Ten Questions in Tropical Geometry

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December 13, 2006

On December 13, 2006, the nine of us met at MSRI Berkeley and discussed ongoing projects and future directions in tropical geometry. Our discussion led to the conclusion that the following ten research problems are particularly important and their resolution would be likely to lead to significant new insights.

Question 1. *Is it possibly true that every tropical variety is shellable?*

It would be amazing, if true. Some tropical varieties have been shown to be shellable (or to have links homotopic to a wedge of spheres), and tropical varieties are nice polyhedral complexes in general. Under some genericity assumptions, tropical varieties have only top homology (Hacking). Even if this is not true, it would be nice to know under what conditions these are shellable.

Question 2. *Give a good formulation for the moduli space of curves of degree d and genus g lying in $\mathbb{TP}^2$ This is unknown even for small d and g (e.g. $(2,0)$.)*

Note that for some purposes (Hannah's) it proves more convenient to think of curves as embedded graphs, where each (bounded or unbounded) edge is given a length and a slope. With this formulation, you may end up with components consisting of "virtual" curves (with edges of zero slope, or fake vertices.) You will get the same practical answers either way due to multiplicity.

Question 3. *Investigate matroid subdivisions. Which f-vectors are possible? Are series-parallel ones extreme in this category? Can you do this for $\Delta(8,4)$, which would be tantamount to figuring out the tropical Grassmannian?*

Related to this is David Speyer's conjecture on f-vectors of tropical linear spaces; this is tantamount to asking for the "loopless f-vectors" of matroid subdivisions. Other things we might like to know are: what matroid properties are inherited in some nice way under matroid subdivisions? What are the face posets of matroid subdivisions? The positive part of this Grassmannian is related to cluster algebras of infinite type and hence to representation theory.

Question 4. *Classify all "root system polytopes", i.e. polytopes given by inequalities of the form $x_i - x_j \leq b_{ij}$. This would be interesting even for dimension 4, and probably too easy to be publishable for dimension 3.*

These are precisely the polytopes (both tropically and ordinarily convex) which arise as bounded regions in tropical linear spaces, and they also form the building blocks of tropical polytopes.

Question 5. *Compute the (positive) tropical flag variety GL_4/B in its Plücker embedding, i.e. classify all flags $\{point \subset line \subset plane\}$ in $\mathbb{TP}^3$.*

This is related to Kamnitzer's work on MV-polytopes.

Question 6. *What do the face lattices of tropical polytopes look like? They conjecturally have many nice properties, such as existing, being homotopy spheres, etc.. The faces themselves should be contractible. Also, it would be nice to have a practical algorithm to compute the faces of tropical polytopes.*

It is extremely likely that these include face lattices of ordinary polytopes (should be trivial to check); what else do they include?

Question 7. *When does tropicalization commute with intersection? Is it enough that the intersection of the tropicalizations is dimensionally transverse; in other words, has the right codimension (i.e. the sum of the codimensions of the varieties)?*

We know that this is true when the actual intersection of the tropical varieties is transverse, but the tropical varieties could intersect in the right dimension but not transversely. This is true for the "snowflake example" also, which is not of this form. Finally, this ought to be true locally (i.e. should be true even if the varieties intersect badly somewhere else.)

Question 8. *What is the best axiomatization of tropical oriented matroids? This is the same as giving matroid-type axioms which govern the possible type sets in a tropical polytope/hyperplane arrangement, in the abstract.*

Ideally, these should be in bijection with the set of all subdivisions of a product of two simplices (not just regular ones) as well as with tropical pseudohyperplane arrangements, whatever those are.

Question 9. *Is the tropical discriminant of a defective point configuration a subfan of the secondary fan?*

This is known to be true for the non-defective case, and an incorrect proof appeared previously.

Question 10. *How can you tell if something is a tropical variety? (If I give you a geometric object, what is the dimension of the space of weights which can be placed on the faces to make it into a tropical variety?)*

There are many possible relevant meanings of this question. For instance, if you have a geometric object (polyhedral complex) in $\mathbb{TP}^{n-1}$, how can you tell if it is a tropical variety? If you have a combinatorial object (polyhedral complex) in the abstract, perhaps with weights, can you tell if it is realizable (in the natural sense) as some tropical variety in some tropical projective space?