A DESINGULARIZATION OF REAL DIFFERENTIABLE ACTIONS OF FINITE GROUPS

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ABSTRACT. We provide abelianizations of differentiable actions of finite groups on smooth real manifolds. De Concini-Procesi wonderful models for (local) subspace arrangements and a careful analysis of linear actions on real vector spaces are at the core of our construction. In fact, we show that our abelianizations have stabilizers isomorphic to elementary abelian 2-groups, a setting for which we suggest the term *digitalization*. As our main examples, we discuss the resulting digitalizations of the permutation actions of the symmetric group on real linear and projective spaces.

1. Introduction

Abelianizations of finite group actions on complex manifolds appeared prominently in the work of Batyrev [Ba], and a connection to the wonderful arrangement models of De Concini and Procesi was observed by Borisov and Gunnells [BG]. The authors of the present article have previously presented a detailed study of the key example over the reals, the abelianization of the permutation action of the symmetric group $\mathcal{S}_n$ on $\mathbb{R}^n$ given by the maximal De Concini-Procesi model of the braid arrangement, cf. [FK2]. In particular, it was shown that stabilizers of points on the arrangement model are elementary abelian 2-groups. We suggest to call an abelianization with this property a *digitalization* of the given action.

In the present article, we extend our analysis in two steps. First, for any linear action of a finite group on a real vector space, we define an arrangement of linear subspaces whose maximal De Concini-Procesi model we then show to be a digitalization of the given action. Second, we proceed by analysing differentiable actions of finite groups on smooth real manifolds. We propose a locally finite stratification of the manifold by smooth submanifolds and, observing that this stratification is actually a local subspace arrangement, we show that the associated maximal De Concini-Procesi model is a digitalization of the given action.

We present examples in the linear and in the non-linear case. First, we consider the permutation action of the symmetric group $\mathcal{S}_n$ on $\mathbb{R}^n$, and we find that our arrangement construction yields the rank 2 truncation of the braid arrangement. The resulting digitalization is the one discussed in [FK2]. It would be interesting to extend this result to the action of an arbitrary reflection group. As a non-linear example, we consider the action of $\mathcal{S}_n$ on $\mathbb{RP}^{n-1}$ given by projectivizing the real permutation action on $\mathbb{R}^n$. We show that our manifold stratification, in this case, coincides with the rank 2 truncation of the projectivized braid arrangement. The resulting digitalization thus is the maximal projective De Concini-Procesi model for the braid arrangement.

We give a short overview on the material presented in this article: In Section 2 we provide a review of De Concini-Procesi arrangement models in an attempt to keep this exposition fairly self-contained. Our main results are presented in Section 3. In 3.1 we propose a digitalization for any given linear action of a finite group on a real vector space; in 3.2 we extend our setting to differentiable actions of finite groups on smooth real manifolds. Section 4 is focused on examples. We work out the details of the proposed digitalizations for the real permutation action in 4.1, and for the permutation action on real projective spaces in 4.2.

2. A review of De Concini-Procesi arrangement models

2.1. **Arrangement models.** We review the construction of De Concini-Procesi arrangement models as presented in [DP1]. Moreover, we recall an encoding of points in maximal arrangement models from [FK2] that will be crucial for the technical handling of stabilizers (cf. 2.2).

2.1.1. *The model construction.* Let $\mathcal{A}$ be a finite family of linear subspaces in some real or complex vector space V. The combinatorial data of such subspace arrangement is customarily recorded by its *intersection lattice* $\mathcal{L} = \mathcal{L}(\mathcal{A})$, the partially ordered set of intersections among subspaces in $\mathcal{A}$ ordered by reversed inclusion. We agree on the empty intersection to be the full space V, represented by the minimal element $\hat{0}$ in the lattice. We will frequently use $\mathcal{L}_{>\hat{0}}$ to denote $\mathcal{L} \setminus \{\hat{0}\}$.

There is a family of arrangement models each coming from the choice of a certain subset of the intersection lattice, so-called *building sets*. For the moment we restrict our attention to the maximal model among those, which results from choosing the whole intersection lattice as a building set.

