

# ABELIANIZING THE REAL PERMUTATION ACTION VIA BLOWUPS

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## 1. INTRODUCTION

Our object of study is an abelianization of the  $\mathcal{S}_n$  permutation action on  $\mathbb{R}^n$  that is provided by a particular De Concini-Procesi wonderful model for the braid arrangement. Our motivation comes from an analogous construction for finite group actions on complex manifolds, due to Batyrev [B1, B2], and subsequent study of Borisov & Gunnells [BG], where the connection of such abelianizations with De Concini-Procesi wonderful models for arrangement complements was first observed.

Whereas previous studies were restricted to complex manifolds, here we study one of the most natural nontrivial actions of a finite group on a *real* differentiable manifold, namely the permutation action on  $\mathbb{R}^n$ . The locus of non-trivial stabilizers in this case is provided by the braid arrangement  $\mathcal{A}_{n-1}$ . We suggest to blow up intersections of subspaces in  $\mathcal{A}_{n-1}$ , respectively proper transforms of those intersections, in the order of an arbitrary linear extension of the intersection lattice  $\Pi_n$ , so as to exhaust all of the arrangement. That is the same as to take the De Concini-Procesi wonderful model of the arrangement complement with respect to the maximal building set, see [DP].

Not only do we obtain an abelianization of the real permutation action, we even show that stabilizers of points in the arrangement model are isomorphic to direct products of  $\mathbb{Z}_2$ . To this end, we develop a combinatorial framework for explicitly describing the stabilizers in terms of automorphism groups of set diagrams over families of cubes.

Moreover, we observe that the natural nested set stratification on the arrangement model is not stabilizer distinguishing with respect to the  $\mathcal{S}_n$ -action, i.e., stabilizers of points are not in general isomorphic on open strata. Motivated by this structural deficiency, we furnish a new stratification of the De Concini-Procesi arrangement model that distinguishes stabilizers.

Arrangement models have been extensively studied over the last years. They were introduced by De Concini & Procesi in [DP], one of the motivations being to provide rational models for cohomology algebras of arrangement complements. In [FK] the De Concini-Procesi model construction was put in a very general combinatorial context, showing that the notions of building sets and nested sets, coined already by Fulton & MacPherson in [FM], along with the notion of a blowup, have canonical combinatorial counterparts in the theory of semilattices. It was also shown in [FK] that this combinatorial framework actually traces precisely the step-by-step change in the incidence structure of strata during the De Concini-Procesi resolution process.

On the geometric side, wonderful arrangement models were generalized to wonderful conical compactifications by MacPherson & Procesi [MP], and Gaiffi [G2] recently provided a further generalization incorporating mixed real subspace and halfspace arrangements as well as real stratified manifolds as starting points of the construction. Algebraic topological invariants of wonderful models are another focus of interest. Yuzvinsky [Y] provided a monomial basis for the cohomology of wonderful compactifications of hyperplane arrangements that was later generalized by Gaiffi to compactifications of subspace arrangements in [G1].

We give a more detailed outline of our paper: In Section 2, we begin our investigations with a brief review of De Concini-Procesi wonderful models. Moreover, we describe how an action of a finite group on an arrangement extends to an action on the arrangement model. We then turn to our specific situation, observing that when blowing up the entire locus of non-trivial stabilizers for  $\mathcal{S}_n$  acting on  $\mathbb{R}^n$ , i.e., the entire braid arrangement, the nested set stratification is not sufficient to distinguish stabilizers. That is, we may have two points lying on the same stratum, but having non-isomorphic stabilizers. In fact, this happens already for  $n = 3$ .

In Section 4, we study the nested set stratification and group actions on De Concini-Procesi models in some detail, so that finally, in Section 5, we are able to rectify the situation: We define a different stratification on the De Concini-Procesi model such that, on one hand, this stratification is naturally arrived at by tracing a certain, interesting on its own right, subspace arrangement in  $\mathbb{R}^n$ , on the other hand, this new stratification is *stabilizer distinguishing*.

In Section 6 we turn to the detailed study of the isomorphism types of stabilizers of points in the De Concini-Procesi resolution of the braid arrangement. Relying on our analysis in the previous sections, we know that the stabilizer of a point in the arrangement model is the intersection of a number of stabilizers of lines and of the stabilizer of one single point in  $\mathbb{R}^n$ . We develop a combinatorial language to describe stabilizers of points and lines in  $\mathbb{R}^n$ , namely by representing them as automorphism groups of set diagrams over families of cubes. The crucial property of this representation is that taking intersections of a number of automorphism groups of such diagrams will again yield an automorphism group over a diagram. This new diagram can be combinatorially read off from the original diagrams. Thus, we succeed to represent the stabilizer of a point in the arrangement model as an automorphism group of a set diagram over a family of cubes. By further analysis of this diagram, we are finally able to prove in Section 7 that, beyond the natural initial expectation that the stabilizers ought to be abelian, they in fact are isomorphic to direct products of  $\mathbb{Z}_2$ , with the number of factors in each product at most  $\lfloor \frac{n}{2} \rfloor$ .

## 2. DE CONCINI-PROCESI ARRANGEMENT MODELS

In this section we briefly review the construction and main characteristics of wonderful arrangement models as introduced by De Concini & Procesi in [DP]. We first remind the notions of building sets and nested sets since they guide the explicit construction and capture the underlying incidence combinatorics of a natural stratification. Moreover, we

comment on actions of finite groups on De Concini-Procesi models that are induced from group actions on the arrangement.

**2.1. Building sets and nested sets.** Let  $\mathcal{A}$  be an arrangement of linear subspaces in a finite dimensional real or complex vector space, and denote by  $\mathcal{L} = \mathcal{L}(\mathcal{A})$  the lattice of intersections of spaces in  $\mathcal{A}$  ordered by reverse inclusion, customarily called the **intersection lattice** of  $\mathcal{A}$ .

**Definition 2.1.** ([DP, §2]) *For  $\mathcal{L} = \mathcal{L}(\mathcal{A})$  the intersection lattice of a complex or real subspace arrangement, let  $\mathcal{L}^*$  denote the lattice formed by the orthogonal complements of intersections in  $\mathcal{A}$  ordered by inclusion.*

- (1) *For  $U \in \mathcal{L}^*$ ,  $U = \bigoplus_{i=1}^k U_i$  with  $U_i \in \mathcal{L}^*$ , is called a **decomposition** of  $U$  if for any  $V \subseteq U$ ,  $V \in \mathcal{L}^*$ ,  $V = \bigoplus_{i=1}^k (U_i \cap V)$  and  $U_i \cap V \in \mathcal{L}^*$ , for  $i = 1, \dots, k$ .*
- (2) *Call  $U \in \mathcal{L}^*$  **irreducible** if it does not admit a non-trivial decomposition.*
- (3)  *$\mathcal{G} \subseteq \mathcal{L}^* \setminus \{\hat{0}\}$  is called a **building set** for  $\mathcal{A}$  if for any  $U \in \mathcal{L}^* \setminus \{\hat{0}\}$  and  $G_1, \dots, G_k$  maximal in  $\mathcal{G}$  below  $U$ ,  $U = \bigoplus_{i=1}^k G_i$  is a decomposition (the  $\mathcal{G}$ -decomposition) of  $U$ .*
- (4) *A subset  $\mathcal{T} \subseteq \mathcal{G}$  is called **nested** if for any set of non-comparable elements  $U_1, \dots, U_k$  in  $\mathcal{T}$ ,  $U = \bigoplus_{i=1}^k U_i$  is the  $\mathcal{G}$ -decomposition of  $U$ . The nested sets in  $\mathcal{G}$  form an abstract simplicial complex, the **nested set complex**  $\mathcal{N}(\mathcal{G})$ .*

We will without further notice consider building sets as subsets of the intersection lattice  $\mathcal{L}$ , and thus let the consideration of  $\mathcal{L}^*$  remain a detour for the sake of providing a transparent definition. Note that for any arrangement  $\mathcal{A}$  the set of irreducible elements in  $\mathcal{L}(\mathcal{A}) \setminus \{\hat{0}\}$  is the minimal building set, whereas  $\mathcal{G} = \mathcal{L}(\mathcal{A}) \setminus \{\hat{0}\}$  is the maximal building set. For the maximal building set the nested set complex coincides with the order complex of the (non-reduced) intersection lattice.

**2.2. Arrangement models and the nested set stratification.** We are now prepared to give the definition of wonderful arrangement models. Let  $\mathcal{A}$  be an arrangement of subspaces in a real or complex vector space  $V$ ,  $\mathcal{L}(\mathcal{A})$  its intersection lattice, and  $\mathcal{G}$  a building set for  $\mathcal{A}$ . On the complement of the arrangement,  $\mathcal{M}(\mathcal{A}) := V \setminus \bigcup \mathcal{A}$ , consider the map

$$(2.1) \quad \Phi : \mathcal{M}(\mathcal{A}) \longrightarrow V \times \prod_{G \in \mathcal{G}} \mathbb{P}(V/G),$$

where in its first coordinate the map is given by inclusion, and in later coordinates by projection to the (real, resp. complex) projectivizations of the respective quotient spaces. Formally,

$$\Phi(x) = (x, (\Phi_G(x))_{G \in \mathcal{G}}),$$

with  $\Phi_G(x) = \langle x, G \rangle / G \in \mathbb{P}(V/G)$ , for  $x \in \mathcal{M}(\mathcal{A})$ , where brackets  $\langle \cdot, \cdot \rangle$  denote the linear span of subspaces or vectors, respectively. This map is an embedding of  $\mathcal{M}(\mathcal{A})$ , the arrangement model  $Y_{\mathcal{G}}$  is defined as the closure of its image in  $V \times \prod_{G \in \mathcal{G}} \mathbb{P}(V/G)$ :

$$Y_{\mathcal{G}} := \text{cl}(\text{Im } \Phi).$$

Alternatively,  $Y_{\mathcal{G}}$  can be described as the result of subsequently blowing up intersections of subspaces in  $\mathcal{A}$ , and proper transforms of such, corresponding to building set elements  $G \in \mathcal{G}$  in some linear extension of the inclusion order.

The arrangement model  $Y_{\mathcal{G}}$  is a smooth variety that contains the arrangement complement  $\mathcal{M}(\mathcal{A})$  as an open subspace. The complement  $D$  of  $\mathcal{M}(\mathcal{A})$  in  $Y_{\mathcal{G}}$  is a divisor with normal crossings, in fact, it is the union of smooth, irreducible components  $D_G$  indexed by building set elements  $G \in \mathcal{G}$ . The intersections of divisors  $D_G$  are smooth and irreducible, naturally, they are indexed with subsets of  $\mathcal{G}$ . One of the main results of De Concini and Procesi, [DP], states that an intersection of divisors is non-empty if and only if it is indexed with a nested set in  $\mathcal{G}$ .

We call the resulting stratification of  $Y_{\mathcal{G}}$  by irreducible divisor components  $D_G$  and their intersections the *nested set stratification* of  $Y_{\mathcal{G}}$ , and denote it by  $(Y_{\mathcal{G}}, \mathfrak{D})$ . Note that the poset of strata for  $(Y_{\mathcal{G}}, \mathfrak{D})$  coincides with the face poset of the nested set complex  $\mathcal{N}(\mathcal{G})$ .

De Concini & Procesi also provide a projective version of their arrangement models obtained by starting out with the projectivization of the arrangement complement and replacing the first factor on the right hand side of (2.1) by  $\mathbb{P}(V)$  accordingly. The properties of the resulting projective model  $\bar{Y}_{\mathcal{G}}$  are similar to those of  $Y_{\mathcal{G}}$ , for details we refer to [DP, §4].

### 2.3. Finite group actions on arrangements and on their wonderful models.

Let us now assume that a finite group  $\Gamma$  acts on our vector space  $V$  by linear transformations, and that the arrangement  $\mathcal{A}$  is invariant under that action. By a standard result from representation theory, any linear action of a finite group is orthogonal [V, 2.3, Thm. 1]. Throughout the paper, we denote the corresponding  $\Gamma$ -invariant positive definite symmetric bilinear form by the usual scalar product.

