{G}(\Omega) \\ B \supset A}} \mu(B) \, , \quad A \in \mathcal{B}(\Omega) \, .$$

Inner regularity is defined similarly with $\mathcal{F}(\Omega)$.

Proposition 4 *Let Ω be a compact metric space, $\mu : \mathcal{B}(\Omega) \to [0,1]$ monotone, $\mu|\mathcal{G}(\Omega)$ σ-continuous from below and μ outer regular. Then for any $A \in \mathcal{B}$ there exist an upper chain measure $\nu \in C^{\sigma}_{uc}(\mu)$ and a σ-additive measure $\alpha \in C^{\sigma}(\overline{\nu})$ such that*

$$\mu(A) = \nu(A) = \alpha(A) \, .$$

Especially we get

$$\mu = \bigwedge_{\nu \in C^{\sigma}_{uc}(\mu)} \bigvee_{\alpha \in C^{\sigma}(\overline{\nu})} \alpha \, .$$

Proof The dual version of [10] Theorem 7 (compactness is not needed there) shows that for any submodular capacity $\nu : \mathcal{B}(\Omega) \to [0,1]$ and any $A \in \mathcal{B}$ there is an $\alpha \in C^\sigma(\overline{\nu})$ such that

$$\nu(A) = \alpha(A)\,.$$

By our regularity assumption there exists for $A \in \mathcal{B}(\Omega)$ a sequence $(G_n)_{n\in\mathbb{N}}$, $G_n \in \mathcal{G}(\Omega)$, $G_n \supset A$, $G_n \supset G_{n+1}\ \forall\, n \in \mathbb{N}$ with $\mu(A) = \lim_{n\to\infty} \mu(G_n)$. Now Lemma 2 provides us with a capacity $\nu \in C_{uc}(\mu)$ with $\mu(G_n) = \nu(G_n)$. Then

$$\mu(A) = \lim_{n\to\infty} \mu(G_n) = \lim_{n\to\infty} \nu(G_n) \geq \nu(A).$$

Since $\mu \leq \nu$ we get altogether

$$\mu(A) = \nu(A)\,,$$

which proves our proposition. □

The representations in Theorem 1 and Proposition 4 are sustained, if the capacities of the upper chain core are replaced by the capacities of the plausibility core.

5 Conclusion

We presented sufficient conditions under which there exist max-min and min-max additive representations of monotone measures and their integrals by means of σ-additive measures. These results are essential tools to generalise products of measures and conditional expectation of random variables to the non-additive case. But some work has still to be done to generalise the respective results of [8] to the non-discrete case. For the case of products a suitable sandwich theorem like Lemma 4.3 in [8] is needed. For generalising conditional expectation the problems raised by the nullsets have to be overcome.

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