We give two alternative descriptions for the maximal De Concini-Procesi model of $\mathcal{A}$. Consider the following map on the complement $\mathcal{M}(\mathcal{A}) := V \setminus \bigcup \mathcal{A}$ of the arrangement,

$$\Psi : \quad \mathcal{M}(\mathcal{A}) \longrightarrow V \times \prod_{X \in \mathcal{L}_{>\hat{0}}} \mathbb{P}(V/X) , \tag{2.1}$$

where Ψ is the natural inclusion into the first factor and the natural projection to the other factors restricted to $\mathcal{M}(\mathcal{A})$. Formally,

$$\Psi(x) \;=\; (\, x \, , (\langle x, X \rangle / X)_{X \in \mathcal{L}_{>\hat{0}}}) \, ,$$

where $\langle \cdot , \cdot \rangle$ denotes the linear span of subspaces or vectors, respectively, and $\langle x, X \rangle / X$ is interpreted as a point in $\mathbb{P}(V/X)$ for any $X \in \mathcal{L}_{>\hat{0}}$.

The map Ψ defines an embedding of $\mathcal{M}(\mathcal{A})$ into the product on the right hand side of (2.1). The closure of its image, $Y_{\mathcal{A}} := \overline{\mathrm{im}\Psi}$, is the *maximal De Concini-Procesi model* of the arrangement $\mathcal{A}$. If we want to stress the ambient space of the original arrangement, we will use the notation $Y_{V,\mathcal{A}}$ for $Y_{\mathcal{A}}$.

Alternatively, one can describe $Y_{\mathcal{A}}$ as the result of successive blowups of strata in V. Consider the stratification of V given by the linear subspaces in $\mathcal{A}$ and their intersections. Choose some linear extension of the opposite order in $\mathcal{L}$. Then, $Y_{\mathcal{A}}$ is the result of successive blowups of strata, respectively proper transforms of strata, corresponding to the subspaces in $\mathcal{L}$ in the chosen linear extension order.

Let us mention here that there is a projective analogue $\overline{Y}_{\mathcal{A}}$ of the affine arrangement model $Y_{\mathcal{A}}$ (cf. [DP1, §4]). In fact, the affine model $Y_{\mathcal{A}}$ is the total space of a line bundle over $\overline{Y}_{\mathcal{A}}$. A description of the integral cohomology algebra of $\overline{Y}_{\mathcal{A}}$ for complex hyperplane arrangements $\mathcal{A}$ in [DP2] gave rise to a class of abstract algebras defined by atomic lattices that bear a wealth of geometric meanings (cf. [FY]).

We will need to refer to projective arrangement models only in one of our examples in Section 4. We therefore stay with the affine setting in the following exposition.

2.1.2. *Normal crossing divisors and nested set stratification.* The term *wonderful* models has been coined for $Y_{\mathcal{A}}$ and its generalizations for other choices of building sets. We summarize the key facts about the maximal model supporting this connotation.

The space $Y_{\mathcal{A}}$ is a smooth algebraic variety with a natural projection onto the original ambient space V, $p : Y_{\mathcal{A}} \longrightarrow V$. The map p is the projection onto the first coordinate of the ambient space of $Y_{\mathcal{A}}$ on the right hand side of (2.1), respectively the concatenation of blowdown maps of the sequence of blowups resulting in $Y_{\mathcal{A}}$. This projection is an isomorphism on $\mathcal{M}(\mathcal{A})$, while the complement $Y_{\mathcal{A}} \setminus \mathcal{M}(\mathcal{A})$ is a divisor with normal crossings with irreducible components indexed by the elements of $\mathcal{L}_{>\hat{0}}$. An intersection of several irreducible components is non-empty (moreover, transversal and irreducible) if and only if the indexing lattice elements form a totally ordered set, i.e., a chain, in $\mathcal{L}$ [DP1, 3.1,3.2]. The stratification by irreducible components of the divisor and their intersections is called the *nested set stratification* of $Y_{\mathcal{A}}$, denoted $(Y_{\mathcal{A}}, \mathfrak{D})$, for reasons that lie in the more general model construction for arbitrary building sets rather than the maximal building set $\mathcal{L}_{>\hat{0}}$.

2.1.3. *An encoding of points in maximal arrangement models.* Points in $Y_{\mathcal{A}}$ can be described as a sequence of a point and a number of lines in the vector space V according to the form of the ambient space for $Y_{\mathcal{A}}$ given on the right hand side of (2.1). However, there is a lot of redundant information in that description. The following compact encoding of points was suggested in [FK2, Sect 4.1].