Since we assume  $\Gamma$  to preserve  $\mathcal{A}$ , the group acts on the intersection lattice of  $\mathcal{A}$ ,

$$\gamma(A_1 \cap \dots \cap A_r) = \gamma(A_1) \cap \dots \cap \gamma(A_r), \quad \text{for all } \gamma \in \Gamma, A_1, \dots, A_r \in \mathcal{A},$$

as well as internally on the corresponding intersections of subspaces. Also,  $\Gamma$  acts on the ambient space of the arrangement model corresponding to the maximal building set, that is on  $V \times \prod_{G \in \mathcal{G}} \mathbb{P}(V/G)$ , where  $\mathcal{G} = \mathcal{L}(\mathcal{A}) \setminus \{\hat{0}\}$ , by

$$\begin{aligned} \gamma(x, (x_G)_{G \in \mathcal{G}}) &= (\gamma(x), (\gamma(x_{\gamma^{-1}(G)}))_{G \in \mathcal{G}}), \\ &\text{for all } \gamma \in \Gamma, (x, (x_G)_{G \in \mathcal{G}}) \in V \times \prod_{G \in \mathcal{G}} \mathbb{P}(V/G). \end{aligned}$$

Moreover, the inclusion map  $\Phi : \mathcal{M}(\mathcal{A}) \longrightarrow V \times \prod_{G \in \mathcal{G}} \mathbb{P}(V/G)$  defined in (2.1) commutes with the action of  $\Gamma$ :

$$\begin{aligned} \gamma(\Phi(x)) &= \gamma(x, (\langle x, G \rangle / G)_{G \in \mathcal{G}}) = (\gamma(x), (\gamma(\langle x, \gamma^{-1}(G) \rangle / \gamma^{-1}(G))_{G \in \mathcal{G}}) \\ &= (\gamma(x), (\langle \gamma(x), G \rangle / G)_{G \in \mathcal{G}}) = \Phi(\gamma(x)), \quad \text{for } \gamma \in \Gamma, x \in \mathcal{M}(\mathcal{A}). \end{aligned}$$

We conclude that, since each element of  $\Gamma$  acts continuously on  $V$ , the closure of  $\text{Im } \Phi$  is  $\Gamma$ -invariant. Hence,  $\Gamma$  acts on the arrangement model  $Y_{\mathcal{G}}$  extending the  $\Gamma$ -action on  $\mathcal{M}(\mathcal{A}) \subseteq Y_{\mathcal{G}}$ .

Note that choosing a  $\Gamma$ -invariant building set  $\mathcal{G} \subsetneq \mathcal{L}(\mathcal{A}) \setminus \{\hat{0}\}$  as well yields an action of  $\Gamma$  on the corresponding arrangement model.

### 3. THE ARRANGEMENT MODEL $Y_{\Pi_n}$

**3.1. A candidate for an abelianization of the permutation action.** We consider the permutation action of the symmetric group  $\mathcal{S}_n$  on  $\mathbb{R}^n$ ,

$$\sigma(x) = (x_{\sigma(1)}, \dots, x_{\sigma(n)}), \quad \text{for all } \sigma \in \mathcal{S}_n, x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

The locus of points in  $\mathbb{R}^n$  with non-trivial stabilizer is a union of hyperplanes  $H_{i,j}$ ,  $H_{i,j} := \ker(x_i - x_j)$  for  $1 \leq i < j \leq n$ . This family of “diagonal hyperplanes” in  $\mathbb{R}^n$  is the *braid arrangement*  $\mathcal{A}_{n-1}$  of rank  $n-1$ , its name referring to the fact that the complement of a complexified version in  $\mathbb{C}^n$  is the classifying space of the pure braid group on  $n$  strands. The braid arrangement is one of the central examples in arrangement theory and has provided a starting point for many investigations and developments in arrangement theory and beyond, see e.g., [OT].

The intersection lattice of  $\mathcal{A}_{n-1}$  is the *partition lattice*  $\Pi_n$ , i.e., the poset of set partitions  $\pi = (\pi_1 | \dots | \pi_r)$  of  $\{1, \dots, n\} =: [n]$ ,  $\pi_i \subseteq [n]$  with  $\bigcup_{i=1}^r \pi_i = [n]$ , ordered by reverse refinement. Clearly, a partition  $\pi = (\pi_1 | \dots | \pi_r)$  in  $\Pi_n$  corresponds to the intersection of hyperplanes  $\bigcap_{(i,j) \in J_\pi} H_{i,j}$  with  $J_\pi = \{(i, j) \mid 1 \leq i < j \leq n, \{i, j\} \subseteq \pi_k, \text{ for some } 1 \leq k \leq r\}$ . We will freely use this correspondence between partitions and intersections of subspaces in the braid arrangement.

For further considerations, we restrict the permutation action to the  $(n-1)$ -dimensional real space

$$V = \{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i = 0\}.$$

The locus of points in  $V$  with non-trivial stabilizers is the intersection of  $\mathcal{A}_{n-1}$  with  $V$ , an essential arrangement with intersection lattice  $\Pi_n$ , which we still call braid arrangement and denote by  $\mathcal{A}_{n-1}$  without further mention.

We propose to study the De Concini-Procesi arrangement model  $Y_{\Pi_n}$  for  $\mathcal{A}_{n-1}$  as a candidate for an abelianization of the permutation action. We allow ourselves here to use the shorthand notation  $Y_{\Pi_n}$  instead of  $Y_{\Pi_n \setminus \{\hat{0}\}}$ . It follows from the general discussion in subsection 2.3 that  $Y_{\Pi_n}$  carries a natural  $\mathcal{S}_n$ -action extending the  $\mathcal{S}_n$ -action on  $\mathcal{M}(\mathcal{A}_{n-1}) \subseteq Y_{\Pi_n}$ . It turns out that rather curious phenomena enter the scene already in low dimensions.

**3.2. The nested set stratification is not stabilizer distinguishing.** Already for  $S_3$  acting on  $\mathbb{R}^3$ , the nested set stratification on the De Concini-Procesi model,  $(Y_{\Pi_3}, \mathfrak{D})$ , is not fine enough to distinguish stabilizers. Let us have a close look at the situation.

As above, we restrict the permutation action to  $V = \{(x_1, x_2, x_3) \mid \sum_{i=1}^3 x_i = 0\} \subseteq \mathbb{R}^3$ . The arrangement model  $Y_{\Pi_3}$  is the result of blowing up  $\{0\}$  in  $V$ . Topologically,  $Y_{\Pi_3}$  is an open Möbius band. As a subspace of  $V \times \mathbb{P}(V)$ ,  $Y_{\Pi_3}$  can be described as follows:

$$Y_{\Pi_3} = \{(x, \langle x \rangle) \mid x \neq 0\} \cup \{(0, l) \mid l \in \mathbb{P}(V)\} \subseteq V \times \mathbb{P}(V).$$

In terms of this pointwise description of  $Y_{\Pi_3}$  the divisors  $D_G$ ,  $G \in \Pi_3$ , read

$$\begin{aligned} D_{\{0\}} &= D_{(1,2,3)} = \{ (0, l) \mid l \in \mathbb{P}(V) \} \\ D_{(1,2)(3)} &= \{ (x, \langle x \rangle) \mid x_1 = x_2 \neq 0 \} \cup \{ (0, \langle (1, 1, -2) \rangle) \}, \end{aligned}$$

with  $D_{(1,3)(2)}$ ,  $D_{(1)(2,3)}$  having analogous descriptions.

Points on  $D_{(1,2)(3)}$  are stabilized by the 2-element subgroup of  $\mathcal{S}_3$  generated by the transposition  $\tau = (1, 2)$ : For a generic point on  $D_{(1,2)(3)}$ ,  $\tau$  fixes the point and thus the generating line. For the single point in  $D_{(1,2)(3)} \cap D_{\{0\}}$ ,  $\tau$  fixes 0 and the line  $\langle (1, 1, -2) \rangle$  pointwise. Analogously, we see that points on  $D_{(1,3)(2)}$  and on  $D_{(1)(2,3)}$  are stabilized by the transpositions  $(1, 3)$  and  $(2, 3)$ , respectively.

On  $D_{\{0\}}$ , however, we find points whose stabilizers the nested set stratification does not distinguish: Stabilizers for points on  $D_{\{0\}}$  are trivial except for those points on the intersections with one of the other three divisors, *and* for 3 additional points

$$\psi_{12} = (0, \langle (1, -1, 0) \rangle) \quad \psi_{13} = (0, \langle (1, 0, -1) \rangle) \quad \psi_{23} = (0, \langle (0, 1, -1) \rangle)$$

The  $\psi_{ij}$  are stabilized by transpositions  $(i, j)$ ,  $1 \leq i < j \leq 3$ , respectively, since the transpositions fix 0 and flip the lines in the second coordinate. In fact, the transposition  $(i, j)$ ,  $1 \leq i < j \leq 3$ , acts on the open Möbius band  $Y_{\Pi_3}$  like a “central symmetry” with fixed point  $\psi_{ij}$ .

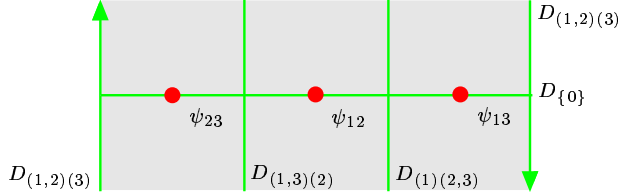


Figure 1. The nested set stratification  $(Y_{\Pi_3}, \mathfrak{D})$ .

We provide here a glance on the already more complicated situation for  $n = 4$ . Our picture below shows the stratification of the exceptional divisor  $D_{\{0\}}$ , a real projective space of dimension 2, as it emerges from the first blowup step in the De Concini-Procesi construction,  $\text{Bl}_{\{0\}}V$ .

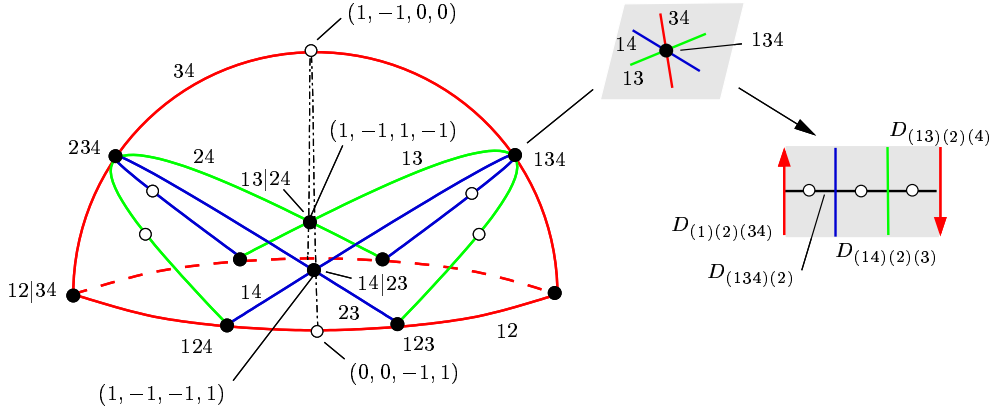


Figure 2. The stratification of  $D_{\{0\}}$  after blowup of  $\{0\}$  in  $V$ .

We choose to place the intersection of  $D_{\{0\}}$  with the hyperplane  $H_{1,2}$  on the equator of the upper hemisphere model, and thus obtain the stratification of  $D_{\{0\}}$  by the braid arrangement as depicted above. The double, respectively, triple intersections of hyperplanes in  $D_{\{0\}}$ , e.g.,  $H_{1,2} \cap H_{3,4}$ , respectively,  $H_{1,3} \cap H_{1,4} \cap H_{3,4}$ , remain to be blown up in later steps, for triple intersections locally producing the situation that we studied above for  $n=3$ .

We mark some points and lines on open strata that ought to be distinguished by a stabilizer distinguishing stratification: For instance, the point on  $D_{\{0\}}$  given by the line that is generated by the vector  $(0, 0, -1, 1)$  in  $H_{1,2}$  should be distinguished from the open stratum corresponding to  $H_{1,2}$ , since not only the transposition  $\tau = (1, 2)$  but also  $\sigma = (3, 4)$  stabilizes this line. The same goes for the (dashed) line obtained on  $D_{\{0\}}$  as the intersection with the plane spanned by the vectors  $(1, -1, 0, 0)$  and  $(0, 0, -1, 1)$ .

#### 4. THE NESTED SET STRATIFICATION OF ARRANGEMENT MODELS

**4.1. Points in  $Y_{\mathcal{G}}$ .** Let  $\mathcal{A}$  be an arrangement of subspaces in a real vector space  $V$ ,  $\mathcal{L}(\mathcal{A})$  its intersection lattice and  $\mathcal{G} = \mathcal{L}(\mathcal{A}) \setminus \{\hat{0}\}$  the maximal building set for  $\mathcal{A}$ . We will encode points in the arrangement model  $Y_{\mathcal{G}}$  into tuples of points and lines in  $V$ , a description that will prove to be favorable for technical purposes.

A point  $\omega$  in  $Y_{\mathcal{G}}$  will be written as

$$(4.1) \quad \omega = (x, H_1, l_1, H_2, l_2, \dots, H_t, l_t),$$

where  $x$  is a point in  $V$ , the  $H_i$  are elements in  $\mathcal{G} = \mathcal{L} \setminus \{\hat{0}\}$ , and the  $l_i$  are lines in  $V$ . The point  $x$  is the first coordinate of  $\omega$  when written as an element in the product space on the right hand side of (2.1).  $H_1$  is the maximal lattice element that, as a subspace of  $V$ , contains  $x$ . The line  $l_1$  is orthogonal to  $H_1$  and corresponds to the coordinate entry of  $\omega$  indexed by  $H_1$  in  $\mathbb{P}(V/H_1)$ . The lattice element  $H_2$ , in turn, is the maximal lattice element that contains both  $H_1$  and  $l_1$ . The specification of lines  $l_i$ , i.e., lines that correspond to coordinates of  $\omega$  in  $\mathbb{P}(V/H_i)$ , and the construction of lattice elements  $H_{i+1}$ , continues analogously for  $i \geq 2$  until a last line  $l_t$  is reached whose span with  $H_t$  is not contained in any lattice element other than the full ambient space  $V$ . Note, that if  $H_t$  is a hyperplane, then the line  $l_t$  is uniquely determined. The whole space  $V$  can be thought of as  $H_{t+1}$ .

Observe that the lattice elements  $H_i$  are determined by the point and the sequence of lines; we still choose to include the  $H_i$  in order to keep the notation more transparent.

To see that the description (4.1) of a point  $\omega$  in the arrangement model  $Y_{\mathcal{G}}$  is sufficient, we need to see that the rest of the coordinates can be read off uniquely from the coordinates  $x, l_1, \dots, l_t$ . The reconstruction can be explicitly done as follows. Fixing  $H_0 := 0$  and  $l_0 := \langle x \rangle$ , the first coordinate of  $\omega$  is  $x$ , and the coordinate of  $\omega$  indexed with  $H \in \mathcal{G}$ ,  $\omega_H$ , can be read from (4.1) as

$$(4.2) \quad \omega_H = \langle l_j, H \rangle / H \in \mathbb{P}(V/H),$$

where  $j$  is chosen from the index set  $\{1, \dots, t\}$  such that  $H \leq H_j$ , but  $H \not\leq H_{j+1}$ .

To prove (4.2) we need the following technical lemma.

**Lemma 4.1.** *Let  $V$  be a vector space and  $\tilde{H}$ ,  $H$  vector subspaces of  $V$ , such that  $\tilde{H} \subseteq H$ . Let furthermore  $(x_i)_{i=1}^\infty$  be a sequence of points in  $V \setminus H$  such that the limit  $\lim_{i \rightarrow \infty} \langle x_i, \tilde{H} \rangle = \Sigma$  exists in the corresponding Grassmannian.*

*Assume that  $\Sigma \not\subseteq H$ , then  $\lim_{i \rightarrow \infty} \langle x_i, H \rangle = \langle \Sigma, H \rangle$ ; again the limit is understood with respect to the topology of the appropriate Grassmannian.*

**Proof.** Let us split  $V$  into the direct sum of linear subspaces:

$$V = \tilde{H} \oplus (\tilde{H}^\perp \cap H) \oplus H^\perp,$$

where  $\tilde{H}^\perp$ , resp.  $H^\perp$ , denotes the orthogonal complement of  $\tilde{H}$ , resp. of  $H$ .

Since  $x_i \notin \tilde{H}$ , we have  $\dim \langle x_i, \tilde{H} \rangle = \dim \tilde{H} + 1$ , hence  $\dim \Sigma = \dim \tilde{H} + 1$ , and therefore there exists  $v \in \tilde{H}^\perp$ ,  $v \neq 0$ , such that  $\Sigma = \langle \tilde{H}, v \rangle$ .

Writing  $x_i = a_i + b_i + c_i$ , where  $a_i \in \tilde{H}$ ,  $b_i \in \tilde{H}^\perp \cap H$ , and  $c_i \in H^\perp$ , for all  $i$ , we have

$$(4.3) \quad \langle x_i, \tilde{H} \rangle = \langle b_i + c_i, \tilde{H} \rangle.$$

Note that  $b_i + c_i \in \tilde{H}^\perp$ , and  $b_i + c_i \neq 0$ . We can scale  $x_i$ , such that  $|b_i + c_i| = 1$ , and, after scaling  $v$  and changing  $x_i$  to  $-x_i$  for some appropriately chosen  $i$ , we get that  $\lim_{i \rightarrow \infty} (b_i + c_i) = v$ . Denote  $\lim_{i \rightarrow \infty} b_i = v_1$  and  $\lim_{i \rightarrow \infty} c_i = v_2$ ; these limits exist since  $b_i$  and  $c_i$  are chosen in mutually orthogonal linear subspaces. We certainly have  $\lim_{i \rightarrow \infty} (b_i + c_i) = \lim_{i \rightarrow \infty} b_i + \lim_{i \rightarrow \infty} c_i = v_1 + v_2$ , and  $v_1 \in \tilde{H}^\perp \cap H$ ,  $v_2 \in H^\perp$ . Since  $v \notin H$ , we have  $v_2 \neq 0$ , hence, for large  $i$ ,  $|c_i| \geq |v_2|/2 > 0$ .

We finish the proof by writing down two sequences of identities. First,

$$\langle \Sigma, H \rangle = \langle \tilde{H}, v, H \rangle = \langle v, H \rangle = \langle v_1 + v_2, H \rangle = \langle v_2, H \rangle,$$

where the second equality follows from  $\tilde{H} \subseteq H$ , and the fourth equality follows from  $v_1 \in H$ . Second,

$$\lim_{i \rightarrow \infty} \langle x_i, H \rangle = \lim_{i \rightarrow \infty} \langle c_i, H \rangle = \langle \lim_{i \rightarrow \infty} c_i, H \rangle = \langle v_2, H \rangle,$$

where the first equality follows from (4.3) and the fact that  $b_i \in H$ . The second equality is the most interesting one, it follows from the fact that the points  $c_i$  lie in  $H^\perp$ , and that the projectivization map  $\gamma : H^\perp \setminus \{0\} \rightarrow \mathbb{P}(H^\perp)$ , mapping a point to the line which it spans, is continuous.  $\square$

**Proof of (4.2).** Choose a sequence  $(x_i)_{i=1}^\infty$ ,  $x_i \in \mathcal{M}(\mathcal{A})$ , such that  $\lim_{i \rightarrow \infty} \Phi(x_i) = w$  in  $V \times \prod_{G \in \mathcal{G}} \mathbb{P}(V/G)$ . This translates into

$$\begin{cases} x = \lim_{i \rightarrow \infty} x_i, \\ \omega_G = \lim_{i \rightarrow \infty} \Phi_G(x_i) = \lim_{i \rightarrow \infty} \langle x, G \rangle / G. \end{cases}$$

Let us choose  $H \in \mathcal{G}$ , and  $j \in \{1, \dots, t\}$ , such that  $H \leq H_j$ , but  $H \not\leq H_{j+1}$ . The identity (4.2) follows now from the following computation:

$$\lim_{i \rightarrow \infty} \langle x_i, H \rangle = \lim_{i \rightarrow \infty} \langle \langle x_i, H_j \rangle, H \rangle = \lim_{i \rightarrow \infty} \langle \langle l_j, H_j \rangle, H \rangle = \lim_{i \rightarrow \infty} \langle l_j, H \rangle,$$

where the first and the third equality are consequences of  $H_j \subseteq H$ , while the second one follows from Lemma 4.1.  $\square$



**4.2. Stabilizers of points in  $Y_{\mathcal{G}}$ .** We now assume that our subspace arrangement carries the action of a finite group  $\Gamma$ . As we discussed above, the action extends to the arrangement model  $Y_{\mathcal{G}}$ . When considering stabilizers of the various actions we will include indices into the notation that indicate the set on which the full group is acting, e.g., we will write  $\text{stab}_V(y)$ ,  $\text{stab}_{Y_{\mathcal{G}}}(y)$  for the stabilizers of  $y$  with respect to the  $\Gamma$ -actions on  $V$  and on  $Y_{\mathcal{G}}$ , respectively.

We take up the encoding of points in  $Y_{\mathcal{G}}$  from subsection 4.1, and derive a description for the stabilizer of a point in  $Y_{\mathcal{G}}$ :

**Proposition 4.2.** *Let an arrangement model  $Y_{\mathcal{G}}$  be equipped with a group action stemming from the action of a finite group  $\Gamma$  on the arrangement. Then for stabilizers of points  $\omega = (x, H_1, l_1, H_2, l_2, \dots, H_t, l_t)$  in  $Y_{\mathcal{G}}$  the following description holds:*

$$(4.4) \quad \text{stab}_{Y_{\mathcal{G}}}(\omega) = \text{stab}_V(x) \cap \text{stab}_V(l_1) \cap \dots \cap \text{stab}_V(l_t),$$

where  $\text{stab}_V(l_i)$ ,  $i = 1, \dots, t$ , denotes the subgroup of elements  $\gamma \in \Gamma$  with  $\gamma(l_i) = l_i$ , i.e., elements preserving  $l_i$  without necessarily fixing the line pointwise.

**Proof.** Using the description of points in  $Y_{\mathcal{G}}$  given in subsection 4.1, and the definition of the group action, we can describe the stabilizer of a point  $\omega \in Y_{\mathcal{G}}$  as follows:

$$(4.5) \quad \text{stab}_{Y_{\mathcal{G}}}(\omega) = \text{stab}_V(x) \cap \text{stab}_{\mathbb{P}(V/H_1)}(l_1) \cap \dots \cap \text{stab}_{\mathbb{P}(V/H_t)}(l_t),$$

where  $\text{stab}_{\mathbb{P}(V/H_i)}(l_i)$ ,  $i = 1, \dots, t$ , translating from the projective to the original linear setting, means elements  $\gamma \in \Gamma$  under which *both*  $H_i$  and  $l_i$  are invariant:

$$\text{stab}_{\mathbb{P}(V/H_i)}(l_i) := \text{stab}_V(H_i) \cap \text{stab}_V(l_i).$$

Again,  $\text{stab}_V(H_i)$  denotes group elements that preserve  $H_i$  but do not necessarily fix  $H_i$  pointwise.

We show that

$$\begin{aligned} \text{stab}_V(x) &\subseteq \text{stab}_V(H_1), \quad \text{and} \\ \text{stab}_V(H_i) \cap \text{stab}_V(l_i) &\subseteq \text{stab}_V(H_{i+1}), \quad \text{for } i = 1, \dots, t-1, \end{aligned}$$

which, successively applied for  $i = t-1$ ,  $i = t-2$ , etc., reduces the right hand side of (4.5) to the right hand side of (4.4), since  $A \cap B = A$ , for any two sets  $A$  and  $B$ , such that  $A \subseteq B$ .

For  $\gamma \in \text{stab}_V(x)$ ,  $x$  is contained in  $\gamma(H_1) \cap H_1$ . But  $H_1 \supseteq \gamma(H_1) \cap H_1$  is assumed to be maximal in  $\mathcal{G} = \mathcal{L} \setminus \{\hat{0}\}$  containing  $x$ , thus, it follows from the fact that  $\mathcal{G}$  is closed under taking intersections, that  $\gamma(H_1) = H_1$ . Similarly for  $\gamma \in \text{stab}_V(H_i) \cap \text{stab}_V(l_i)$ :  $\gamma(H_{i+1}) \cap H_{i+1}$  contains both  $H_i$  and  $l_i$ , but  $H_{i+1}$  should be maximal in  $\mathcal{G} = \mathcal{L} \setminus \{\hat{0}\}$  with this property, hence  $\gamma(H_{i+1}) = H_{i+1}$ .