Proposition 2.1. *Let ω be a point in the maximal wonderful model $Y_{\mathcal{A}}$ for a subspace arrangement $\mathcal{A}$ in complex or real space V. Then ω can be uniquely written as*

$$\omega = (x, H_1, \ell_1, H_2, \ell_2, \ldots, H_t, \ell_t) = (x, \ell_1, \ell_2, \ldots, \ell_t)\,, \tag{2.2}$$

where x is a point in V, the $H_1, \ldots, H_t$ form a descending chain of subspaces in $\mathcal{L}_{>\hat{0}}$, and the ℓ_i are lines in V, all subject to a number of additional conditions.

More specifically, $x = p(\omega)$, and the linear space H_1 is the maximal lattice element that, as a subspace of V, contains x. The line ℓ_1 is orthogonal to H_1 and corresponds to the coordinate entry of ω indexed by H_1 in $\mathbb{P}(V/H_1)$. The lattice element H_2, in turn, is the maximal lattice element that contains both H_1 and ℓ_1. The specification of lines ℓ_i, i.e., lines that correspond to coordinates of ω in $\mathbb{P}(V/H_i)$, and the construction of lattice elements H_{i+1}, continues analogously for $i \geq 2$ until a last line ℓ_t is reached whose span with H_t is not contained in any lattice element other than the full ambient space V. Note that, if H_t is a hyperplane, then the line ℓ_t is uniquely determined. The whole space V can be thought of as H_{t+1}. Observe that the H_i are determined by x and the sequence of lines ℓ_i; we choose to include the H_i at times in order to keep the notation more transparent.

The full coordinate information on ω can be recovered from (2.2) by setting $H_0 = \bigcap \mathcal{A}$, $\ell_0 = \langle x \rangle$, and retrieving the coordinate ω_H indexed by $H \in \mathcal{L}_{>\hat{0}}$ as

$$\omega_H \;=\; \langle \ell_j, H \rangle / H \;\in\; \mathbb{P}(V/H)\,, \tag{2.3}$$

where j is chosen from $\{1, \ldots, t\}$ such that $H \leq H_j$, but $H \not\leq H_{j+1}$.

For completeness, let us mention here that we can tell the open stratum in the nested set stratification $(Y_{\mathcal{A}}, \mathfrak{D})$ that contains a given point ω from its point/line encoding stated in Proposition 2.1.

Proposition 2.2. ([FK2, Prop 4.5]) *A point ω in a maximal arrangement model $Y_{\mathcal{A}}$ is contained in the open stratum of $(Y_{\mathcal{A}}, \mathfrak{D})$ indexed by the chain $H_1 > H_2 > \ldots > H_t > \hat{0}$ in $\mathcal{L}$ if and only if its point/line description* (2.2) *reads $\omega = (x, H_1, \ell_1, H_2, \ell_2, \ldots, H_t, \ell_t)$.*

2.2. **Group actions on arrangement models and a description of stabilizers.** Provided an arrangement is invariant under the action of a finite group, this action extends to the maximal arrangement model. We review the details, and recall a description for stabilizers of points in the model from [FK2].

2.2.1. *Group actions on $Y_{\mathcal{A}}$.* Let $\mathcal{A}$ be an arrangement that is invariant under the linear action of a finite group G on the real or complex ambient space V. Without loss of generality, we can assume that this action is orthogonal [V, 2.3, Thm 1]. We denote the corresponding G-invariant positive definite symmetric bilinear form by the usual scalar product.

The group G acts on the ambient space of the arrangement model $Y_{\mathcal{A}}$, i.e., for $(x, (x_X)_{X \in \mathcal{L}_{>\hat{0}}}) \in V \times \prod_{X \in \mathcal{L}_{>\hat{0}}} \mathbb{P}(V/X)$ and $g \in G$, we have

$$g\,(x, (x_X)_{X \in \mathcal{L}_{>\hat{0}}}) \;=\; (\,g(x), (\,g(x_{g^{-1}(X)})\,)_{X \in \mathcal{L}_{>\hat{0}}})\,,$$

where $g(x_{g^{-1}(X)}) \in \mathbb{P}(V/X)$ for $X \in \mathcal{L}_{>\hat{0}}$. Since the inclusion map Ψ of (2.1) commutes with the G-action, and G acts continuously on V, we conclude that $Y_{\mathcal{A}} = \overline{\mathrm{Im}\Psi}$ is as well G-invariant. In particular, the G-action on $Y_{\mathcal{A}}$ extends the G-action on the complement of $\mathcal{A}$.

2.2.2. *Stabilizers of points on $Y_{\mathcal{A}}$.* The point/line description for points in the arrangement model $Y_{\mathcal{A}}$ given in 2.1.3 allows for a concise description of stabilizers with respect to the G-action on $Y_{\mathcal{A}}$.