Note additionally, that if  $H_t$  is a hyperplane, then  $\text{stab}_V(H_t) = \text{stab}_V(l_t)$ , hence, in this case,  $\text{stab}_V(l_t)$  can be removed from the right hand side of (4.4) without changing the expression.  $\square$

**4.3. The divisors  $D_G$ ,  $G \in \mathcal{G}$ .** Recall from Section 2 that the nested set stratification  $(Y_{\mathcal{G}}, \mathfrak{D})$  on an arrangement model  $Y_{\mathcal{G}}$  is given by irreducible components of divisors and their intersections. Our objective is to provide, in our special setting, a description of the divisors  $D_G$ ,  $G \in \mathcal{G}$ , that enables us to tell for a given point in the arrangement model on which of these divisors it lies.

De Concini & Procesi give a description of the divisors in terms of affine and projective arrangement models for “smaller” arrangements. To keep track of the respective settings, we provide arrangement models with an additional index that specifies the ambient space of the original arrangement, and we indicate projective models by a bar, e.g., in presence of other arrangement models we will now write  $Y_{V,\mathcal{G}}$  for the affine and  $\bar{Y}_{V,\mathcal{G}}$  for the projective model of the previously considered arrangement.

In our special setting the description of divisors by De Concini & Procesi reads as follows:

**Proposition 4.3.** [DP, Thm.4.3, Rem.4.3.(1)] *Let  $\mathcal{A}$  be an essential arrangement of subspaces,  $\mathcal{G}$  the maximal building set,  $\mathcal{G} = \mathcal{L}(\mathcal{A}) \setminus \{\hat{0}\}$ , and  $Y_{V,\mathcal{G}}$  the corresponding arrangement model. For the irreducible divisors  $D_G$ ,  $G \in \mathcal{G}$ , there are natural isomorphisms:*

$$(4.6) \quad D_{\{0\}} \cong \bar{Y}_{V,\mathcal{G}},$$

$$(4.7) \quad D_{\{G\}} \cong \bar{Y}_{V/G,\mathcal{G}_{\leq G}} \times Y_{G,\mathcal{G}_{>G}}, \quad \text{for } G \neq \{0\}.$$

Here,  $\bar{Y}_{V/G,\mathcal{G}_{\leq G}}$  is the projective model for the quotient arrangement  $\mathcal{A}/G := \{H/G \mid H \in \mathcal{A}, H \supseteq G\}$  with (maximal) building set  $\mathcal{G}_{\leq G} = \{H \in \mathcal{G} \mid H \leq G\}$ , and  $Y_{G,\mathcal{G}_{>G}}$  is the affine model for the restricted arrangement  $\mathcal{A} \cap G := \{H \cap G \mid H \in \mathcal{A}\}$  with (maximal) building set  $\mathcal{G}_{>G} = \{H \in \mathcal{G} \mid H > G\}$ .

The projective model  $\bar{Y}_{V,\mathcal{G}}$ , in fact, is isomorphic to the inverse image of  $\{0\}$  when projecting  $Y_{V,\mathcal{G}}$  to  $V$ , the first coordinate of its ambient space [DP, Thm.4.1]. Hence,  $\omega \in D_{\{0\}}$  if and only if  $\omega_{\{0\}} = 0$ , in other words

$$(4.8) \quad \omega \in D_{\{0\}} \Leftrightarrow \omega \in Y_{V,\mathcal{G}} \cap \left( \{0\} \times \prod_{G \in \mathcal{G}} \mathbb{P}(V/G) \right).$$

It is a description of this type that we want to achieve for the other divisors,  $D_G$ ,  $G \neq \{0\}$ , as well.

To this end, note that the right hand side of (4.7) can be considered as a subspace of

$$\{0\} \times \prod_{H \in \mathcal{G}_{\leq G}} \mathbb{P}(V/G/H/G) \times G \times \prod_{H \in \mathcal{G}_{>G}} \mathbb{P}(G/H).$$

For  $K \in \mathcal{G}_{>G}$ , we can “expand” the factor  $\mathbb{P}(G/K)$  by a diagonal map

$$\mathbb{P}(G/K) \longrightarrow \prod_{\substack{H \in \mathcal{G} \\ H \vee G = K}} \mathbb{P}(G/(H \vee G)),$$

and thus interpret  $D_G$  as a subset of

$$U_G := G \times \prod_{H \in \mathcal{G}_{\leq G}} \mathbb{P}(G/(H \vee G)) \times \prod_{H \in \mathcal{G}_{\leq G}} \mathbb{P}(V/H).$$

With  $G/(H \vee G) \cong \langle G, H \rangle / H$ ,  $U_G$  can be considered a subspace of the ambient space  $V \times \prod_{H \in \mathcal{G}} \mathbb{P}(V/G)$  of the arrangement model.

We thus can state our description of divisors  $D_G$ :

**Proposition 4.4.** *Let  $\mathcal{A}$  be an essential arrangement of subspaces,  $\mathcal{G}$  the maximal building set,  $\mathcal{G} = \mathcal{L}(\mathcal{A}) \setminus \{\hat{0}\}$ , and  $Y_{\mathcal{G}}$  the corresponding arrangement model. The irreducible divisors  $D_G$ ,  $G \in \mathcal{G}$ , are intersections of  $Y_{\mathcal{G}}$  with the product spaces  $U_G$ , where the  $U_G$  are obtained by restricting those factors of the original ambient space of  $Y_{\mathcal{G}}$  which are indexed with  $H \in \mathcal{G}_{\not\leq G}$ :*

$$\begin{aligned} D_G &= Y_{\mathcal{G}} \cap U_G \\ &= Y_{\mathcal{G}} \cap \left( G \times \prod_{H \in \mathcal{G}_{\not\leq G}} \mathbb{P}(\langle G, H \rangle / H) \times \prod_{H \in \mathcal{G}_{\leq G}} \mathbb{P}(V/H) \right). \end{aligned}$$

**Proof.** Observe first that the description for  $D_{\{0\}}$  given in (4.8) coincides with the one stated in the Proposition: intersecting  $Y_{\mathcal{G}}$  with  $U_{\{0\}}$  restricts the first coordinate to 0.

For  $G \neq \{0\}$ , we start with the description of  $D_G$  in (4.7) and see from the reasoning above that any element in  $D_G$  is contained in  $U_G$ . For the converse, let  $\omega = (x, H_1, l_1, H_2, l_2, \dots, H_t, l_t)$  be contained in  $Y_{\mathcal{G}} \cap U_G$ . From  $\omega \in U_G$  we conclude that  $x \in G$ , hence  $H_1 \geq G$ . Assuming for the moment that  $H_1 \gneq G$ , we look at the component of  $\omega$  indexed by  $H_1$ . Using the expansion of  $\omega$  from (4.2) and the fact that  $\omega \in U_G$ , we see that

$$\omega_{H_1} = \langle l_1, H_1 \rangle / H_1 \in \mathbb{P}(G/H_1),$$

hence  $l_1 \subseteq G$ . This implies that  $H_2$  is larger or equal  $G$ , for, if it were not,  $H_2 \vee G \gneq H_2$  would contain both  $H_1$  and  $l_1$  in contradiction to  $H_2$  being maximal with this property.

We conclude that there is an index  $k \in \{1, \dots, t\}$  with  $H_k = G$ , and can thus split the point/lines description of  $\omega$  into

$$\omega = \left( (x, H_1, l_1, H_2, l_2, \dots, l_{k-1}, G), (l_k, H_{k+1}, \dots, H_t, l_t) \right).$$

The first tuple clearly describes an element in  $Y_{G, \mathcal{G}_{>G}}$ . We rewrite the second tuple as follows:

$$(0_{V/G}, l_k, H_{k+1}/G, \dots, H_t/G, l_t).$$

With  $l_j$  being orthogonal to  $G$ , hence  $l_j \in \mathbb{P}(V/G)$ , we can then interpret it as an element of  $\bar{Y}_{V/G, \mathcal{G}_{\leq G}}$ . With (4.7) we thus conclude that  $\omega \in D_G$ .  $\square$

**4.4. Open strata of the nested set stratification.** We will provide a characterization of points on open strata of the nested set stratification of  $Y_{\mathcal{G}}$  in terms of their point/line encoding described in subsection 4.1.

To fix some notation, let us denote by  $D_{G_1, \dots, G_m}^\circ$  the open stratum in  $(Y_{\mathcal{G}}, \mathfrak{D})$  that lies in the intersection of divisors  $D_{G_1}, \dots, D_{G_m}$ , but on no other divisors indexed with building set elements. Recall that the index set  $\{G_1, \dots, G_m\}$  is  $\mathcal{G}$ -nested, which in our context, i.e., for the maximal building set, means that it is a chain in  $\mathcal{L}(\mathcal{A})$ . We tacitly assume that the  $G_i$  are listed in a descending order:  $G_1 > \dots > G_m$ .

**Proposition 4.5.** *Let  $Y_G$  be an arrangement model with nested set stratification  $\mathfrak{D}$ . A point  $\omega \in Y_G$  is contained in the open stratum of  $\mathfrak{D}$  indexed with the nested set  $\mathcal{T} = \{G_1, \dots, G_m\}$  if and only if the spaces in  $\mathcal{T}$  coincide with the spaces occurring in the point/line description of  $\omega$ :*

$$\omega \in D_{G_1, \dots, G_m}^\circ \iff \omega = (x, G_1, l_1, \dots, G_m, l_m),$$

where on the right hand side the usual restrictions for coordinates of a point/line tuple as in (4.1) apply.

**Proof.** First observe that the claim holds for points  $\omega$  in the big open stratum  $Y_G \setminus D = \mathcal{M}(\mathcal{A})$ , that is for  $m = 0$ : The indexing nested set is empty, and the point/line description for  $\omega$  reduces to the point entry  $x \in \mathcal{M}(\mathcal{A})$ .

We can thus assume that  $\omega \in D$ , in particular,  $\omega$  is contained in some open stratum in  $D$ , say

$$\omega \in D_{G_1, \dots, G_m}^\circ,$$

where we remind that the  $G_i$  are indexed in descending order, and  $m \geq 1$ .

At the same time,  $\omega$  has a point/line description, say

$$\omega = (x, H_1, l_1, \dots, H_t, l_t),$$

where  $H_1, \dots, H_t \in \mathcal{G}$ ,  $x \in H_1$ , and  $l_i \in \mathbb{P}(V/H_i)$ , for  $i = 1, \dots, t$ . We show in the following that the descending chains  $G_1 > \dots > G_m$  and  $H_1 > \dots > H_t$  coincide, in particular implying  $m = t$ .

**Step 1:** *The maximal elements of the chains coincide:  $H_1 = G_1$ .*

With  $\omega \in D_{G_1}$ , we know by Proposition 4.4 that  $x \in G_1$ ; but  $H_1$  is maximal with this property, hence,  $H_1 \geq G_1$ .

We want to see, that  $\omega \in D_{H_1}$ . Using again Proposition 4.4 and the expansion of  $\omega$  in (4.2), we have to check that  $x \in H_1$ , and that for any  $H \not\leq H_1$  the coordinate  $\omega_H = \langle \langle x, H \rangle, H \rangle / H$  is a point in  $\mathbb{P}(\langle \langle H_1, H \rangle, H \rangle / H)$ . With  $\langle x \rangle \subseteq H_1$  this is obviously the case.

We conclude that  $H_1 \in \mathcal{T}$ , hence,  $H_1 \leq G_1$  by maximality of  $G_1$  in  $\mathcal{T}$ . This yields our claim. In particular, we see that  $t \geq 1$ .

**Step 2:** *Assume  $H_j = G_j$  for  $j = 1, \dots, i$ , and  $i \leq t$ . Then  $m \geq i+1$  and  $H_{i+1} = G_{i+1}$ .*

Here, we first want to see, that  $\omega \in D_{H_{i+1}}$ . For this we need to check that  $x \in H_{i+1}$ , and that for any  $H \not\leq H_{i+1}$  the coordinate  $\omega_H = \langle l_j, H \rangle / H$  is a point in  $\mathbb{P}(\langle \langle H_{i+1}, H \rangle, H \rangle / H)$ . The line  $l_j$  depends on  $H$  (compare (4.2)), but for any  $H$  in question its index  $j$  is strictly less than  $i+1$ . From the point/line description for  $\omega$  we see that  $x \in H_1 \subseteq H_{i+1}$ . With  $l_j \subset H_{j+1} \subseteq H_{i+1}$  we conclude that  $\langle l_j, H \rangle / H \in \mathbb{P}(\langle \langle H_{i+1}, H \rangle, H \rangle / H)$ , hence  $\omega \in D_{H_{i+1}}$ .

Since  $H_{i+1}$  belongs to the nested set  $\mathcal{T}$ ,  $H_{i+1} < H_i = G_i$ , implies that, in fact,  $m \geq i+1$  and  $H_{i+1} \leq G_{i+1}$ .

To obtain equality we write out the condition on the coordinate of  $\omega$  indexed with  $H_i$  that results from  $\omega \in D_{G_{i+1}}$ :  $\omega_{H_i} = \langle l_i, H_i \rangle / H_i \in \mathbb{P}(\langle \langle G_{i+1}, H_i \rangle, H_i \rangle / H_i) = \mathbb{P}(G_{i+1}/H_i)$ .

We conclude that  $l_i \subseteq G_{i+1}$ . Moreover,  $G_i \subseteq G_{i+1}$  by descending order on  $\mathcal{T}$ . But  $H_{i+1}$  is maximal in  $\mathcal{G}$  containing both  $H_i = G_i$  and  $l_i$ , hence  $H_{i+1} \geq G_{i+1}$ , from which our claim follows.

**Step 3:**  *$m = t$ , and hence the chains coincide.*

From Steps (1) and (2) we conclude that  $m \geq t$ . Let us assume that  $m > t$ , in particular,  $\omega \in D_{G_{t+1}}$ . We conclude from the resulting condition on the coordinate indexed

by  $H_t$ ,  $\omega_{H_t} = \langle l_t, H_t \rangle / H_t \in \mathbb{P}(\langle G_{t+1}, H_t \rangle / H_t) = \mathbb{P}(G_{i+1} / H_i)$ , that both  $l_t$  and  $H_t = G_t$  are contained in  $G_{t+1}$  which contradicts the fact that the point/line description of  $\omega$  was terminated after the  $t$ -th step. Hence  $m = t$ , and the chains  $G_1 > \dots > G_t$  and  $H_1 > \dots > H_t$  coincide.  $\square$

## 5. A STABILIZER DISTINGUISHING STRATIFICATION OF $Y_{\Pi_n}$

**5.1. Adding strata.** On our way to construct a stabilizer distinguishing stratification for  $Y_{\Pi_n}$  we first analyze the locus of *lines* in  $\mathbb{R}^n$  that are stabilized by a given element in  $\mathcal{S}_n$ . Let  $\pi \in \mathcal{S}_n$ , and, restricting the permutation action, consider  $\mathbb{R}^n$  as a representation space of the cyclic group  $\langle \pi \rangle$ . In  $\mathbb{R}^n$  we have, on one hand, the linear subspace  $T_1(\pi) = \text{Fix}(\pi)$ , the locus of lines that are pointwise fixed by  $\pi$ , on the other hand, we have the subspace  $T_{-1}(\pi)$ , the locus of lines that are flipped by  $\pi$ . We can characterize lines in  $\mathbb{R}^n$  that are invariant under  $\pi \in \mathcal{S}_n$  as follows:

**Proposition 5.1.** *Let  $\pi \in \mathcal{S}_n$  and  $S(\pi) := T_1(\pi) \cup T_{-1}(\pi)$ . For a given line  $l$  in  $\mathbb{R}^n$ ,*

$$\pi \in \text{stab}(l) \iff l \subseteq S(\pi).$$

We would like to emphasize that  $S(\pi)$  is defined as a union of  $T_1(\pi)$  and  $T_{-1}(\pi)$ , *not* as their span.

Let us now describe stratifications of the orthogonal complements  $G^\perp$  of subspaces  $G$  in  $\Pi_n$ . For such  $G$ , and for any  $\pi \in \mathcal{S}_n$ , define  $S(\pi, G) := S(\pi) \cap G^\perp$ . Then,

$$\mathfrak{S}_G := \{S(\pi, G)\}_{\pi \in \mathcal{S}_n}$$

is a stratification of  $G^\perp$ . Unlike the restriction of the braid arrangement stratification to  $G^\perp$ , it distinguishes stabilizers of points as well as stabilizers of lines.

We propose a construction for subsets in real arrangement models  $Y_{\mathcal{G}}$  that takes unions of linear subspaces in  $\mathbb{R}^n$  as input data. It is inspired by the description of divisors  $D_G$ ,  $G \in \mathcal{G}$ , that we presented in Proposition 4.4. Taking spaces  $S(\pi, G) \times G$ ,  $G \in \mathcal{G}$ ,  $\pi \in \mathcal{S}_n$ , with  $S(\pi, G)$  as defined above, our construction will provide us with the additional maximal strata in  $Y_{\Pi_n}$  for obtaining a stabilizer distinguishing stratification.

**Definition 5.2.** *Let  $Y_{V, \mathcal{G}}$  be an arrangement model, and  $W = \{W_1, \dots, W_m\}$  a family of real linear subspaces in  $V$ . Define a subset  $B(W)$  in  $Y_{\mathcal{G}}$  by*

$$B(W) := Y_{\mathcal{G}} \cap \left( \bigcup W \times \prod_{\substack{H \in \mathcal{G}, H \not\supseteq W_i \\ \text{for any } W_i \in W}} \mathbb{P}(\langle W, H \rangle / H) \times \prod_{\substack{H \in \mathcal{G}, H \supseteq W_i \\ \text{for some } W_i \in W}} \mathbb{P}(V/H) \right),$$

where  $\mathbb{P}(\langle W, H \rangle / H)$  stands for the projectivization of  $\bigcup_{i=1}^m \langle W_i, H \rangle / H$ .

We now can refine the nested set stratification  $\mathfrak{D}$  of  $Y_{\Pi_n}$  so as to obtain a stabilizer distinguishing stratification. As before, we describe the stratification by listing its maximal strata:

$$(5.1) \quad \mathfrak{B} := \left\{ (D_G)_{G \in \Pi_n}, (B(S(\pi, G) \times G))_{G \in \Pi_n, \pi \in \mathcal{S}_n} \right\},$$

where in the second family of strata we only consider those with  $\{0\} \subsetneq S(\pi, G) \subseteq G^\perp$ .

5.2.  $(Y_{\Pi_n}, \mathfrak{B})$  is **stabilizer distinguishing**. We can now state one of the main results of this article:

**Theorem 5.3.** *The stratification  $\mathfrak{B}$  for the arrangement model  $Y_{\Pi_n}$  defined in (5.1) is stabilizer distinguishing, i.e., the stabilizer of a point  $\omega \in Y_{\Pi_n}$  is completely determined by the open stratum of  $\mathfrak{B}$  that contains  $\omega$ .*

**Proof.** We pick a point  $\omega = (x, G_1, l_1, \dots, G_t, l_t)$  in  $Y_{\Pi_n}$ , and assume that we have the complete list of maximal strata in  $\mathfrak{B}$  which contain  $\omega$ . We want to show that the stabilizer of  $\omega$  is fully determined by this list.

Note first that by Proposition 4.5 our list of strata contains the divisors  $D_{G_1}, \dots, D_{G_t}$ , and no other divisors of this type. This means that we can read off from the list the elements  $G_1, \dots, G_t$  for the point/line description of  $\omega$ .

Assume  $\omega \in B(S(\pi, G_i) \times G_i)$ , for some  $G_i$ ,  $i \in \{1, \dots, t\}$ . With Definition 5.2, and  $S(\pi, G_i) \times G_i \supseteq G_i$ , this puts the following restriction on the coordinate of  $\omega$  that is indexed by  $G_i$ :

$$\omega_{G_i} = \langle l_i, G_i \rangle / G_i \in \mathbb{P}(\langle S(\pi, G_i) \times G_i, G_i \rangle / G_i).$$

We conclude that  $l_i \subseteq S(\pi, G_i)$ , in particular,  $\pi$  stabilizes  $l_i$ .

From the strata  $B(S(\pi, G_i) \times G_i)$ , that occur on our list for a fixed space  $G_i$ ,  $i \in \{1, \dots, t\}$ , we can read off a subset  $\Gamma_i$  of  $\text{stab}(l_i)$ . Namely, for each  $i \in \{1, \dots, t\}$ ,  $\Gamma_i$  consists of all  $\pi$  such that  $\omega \in B(S(\pi, G_i) \times G_i)$ .

Let us assume that, when constructing  $\Gamma_i$  from our list of strata for  $\omega$ , we actually missed some elements of  $\text{stab}(l_i)$ : let  $\sigma \in \text{stab}(l_i) \setminus \Gamma_i$ . Then  $l_i \subseteq S(\sigma, G_i)$ , but  $\omega \notin B(S(\sigma, G_i) \times G_i)$ . By definition of the additional maximal strata we conclude that there exists a subspace  $H \in \Pi_n$ , which does not contain any of the spaces in  $S(\sigma, G_i) \times G_i$ , such that

$$(5.2) \quad \omega_H = \langle l_j, H \rangle / H \notin \mathbb{P}(\langle S(\sigma, G_i) \times G_i, H \rangle / H).$$

The line index  $j$  depends on  $H$ , but in any case,  $j > i$ : for  $j < i$ ,  $l_j \subseteq G_i$ , and for  $j = i$ ,  $l_i \subseteq S(\sigma, G_i)$ , and the condition on  $\omega_H$  for  $\omega$  being contained in  $B(S(\sigma, G_i) \times G_i)$  would be fulfilled.

It follows from (5.2) that  $l_j \not\subseteq S(\sigma, G_i)$ . Since  $l_j$  is orthogonal to  $G_i$ , it implies  $\sigma \notin \text{stab}(l_j)$ , and, in particular,  $\sigma \notin \bigcap_{i=1}^t \text{stab}(l_i)$ . Hence, even if for some  $i$ ,  $\Gamma_i \subsetneq \text{stab}(l_i)$ , once the full intersection is taken, this is rectified:

$$\bigcap_{i=1}^t \Gamma_i = \bigcap_{i=1}^t \text{stab}(l_i).$$

With the description of  $\text{stab}(\omega)$  from Proposition 4.2, and  $\text{stab}(x)$  being determined by the partition pattern of  $x$ , hence by  $G_1$ , we can conclude that the list of strata in  $\mathfrak{B}$  containing  $\omega$  actually determines the stabilizers of  $\omega$ .  $\square$

5.3.  $Y_{\Pi_3}$  **revisited**. Let us have a look at the stratification  $\mathfrak{B}$  on  $Y_{\Pi_3}$  and see how it resolves the problem raised in 3.2, namely to distinguish stabilizers of points by means of a stratification.

To start with, we have to identify those spaces  $S(\pi, G) \times G$  for  $G \in \Pi_3$ ,  $\pi \in \mathcal{S}_3$ , that give raise to new strata  $B(S(\pi, G) \times G)$ . We claim that the only interesting case occurs for  $\pi$  a transposition,  $\pi = (i, j)$ ,  $1 \leq i < j \leq 3$ , and  $G = \{0\}$ .

We have  $S(\pi) = H_{i,j} \cup H_{i,j}^-$ , where we denote hyperplanes of  $\mathcal{A}_{n-1}$  in  $V$  by  $H_{i,j}$ , just as for the original (non-essential) arrangement in  $\mathbb{R}^3$ , and their orthogonal complements by  $H_{i,j}^-$ . With  $S(\pi, \{0\}) = S(\pi)$ , we obtain new strata

$$B_{(i,j)} = B(S((i, j), \{0\}) \times \{0\}) = Y_{\Pi_3} \cap ((H_{i,j} \cup H_{i,j}^-) \times (\mathbb{P}(H_{i,j}) \cup \mathbb{P}(H_{i,j}^-))).$$

In terms of the pointwise description for  $Y_{\Pi_3}$  that we gave in 3.2 this reads

$$\begin{aligned} B_{(1,2)} = & \{ (x, \langle x \rangle) \mid x_1 = x_2 \neq 0 \text{ or } x_1 = -x_2 \neq 0 \} \\ & \cup \{ (0, \langle (1, 1, -2) \rangle), (0, \langle (1, -1, 0) \rangle) \}, \end{aligned}$$

analogously for  $B_{(1,3)}$ ,  $B_{(2,3)}$ . Hence, as opposed to the nested set stratification  $\mathfrak{D}$ , the stratification  $\mathfrak{B} = \{(D_G)_{G \in \Pi_3}, B_{(1,2)}, B_{(1,3)}, B_{(2,3)}\}$  distinguishes the points  $\psi_{i,j}$ ,  $1 \leq i < j \leq 3$  from the rest of the divisor  $D_{\{0\}}$ .