Proposition 2.3. ([FK2, Prop 4.2]) *For a maximal arrangement model $Y_{\mathcal{A}}$ that is equipped with the action of a finite group G stemming from a linear action of G on the arrangement, the stabilizer of a point $\omega = (x, H_1, \ell_1, H_2, \ell_2, \ldots, H_t, \ell_t)$ in $Y_{\mathcal{A}}$ is of the form*

$$\mathrm{stab}_{Y_{\mathcal{A}}}(\omega) \;=\; \mathrm{stab}_V(x) \cap \mathrm{stab}_V(\ell_1) \cap \ldots \cap \mathrm{stab}_V(\ell_t)\,, \tag{2.4}$$

where, for $i = 1, \ldots, t$, $\mathrm{stab}_V(\ell_i)$ denotes the elements in G that preserve the line ℓ_i in V as a set.

2.3. **Models for local subspace arrangements.** The arrangement model construction of De Concini & Procesi generalizes to the context of local subspace arrangements.

Definition 2.4. *Let X be a smooth d-dimensional real or complex manifold and $\mathcal{A}$ a family of smooth real or complex submanifolds in X such that all non-empty intersections of submanifolds in $\mathcal{A}$ are connected, smooth submanifolds. The family $\mathcal{A}$ is called a* local subspace arrangement *if for any $x \in \bigcup \mathcal{A}$ there exists an open neighborhood U of x in X, a subspace arrangement $\tilde{\mathcal{A}}$ in a real or complex d-dimensional vector space V and a diffeomorphism $\phi : U \longrightarrow V$, mapping $\mathcal{A}$ to $\tilde{\mathcal{A}}$.*

Local subspace arrangements fall into the class of conically stratified manifolds as appearing in work of MacPherson & Procesi [MP] in the complex and in work of Gaiffi [Ga] in the real setting.

A generalization of the arrangement model construction of De Concini & Procesi by sequences of blowups of smooth strata for conically stratified complex manifolds is given in [MP]. Details are provided for blowing up so-called irreducible strata, the more general construction for an arbitrary building set of the stratification is outlined in Sect. 4 of [MP].

In this article, we will be concerned with *maximal wonderful models* for conically stratified real manifolds X, in the special case of local subspace arrangements $\mathcal{A}$. The maximal model $Y_{\mathcal{A}} = Y_{X,\mathcal{A}}$ results from successive blowups of *all* initial strata, respectively their proper transforms, according to some linear order on strata which is non-decreasing in dimension.

In fact, local subspace arrangements consisting of a *finite* number of submanifolds implicitly appear already in the arrangement model construction of De Concini & Procesi [DP1]. A single blowup in a subspace arrangement leads to the class of local arrangements, and it is due to the choice of blowup order on building set strata that this class is closed under blowups that occur in the inductive construction of the arrangement models (cf. the discussion in [FK1, 4.1.2], in particular, Example 4.6).

We will encounter the case of local subspace arrangements $\mathcal{A}$ in a smooth real manifold X that are invariant under the differentiable action of a finite group G on X. The G-action can be extended to the maximal model $Y_{\mathcal{A}}$, observing that we can simultaneously blow up orbits of strata, thereby lifting the G-action step by step through the construction process. In particular, the concatenation of blowdown maps $p : Y_{\mathcal{A}} \to X$ is G-equivariant.

3. Digitalizing finite group actions

3.1. **Finite linear actions on $\mathbb{R}^n$.** In this subsection we assume G to be a finite subgroup of the orthogonal group $O(n)$ acting effectively on $\mathbb{R}^n$. As pointed out before, assuming the action to be orthogonal is not a restriction (cf. 2.2.1).

We construct an abelianization of the given action. For any subgroup H in G (we use the notation $H \leq G$ in the sequel), define

$$L(H) \ := \ \langle \, \ell \, | \, \ell \text{ line in } \mathbb{R}^n \text{ with } \ h \cdot \ell = \ell \text{ for all } \ h \in H \, \rangle \, ,$$

the linear span of lines in $\mathbb{R}^n$ that are invariant under H, i.e., the span of lines that are either fixed or flipped by any element h in H. Denote by $\mathcal{A}$ the arrangement given by the *proper* subspaces $L(H) \subsetneq \mathbb{R}^n$, H subgroup in G. Set $Y_{\mathcal{A}}$ denote the maximal

De Concini-Procesi wonderful model for $\mathcal{A}$ as discussed in 2.1. If we want to stress the particular group action that gives rise to the arrangement $\mathcal{A}$ we write $\mathcal{A}(G)$ and $Y_{\mathcal{A}(G)}$, or $\mathcal{A}(G \circlearrowright \mathbb{R}^n)$ and $Y_{\mathcal{A}(G\circlearrowright\mathbb{R}^n)}$, respectively.