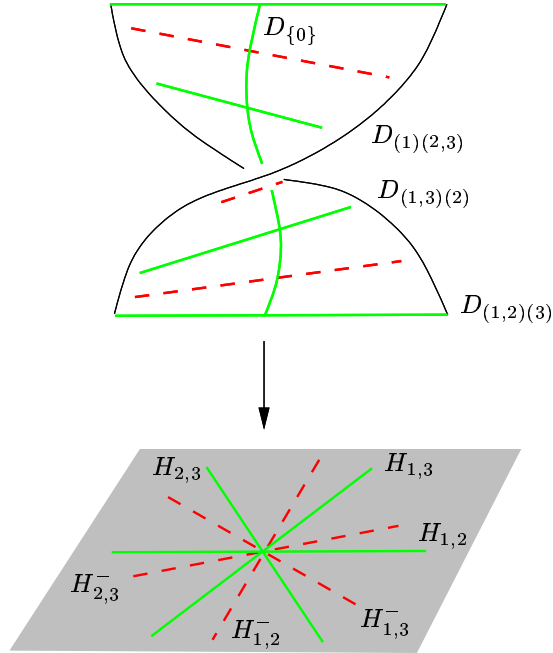


Figure 3. The stratification  $(Y_{\Pi_3}, \mathfrak{B})$ .

## 6. A COMBINATORIAL FRAMEWORK FOR DESCRIBING STABILIZERS

In this section we develop a combinatorial framework for describing stabilizers of points on the De Concini-Procesi arrangement model  $Y_{\Pi_n}$  with respect to the  $\mathcal{S}_n$ -action. In Section 7 we will use this description to prove that the stabilizers of points of  $Y_{\Pi_n}$  are isomorphic to direct products of  $\mathbb{Z}_2$ .

### 6.1. Diagrams over families of cubes.

#### Definition 6.1.

- (1) Let  $I$  be a finite, possibly empty set of positive integers. We call the collection of all subsets of  $I$  (including the empty subset) an  $I$ -**cube**. Reversely, given an  $I$ -cube  $K$ , we call  $I$  the **index set** of  $K$ .
- (2) Let  $t$  be a positive integer. A  $t$ -**family of cubes** is a collection  $\mathcal{C} = \{K_1, \dots, K_p\}$ , where, for each  $j = 1, \dots, p$ ,  $K_j$  is an  $I(j)$ -cube, for some  $I(j) \subseteq \{1, \dots, t\}$ .

One can make use of geometric intuition by thinking of an  $I$ -cube as a coordinate 0/1-cube with  $I$  indexing the set of “directions” of the cube. The  $\emptyset$ -cube is simply the point at the origin. For every  $n \geq \max(I)$ , the  $I$ -cube can be imbedded as a coordinate 0/1-cube in  $\mathbb{R}^n$ , and our object is the equivalence class of all these imbeddings.

Let  $K$  be an  $I$ -cube, to discriminate from other  $I$ -cubes, we write elements of  $K$  as pairs  $(K, S)$ , for  $S \subseteq I$ . We denote  $\text{vert}(K) = \{(K, S) \mid S \subseteq I\}$ , and refer to its elements as vertices of  $K$ . When it is clear which cube we are in, we may choose to skip  $K$ , and call  $S$  itself a vertex of  $K$ .

Note also that a  $t$ -family of cubes is simply specified by a function  $I : [p] \rightarrow 2^{[t]}$ , and that if  $\tilde{t} > t$ , then every  $t$ -family of cubes is also a  $\tilde{t}$ -family. For  $\mathcal{C} = \{K_1, \dots, K_p\}$  we denote  $\text{vert}(\mathcal{C}) = \bigcup_{i=1}^p \text{vert}(K_i)$ , and refer to its elements as vertices of  $\mathcal{C}$ .

#### Definition 6.2.

- (1) Let  $\mathcal{C}$  be a  $t$ -family of cubes,  $\mathcal{C} = \{K_1, \dots, K_p\}$ , and let  $n$  be a positive integer. An  $n$ -**diagram**  $\mathcal{D}$  over  $\mathcal{C}$  is a partition of the set  $[n]$  into  $|\text{vert}(\mathcal{C})|$  blocks, some blocks may be empty, and an assignment of the blocks of this partition to vertices of  $\mathcal{C}$ ; in other words, it is a function

$$(6.1) \quad \begin{array}{ccc} \mathcal{D} & : & [n] \longrightarrow \text{vert}(\mathcal{C}), \\ & & k \mapsto (K_{\alpha(k)}, v_k), \end{array}$$

where  $\alpha(k) \in [p]$  specifies the index of the cube and  $v_k \subseteq I(\alpha(k))$  the vertex of  $K_{\alpha(k)}$  assigned to  $k$ .

- (2) For a vertex  $(K, v)$  of  $\mathcal{C}$ , we call the set  $\mathcal{D}^{-1}(K, v)$  the **fiber** of  $\mathcal{D}$  over  $(K, v)$ . For an  $I$ -cube  $K$  in  $\mathcal{C}$ , the fiber of  $\mathcal{D}$  over  $K$  is defined as the union of the fibres of the vertices of  $K$ :

$$\mathcal{D}^{-1}(K) := \bigcup_{v \subseteq I} \mathcal{D}^{-1}(K, v).$$

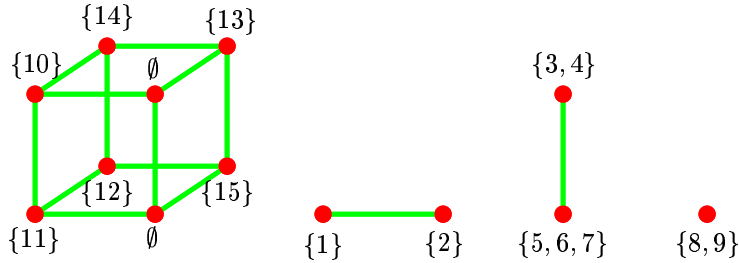


Figure 4. An example of a 15-diagram over a 3-family of cubes.



As yet another piece of notation, let  $\rho(\mathcal{D}) \vdash n$  be the set partition with blocks being the fibers of  $\mathcal{D}$  over the vertices of  $\mathcal{C}$ , i.e.,  $\rho(\mathcal{D}) = \{\mathcal{D}^{-1}(K, v)\}_{(K,v) \in \text{vert}(\mathcal{C})}$ , where we disregard all the empty blocks in the set on the right hand side.

**6.2. Automorphism groups.** There is a standard  $\mathbb{Z}_2^n$ -action on an  $[n]$ -cube: it is generated by reflections with respect to  $n$  hyperplanes, which are parallel to the facets of the cube, and which go through the center of the cube. A technically convenient way to describe this action is to think of the vertices of an  $[n]$ -cube as vectors in an  $n$ -dimensional vector space over the field  $\mathbb{F}_2$ , again denoted  $\mathbb{Z}_2^n$ , and the action as parallel translations by vectors in  $\mathbb{Z}_2^n$  (i.e., generated by parallel translations with respect to the coordinate vectors).

For a subset  $I \subseteq [n]$ , let  $\mathbb{Z}_2^I$  denote the corresponding coordinate subspace of  $\mathbb{Z}_2^n$ , and let  $\text{proj}_I : \mathbb{Z}_2^n \rightarrow \mathbb{Z}_2^I$  denote the projection onto  $\mathbb{Z}_2^I$  which simply "forgets" the coordinates with indices outside of  $I$ .

The following definition generalizes these actions to the case of diagrams over families of cubes.

**Definition 6.3.** *Let  $\mathcal{D}$  be an  $n$ -diagram over a  $t$ -family of cubes  $\mathcal{C} = \{K_1, \dots, K_p\}$ . We define the **group of automorphisms** of  $\mathcal{D}$ , which we denote  $\text{Aut}(\mathcal{D})$ , as follows:  $\text{Aut}(\mathcal{D})$  consists of all permutations  $\pi \in \mathcal{S}_n$ , such that*

- i)  $\pi_{\mathcal{D}^{-1}(K_j)} \in \mathcal{S}_{\mathcal{D}^{-1}(K_j)}$ , for all  $j = 1, \dots, p$ , i.e.,  $\pi$  preserves the fibers over cubes;
- ii) there exists (not necessarily unique)  $\sigma \in \mathbb{Z}_2^t$ , such that

$$(6.2) \quad v_{\pi(k)} = \sigma_{\alpha(k)}(v_k), \quad \text{for all } k \in \{1, \dots, n\},$$

where  $\sigma_j = \text{proj}_{I(j)}(\sigma)$ , for all  $j \in \{1, \dots, p\}$ , and where  $v_k$  and  $\alpha(k)$  are as in (6.1). In other words,  $\pi$  maps fibers to fibers according to a uniform scheme obtained by restricting  $\sigma$  to the cubes in the family  $\mathcal{C}$ .

**Remark 6.4.** *Maps between fibers of an  $n$ -diagram  $\mathcal{D}$  over a  $t$ -family of cubes  $\mathcal{C}$ , which are induced by an element  $\pi \in \text{Aut}(\mathcal{D})$ , must be bijections.*

Indeed, let  $K$  be an  $I$ -cube in  $\mathcal{C}$ , let  $v \subseteq I$ , and let  $\sigma \in \mathbb{Z}_2^t$  be associated to  $\pi$  by Definition 6.3 ii), then, by (6.2), we have

$$\pi(\mathcal{D}^{-1}(K, v)) \subseteq \mathcal{D}^{-1}(K, \text{proj}_I(\sigma)(v)),$$

while

$$\pi(\mathcal{D}^{-1}(K, \text{proj}_I(\sigma)(v))) \subseteq \mathcal{D}^{-1}(K, \text{proj}_I(\sigma)^2(v)) = \mathcal{D}^{-1}(K, v).$$

Since  $\pi$  is injective, its restrictions are injective as well, hence we can conclude that  $\pi$  restricts to a bijection between  $\mathcal{D}^{-1}(K, v)$  and  $\mathcal{D}^{-1}(K, \text{proj}_I(\sigma)(v))$ .

**Lemma 6.5.**

- (1) *For  $x \in \mathbb{R}^n$ , the stabilizer of  $x$  under the  $\mathcal{S}_n$ -action is the Young subgroup of  $\mathcal{S}_n$  indexed by the set partition of  $[n]$ , which is induced by the coordinates of  $x$ . One can represent this Young subgroup as an automorphism group of an  $n$ -diagram over a 0-family of cubes.*
- (2) *For a line  $l \subseteq \mathbb{R}^n$ , the stabilizer of  $l$  under the  $\mathcal{S}_n$ -action can be represented as an automorphism group of an  $n$ -diagram over a 1-family of cubes.*

**Proof.** (1) The first part of the statement is immediate. To construct the necessary  $n$ -diagram, group together all the coordinates of  $x$  that are equal and assign the corresponding sets of indices to different 0-cubes. This yields an  $n$ -diagram  $\mathcal{D}$  over a 0-family of cubes, and, obviously,  $\text{Aut}(\mathcal{D})$  is exactly the  $\mathcal{S}_n$ -stabilizer of  $x$  in  $\mathbb{R}^n$ .

(2) Take a nonzero vector  $v \in l$ . Group together all the equal coordinates of  $v$ , and assign corresponding sets of indices to 0-cubes, just like we did for  $x$ . Now, whenever there are two groups of coordinates, such that these groups are of equal cardinality, and the coordinates in the two groups are negatives of each other, we connect the two corresponding 0-cubes with an edge, to form a 1-cube. We orient all these cubes in the same coordinate direction. Clearly, this yields an  $n$ -diagram  $\mathcal{D}$  over a 1-family of cubes.

Assume first that our diagram consists of a number of 1-cubes and at most one 0-cube, with the fiber over this 0-cube consisting of all the indices of the coordinates of  $v$  which are equal to 0. The elements of the group  $\text{Aut}(\mathcal{D})$  are of two sorts, depending on which of the two elements of  $\mathbb{Z}_2$  they are associated to. We easily verify that those elements of  $\text{Aut}(\mathcal{D})$ , which are associated to  $0 \in \mathbb{Z}_2$ , are exactly those  $\pi \in \mathcal{S}_n$ , which fix  $v$ , while those elements of  $\text{Aut}(\mathcal{D})$ , which are associated to  $1 \in \mathbb{Z}_2$ , are exactly those  $\pi \in \mathcal{S}_n$ , which map  $v$  to  $-v$ . Since these are the only two options for mapping  $v$ , if  $l$  is to be preserved by the element  $\pi$ , we have proven the lemma in this case.

Assume now that  $\mathcal{D}$  is a diagram of some other form. Then, there exist no  $\pi \in \mathcal{S}_n$  such that  $\pi(v) = -v$ , i.e., each element of  $\text{stab}(l)$  fixes  $l$  pointwise. In this case,  $\text{stab}(l) = \text{stab}(v)$ , thus we are back to case (1) and the diagram can be obtained by splitting all the 1-cubes into 0-cubes.  $\square$

**6.3. Intersections of diagrams.** Let  $\mathcal{C}_1 = \{K_1, \dots, K_p\}$ , resp.  $\mathcal{C}_2 = \{L_1, \dots, L_q\}$ , be a  $t_1$ -, resp.  $t_2$ -family of cubes, where  $K_i$  is an  $I_1(i)$ -cube, and  $L_j$  is an  $I_2(j)$ -cube, for all  $i \in [p]$ ,  $j \in [q]$ .

Let  $\mathcal{D}_1$ , resp.  $\mathcal{D}_2$ , be  $n$ -diagrams over  $\mathcal{C}_1$ , resp.  $\mathcal{C}_2$ :

$$\begin{aligned} \mathcal{D}_1 &: [n] \longrightarrow \text{vert}(\mathcal{C}_1), \\ &k \mapsto (K_{\alpha_1(k)}, v_k^{(1)}), \end{aligned}$$

$$\begin{aligned} \mathcal{D}_2 &: [n] \longrightarrow \text{vert}(\mathcal{C}_2), \\ &k \mapsto (L_{\alpha_2(k)}, v_k^{(2)}). \end{aligned}$$

**Definition 6.6.** *The intersection of diagrams  $\mathcal{D}_1$  and  $\mathcal{D}_2$ , denoted  $\mathcal{D} = \mathcal{D}_1 \cap \mathcal{D}_2$ , is an  $n$ -diagram over a  $(t_1 + t_2)$ -family of cubes  $\mathcal{C}$  defined as follows:*

$$\mathcal{C} = \{M_{i,j}\}_{i \in [p], j \in [q]}, \quad I(i, j) = I_1(i) \cup \{x + t_1 \mid x \in I_2(j)\},$$

here  $M_{i,j}$  is an  $I(i, j)$ -cube, furthermore

$$\begin{aligned} \mathcal{D} &: [n] \longrightarrow \text{vert}(\mathcal{C}), \\ &k \mapsto (M_{\alpha_1(k), \alpha_2(k)}, v_k^{(1)} \cup \{x + t_1 \mid x \in v_k^{(2)}\}). \end{aligned}$$

Note that the fibers over the vertices and cubes of  $\mathcal{D}$  are determined by the fibers of  $\mathcal{D}_1$  and  $\mathcal{D}_2$  as follows:

$$\mathcal{D}^{-1}(M_{i,j}) = \mathcal{D}_1^{-1}(K_i) \cap \mathcal{D}_2^{-1}(L_j),$$

and

$$(6.3) \quad \mathcal{D}^{-1}(M_{i,j}, v) = \mathcal{D}_1^{-1}(K_i, I_1(i) \cap v) \cap \mathcal{D}_2^{-1}(L_j, \{x - t_1 \mid x \in v, x > t_1\}),$$

for each  $v \subseteq I(i, j)$ .

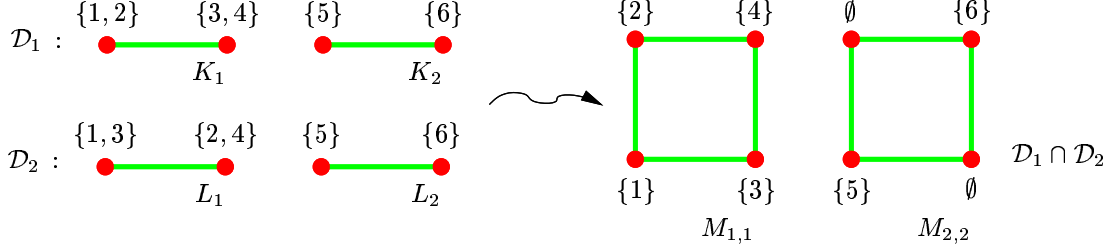


Figure 5. An example of an intersection of two diagrams.

In the above example, observe that  $\mathcal{D}_1 \cap \mathcal{D}_2$  actually contains two more cubes,  $M_{1,2}$  and  $M_{2,1}$ , with 2-element index sets  $I(1, 2)$  and  $I(2, 1)$ , whose fibers, however, are all empty.

**Lemma 6.7.** *For two  $n$ -diagrams  $\mathcal{D}_1$  and  $\mathcal{D}_2$ , we have  $\rho(\mathcal{D}_1 \cap \mathcal{D}_2) = \rho(\mathcal{D}_1) \wedge \rho(\mathcal{D}_2)$ , where  $\wedge$  denotes the operation of common refinement of the set partitions.*

**Proof.** By (6.3), the blocks of  $\rho(\mathcal{D}_1 \cap \mathcal{D}_2)$  are all nonempty intersections of the blocks of  $\rho(\mathcal{D}_1)$  with the blocks of  $\rho(\mathcal{D}_2)$ , which is precisely the definition of the common refinement operation.  $\square$

We shall prove two structural theorems about  $n$ -diagrams. The first one asserts that taking intersections of diagrams commutes with passing to the automorphism group.

**Theorem 6.8.** *For two  $n$ -diagrams  $\mathcal{D}_1$  and  $\mathcal{D}_2$  as above, and  $\mathcal{D} = \mathcal{D}_1 \cap \mathcal{D}_2$  their intersection, we have*

$$(6.4) \quad \text{Aut}(\mathcal{D}_1) \cap \text{Aut}(\mathcal{D}_2) = \text{Aut}(\mathcal{D}).$$

**Proof.** First we prove that the set on the left hand side of (6.4) is a subset of the set on the right hand side.

Let  $\pi \in \text{Aut}(\mathcal{D}_1) \cap \text{Aut}(\mathcal{D}_2)$ . By Definition 6.3 i) we know that  $\pi$  preserves the fibers  $\mathcal{D}_1^{-1}(K_i)$ , for all  $i \in [p]$ , and  $\pi$  preserves the fibers  $\mathcal{D}_2^{-1}(L_j)$ , for all  $j \in [q]$ . Hence  $\pi$  preserves  $\mathcal{D}_1^{-1}(K_i) \cap \mathcal{D}_2^{-1}(L_j) = \mathcal{D}^{-1}(M_{i,j})$ , for all  $i \in [p], j \in [q]$ , and so property i) of Definition 6.3 is valid for  $\pi$ .

By Definition 6.3 ii), there exist  $\sigma^{(1)} \in \mathbb{Z}_2^{t_1}$ , and  $\sigma^{(2)} \in \mathbb{Z}_2^{t_2}$ , such that

$$\sigma_{\alpha_1(k)}^{(1)}(v_k^{(1)}) = v_{\pi(k)}^{(1)}, \quad \text{and} \quad \sigma_{\alpha_2(k)}^{(2)}(v_k^{(2)}) = v_{\pi(k)}^{(2)},$$

for all  $k \in [n]$ , where  $\sigma_{\alpha_1(k)}^{(1)} = \text{proj}_{I_1(\alpha_1(k))}(\sigma)$ , and  $\sigma_{\alpha_2(k)}^{(2)} = \text{proj}_{I_2(\alpha_2(k))}(\sigma)$ .

Define  $\sigma \in \mathbb{Z}_2^{t_1+t_2}$  as a concatenation  $\sigma = (\sigma^{(1)}, \sigma^{(2)})$ , that is the first  $t_1$  coordinates of  $\sigma$  are equal to  $\sigma^{(1)}$ , and the last  $t_2$  coordinates of  $\sigma$  are equal to  $\sigma^{(2)}$ . Let  $k \in [n]$ , and decompose  $v_k \subseteq [t_1 + t_2]$  as  $v_k = v_k^{(1)} \cup \tilde{v}_k^{(2)}$ , where  $v_k^{(1)} = v_k \cap \{1, \dots, t_1\}$ , and

$\tilde{v}_k^{(2)} = v_k \cap \{t_1 + 1, \dots, t_1 + t_2\}$ . Then, we have

$$\begin{aligned} \sigma_{\alpha_1(k), \alpha_2(k)}(v_k) &= \sigma_{\alpha_1(k), \alpha_2(k)}(v_k^{(1)} \cup \tilde{v}_k^{(2)}) \\ &= \sigma_{\alpha_1(k)}(v_k^{(1)}) \cup \tilde{\sigma}_{\alpha_2(k)}(\tilde{v}_k^{(2)}) = v_{\pi(k)}^{(1)} \cup \tilde{v}_{\pi(k)}^{(2)} = v_{\pi(k)}, \end{aligned}$$

where  $\sigma_{\alpha_1(k), \alpha_2(k)} = \text{proj}_{I(\alpha_1(k), \alpha_2(k))}(\sigma)$ ,  $\tilde{\sigma}_{\alpha_2(k)}$  is equal to  $\sigma_{\alpha_2(k)}$  in the coordinates  $\{t_1 + 1, \dots, t_1 + t_2\}$ , and is equal to 0 in the other coordinates, while  $\tilde{v}_{\pi(k)}^{(2)} = \{x + t_1 \mid x \in v_{\pi(k)}^{(2)}\}$ . In other words,  $\tilde{\sigma}_{\alpha_2(k)}$  and  $\tilde{v}_{\pi(k)}^{(2)}$  are the  $t_1$ -shifted versions of  $\sigma_{\alpha_2(k)}$  and  $v_{\pi(k)}^{(2)}$ . So, we have shown that  $\pi \in \text{Aut}(\mathcal{D}_1 \cap \mathcal{D}_2)$ .

Now let us prove that the set on the right hand side of (6.4) is a subset of the set on the left hand side.

Take  $\pi \in \text{Aut}(\mathcal{D})$ , then  $\pi$  preserves  $\mathcal{D}^{-1}(M_{i,j})$ , and therefore  $\pi$  also preserves

$$\begin{aligned} \bigcup_{j=1}^q \mathcal{D}^{-1}(M_{i,j}) &= \bigcup_{j=1}^q \mathcal{D}_1^{-1}(K_i) \cap \mathcal{D}_2^{-1}(L_j) = \mathcal{D}_1^{-1}(K_i) \cap \bigcup_{j=1}^q \mathcal{D}_2^{-1}(L_j) \\ &= \mathcal{D}_1^{-1}(K_i) \cap [n] = \mathcal{D}_1^{-1}(K_i), \quad \text{for any } i \in [p]; \end{aligned}$$

in the same way  $\pi$  preserves  $\mathcal{D}_2^{-1}(L_j)$ , for any  $j \in [q]$ . This checks condition i) of Definition 6.3.

Finally, by condition ii) of Definition 6.3, there exists  $\sigma \in \mathbb{Z}_2^{t_1+t_2}$ , such that for any  $k \in [n]$  we have  $\sigma_{\alpha_1(k), \alpha_2(k)}(v_k) = v_{\pi(k)}$ . As above, we can decompose  $\sigma = (\sigma^{(1)}, \sigma^{(2)})$  and  $v_k = v_k^{(1)} \cup \tilde{v}_k^{(2)}$  as a concatenation of the first  $t_1$  and the last  $t_2$  coordinates. Then, in the notations which we used above, we can derive that

$$\sigma_{\alpha_1(k)}^{(1)}(v_k^{(1)}) = v_{\pi(k)}^{(1)}, \quad \text{and} \quad \tilde{\sigma}_{\alpha_2(k)}^{(2)}(\tilde{v}_k^{(2)}) = \tilde{v}_{\pi(k)}^{(2)}.$$

Shifting the second identity down by  $t_1$ , we get  $\sigma_{\alpha_2(k)}^{(2)}(v_k^{(2)}) = v_{\pi(k)}^{(2)}$ .  $\square$

**6.4. A reduction theorem.** When  $\mathcal{D}$  is an  $n$ -diagram over a  $t$ -family of cubes, not every element  $\sigma \in \mathbb{Z}_2^t$  gives rise to an element  $\pi \in \text{Aut}(\mathcal{D})$ . The natural obstruction is that, by Remark 6.4, fibers with different cardinalities cannot map to each other. It turns out that one can always canonically reduce  $\mathcal{D}$  to another  $n$ -diagram with the same automorphism group, such that in this new  $n$ -diagram all fibers over vertices in the same cube have the same cardinality.

**Theorem 6.9.** *Let  $\mathcal{D}$  be an  $n$ -diagram over a  $t$ -family of cubes  $\mathcal{C} = (K_1, \dots, K_p)$ . Then, there exists an  $n$ -diagram  $\tilde{\mathcal{D}}$  over a  $\tilde{t}$ -family of cubes  $\tilde{\mathcal{C}} = (L_1, \dots, L_q)$ , such that*

- 0)  $\tilde{t} \leq t$ ;
- 1)  $\text{Aut}(\mathcal{D}) = \text{Aut}(\tilde{\mathcal{D}})$ ;
- 2)  $|\mathcal{D}^{-1}(L_j, v)| = |\tilde{\mathcal{D}}^{-1}(L_j, v')|$ , for all  $j \in [q]$ , and for all  $v, v' \subseteq I_2(j)$ , where  $I_2(j)$  is the index set of  $L_j$ .

In the continuation, we shall call an  $n$ -diagram satisfying Condition 2) of Theorem 6.9 a *reduced diagram*.

**Proof of Theorem 6.9.** Let  $G$  be the set of all  $\sigma \in \mathbb{Z}_2^t$ , such that  $\sigma$  occurs as a  $[t]$ -cube symmetry for some  $\pi \in \text{Aut}(\mathcal{D})$ . Clearly,  $G$  is a linear subspace of  $\mathbb{Z}_2^t$ , when both are viewed as vector spaces over the field  $\mathbb{F}_2$ . Hence, there exists  $0 \leq d \leq t$ , such that  $G \cong \mathbb{Z}_2^d$ . Therefore, we can choose an orthogonal linear basis  $\{e_1, \dots, e_t\}$  for  $\mathbb{Z}_2^t$ , such that  $\{e_1, \dots, e_d\}$  is an orthogonal linear basis for  $G$ .

Let us split each cube  $K_i \in \mathcal{C}$  into the orbits of the restriction of the action of  $G$  to  $K_i$ . We can think of cubes  $K_i$  as coordinate subspaces, that is as intersections of coordinate hyperplanes, with respect to the standard basis in the vector space  $\mathbb{Z}_2^t$ . The orbits themselves however are not coordinate subspaces, rather they are intersections of the coordinate subspaces corresponding to cubes with affine linear subspaces of dimension  $d$  obtained from  $G$  by parallel translations. Therefore, if we change the linear basis in  $\mathbb{Z}_2^t$  from the standard one to  $\{e_1, \dots, e_t\}$  at the same time as we split the cubes of  $\mathcal{C}$  into the orbits as described above, we end up with a new  $t$ -family of cubes  $\tilde{\mathcal{C}} = (L_1, \dots, L_q)$ , and an  $n$ -diagram  $\tilde{\mathcal{D}}$  over this family, which is induced from  $\mathcal{D}$ .

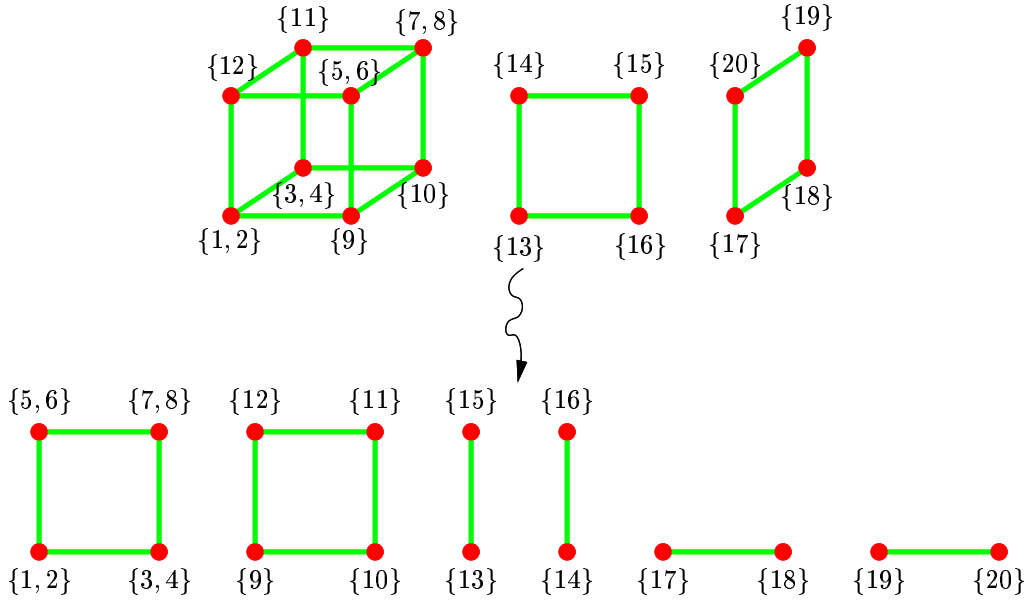


Figure 6. An example of the canonical splitting of a diagram.

By the choice of  $G$  and of the basis  $\{e_1, \dots, e_t\}$ , we see that all the cubes of  $\tilde{\mathcal{C}}$  actually lie within the coordinate subspace of  $\mathbb{Z}_2^t$  corresponding to the first  $d$  coordinates. Thus, we might as well think of  $\tilde{\mathcal{C}}$  as a  $d$ -family of cubes, with  $\mathbb{Z}_2^d$  action induced from the action of  $\mathbb{Z}_2^t$ , from which condition 0) of the theorem follows.

Also, since the action on the ground set  $[n]$  never changed, we still have the equality  $\text{Aut}(\mathcal{D}) = \text{Aut}(\tilde{\mathcal{D}})$ , verifying condition 1) of the theorem.

Finally, since  $G$  acts transitively on each of its orbits, we can conclude that the cardinalities of the fibers are constant for the vertices of the same cube in  $\tilde{\mathcal{C}}$ , thus demonstrating the truth of the last condition, and completing the proof of the theorem.  $\square$

7. STABILIZERS OF POINTS IN  $Y_{\Pi_n}$ 

In this section we show that the stabilizers of points in  $Y_{\Pi_n}$  are not just abelian, but in fact are isomorphic to direct products of  $\mathbb{Z}_2$ . In view of the already proven results, it merely remains to put the puzzle pieces together.

**Theorem 7.1.** *For  $Y_{\Pi_n}$ , the De Concini-Procesi arrangement model of the braid arrangement, and  $\omega \in Y_{\Pi_n}$ , the stabilizer of  $\omega$  with respect to the  $\mathcal{S}_n$ -action on  $Y_{\Pi_n}$  is a direct product of  $\mathbb{Z}_2$ 's:*

$$\text{stab}_{Y_{\Pi_n}}(\omega) \cong \mathbb{Z}_2^h, \quad \text{for some } 0 \leq h \leq \lfloor n/2 \rfloor.$$

**Proof.** By (4.1) a point in  $Y_{\Pi_n}$  can be written as  $\omega = (x, H_1, l_1, H_2, \dots, H_t, l_t)$ , where  $H_i \in \Pi_n \setminus \{\hat{0}\}$ , and there does not exist a subspace  $H \in \Pi_n$ ,  $H \neq \mathbb{R}^n$ , such that  $H \supseteq \langle H_t, l_t \rangle$ . By Proposition 4.2 we know that

$$(7.1) \quad \text{stab}_{Y_{\Pi_n}}(\omega) = \text{stab}_{\mathbb{R}^n}(x) \cap \text{stab}_{\mathbb{R}^n}(l_1) \cap \dots \cap \text{stab}_{\mathbb{R}^n}(l_t).$$

By Lemma 6.5 there exist diagrams  $\mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_t$ , such that

$$(7.2) \quad \text{Aut}(\mathcal{D}_0) = \text{stab}_{\mathbb{R}^n}(x), \quad \text{and} \quad \text{Aut}(\mathcal{D}_i) = \text{stab}_{\mathbb{R}^n}(l_i), \quad \text{for each } i \in [t].$$

Combining (7.1), (7.2), and Theorem 6.8, we find an  $n$ -diagram  $\mathcal{D}$ , such that  $\text{Aut}(\mathcal{D}) = \text{stab}_{Y_{\Pi_n}}(\omega)$ . Moreover, by the Reduction Theorem 6.9, we can assume that  $\mathcal{D}$  is reduced.

If the partition  $\rho(\mathcal{D})$  has a block  $B$  of cardinality at least 3, then, by Lemma 6.7, so do also the partitions  $\rho(\mathcal{D}_0), \rho(\mathcal{D}_1), \dots, \rho(\mathcal{D}_t)$ . Let  $H$  be the linear subspace of  $\mathbb{R}^n$  of codimension 2 defined by setting the coordinates with indices in  $B$  equal. By construction,  $x \in H$ , and  $l_1 \subseteq H, \dots, l_t \subseteq H$ . Since  $H \in \Pi_n$ , we see that  $x \in H$  implies  $H_1 \subseteq H$ . Further  $l_1 \subseteq H$ , together with  $H_1 \subseteq H$ , implies  $\langle l_1, H_1 \rangle \subseteq H$ . Hence  $H_2 \subseteq H$ , and so on, until we can conclude that  $\langle l_t, H_t \rangle \subseteq H$ . This yields a contradiction, since  $H \neq \mathbb{R}^n$ .

So we proved that all blocks of the partition  $\rho(\mathcal{D})$  are of cardinality at most 2. Assume now there exist two different blocks  $B_1$  and  $B_2$  in  $\rho(\mathcal{D})$ , such that  $|B_1| = |B_2| = 2$ . Let  $H$  be the linear subspace of  $\mathbb{R}^n$  of codimension 2 defined by equations  $x_{i_1} = x_{i_2}, x_{j_1} = x_{j_2}$ , where  $B_1 = \{i_1, i_2\}$ ,  $B_2 = \{j_1, j_2\}$ . Again  $H \in \Pi_n$ , and by an argument completely analogous to the previous one, we can trace the two blocks  $B_1$  and  $B_2$  through the partitions  $\rho(\mathcal{D}_0), \rho(\mathcal{D}_1), \dots, \rho(\mathcal{D}_t)$ , and conclude that  $\langle l_t, H_t \rangle \subseteq H$ . This again yields a contradiction, since  $H \neq \mathbb{R}^n$ .

Now we know that  $\rho(\mathcal{D})$  has at most one block of size 2. In particular, since  $\mathcal{D}$  is reduced, all the fibers over  $I$ -cubes, for  $|I| \geq 1$ , are of cardinality 1. Let us say  $\mathcal{D}$  is an  $n$ -diagram over a  $t$ -family of cubes  $\mathcal{C} = (K_1, \dots, K_q)$ , where  $t$  is minimal possible. If  $\rho(\mathcal{D})$  has no blocks of size 2, then there exists a group isomorphism between  $\text{Aut}(\mathcal{D})$  and  $\mathbb{Z}_2^t$ , since each element  $\pi \in \mathbb{Z}_2^t$  defines the maps between the fibers uniquely. Each  $I$ -cube defines at most  $|I|$  new directions and has  $2^{|I|}$  vertices, hence

$$t \leq |I_1| + \dots + |I_t| \leq 2^{|I_1|-1} + \dots + 2^{|I_t|-1} = n/2.$$

If the partition  $\rho(\mathcal{D})$  has one block  $B$  of size 2, then, since  $\mathcal{D}$  is reduced,  $B$  has to be a fiber over a  $\emptyset$ -cube. With  $t$  chosen as above, it is immediate that  $\text{Aut}(\mathcal{D}) \cong \mathbb{Z}_2^t \times \mathbb{Z}_2$ , where the first factor on the right hand side is the group acting on the  $[t]$ -cube, and the second factor is acting on the set  $B$ . Just as before we get  $t \leq (n-2)/2$ , hence  $t+1 \leq n/2$ .  $\square$

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