We will now propose $Y_{\mathcal{A}}$ as an abelianization of the given linear action. Recall that we use the term *digitalization* for an abelianization with stabilizers that are not merely abelian but elementary abelian 2-groups, i.e., are isomorphic to $\mathbb{Z}_2^k$ for some $k \in \mathbb{N}$.

Theorem 3.1. *Let an effective action of a finite subgroup G of $O(n)$ on $\mathbb{R}^n$ be given. Then the wonderful arrangement model $Y_{\mathcal{A}(G)}$, as defined above, is a digitalization of the given action.*

Proof. As a first step we prove that

$$L(\operatorname{stab}\omega) \;=\; \mathbb{R}^n\,, \qquad \text{for any } \omega \in Y_{\mathcal{A}}\,.$$

Let $\omega \in Y_{\mathcal{A}}$. Using the encoding of points in arrangement models as sequences of point and lines from 2.1.3, we have $\omega = (x, \ell_1, \dots, \ell_t)$, the associated sequence of building set spaces being $V_1, \dots, V_t$. The description of $\operatorname{stab}\omega$ from Proposition 2.3,

$$\operatorname{stab}\omega \;=\; \operatorname{stab}x \cap \operatorname{stab}\ell_1 \cap \ldots \cap \operatorname{stab}\ell_t\,,$$

implies that $x \in L(\operatorname{stab}\omega)$, and $\ell_i \subseteq L(\operatorname{stab}\omega)$ for $i = 1, \dots, t$.

The building set element V_1 is the smallest subspace among intersections of spaces $L(H)$ in $\mathcal{A}$ such that $x \in V_1$, in particular, $V_1 \subseteq L(\operatorname{stab}\omega)$. Similarly, the building set element V_2 is the smallest subspace among intersections of spaces $L(H)$ in $\mathcal{A}$ such that $\langle V_1, \ell_1\rangle \subseteq V_2$; since $\langle V_1, \ell_1\rangle \subseteq L(\operatorname{stab}\omega)$, so is V_2: $V_2 \subseteq L(\operatorname{stab}\omega)$.

By analogous arguments we conclude that $V_3, \dots, V_{t+1} \subseteq L(\operatorname{stab}\omega)$. However, by the description of ω as a sequence of point and lines we know that $V_{t+1} = \mathbb{R}^n$, which proves our claim.

With $L(\operatorname{stab}\omega) = \mathbb{R}^n$, we can now choose a basis $v_1, \dots, v_n$ in $\mathbb{R}^n$ such that any $\langle v_i\rangle$, for $i = 1, \dots, n$, is invariant under the action of $\operatorname{stab}\omega$.

Consider the homomorphism

$$\begin{array}{rrcl} \alpha: & \operatorname{stab}\omega & \longrightarrow & \mathbb{Z}_2^n \\ & h & \longmapsto & (\epsilon_1, \dots, \epsilon_n)\,, \end{array}$$

with $\epsilon_i \in \mathbb{Z}_2$ defined by $h(v_i) = \epsilon_i\, v_i$ for $i = 1, \dots, n$. Since we assume the action to be effective, α is injective. Hence $\operatorname{stab}\omega \cong \mathbb{Z}_2^k$ for some $k \le n$. □

3.2. **Finite differentiable actions on manifolds.** We now generalize the results of the previous subsection to differentiable actions of finite groups on smooth manifolds. To this end, we first propose a stratification of the manifold and show that the stratification locally coincides with the arrangement stratifications on tangent spaces that arise from the induced linear actions as described in the previous section. We can assume, without loss of generality, that the manifold is connected, since we can work with connected components one at a time.

3.2.1. *The $\mathcal{L}$-stratification.* Let X be a smooth manifold, G a finite group that acts differentiably on X. For any point $x \in X$, and any subgroup $H \leq \operatorname{stab} x$, H acts linearly on the tangent space T_xX of X in x. Consider as above

$$L(x,H) \;:=\; \langle\, \ell \,|\, \ell \text{ line in } T_xX \text{ with } h\cdot\ell=\ell \text{ for all } h\in H\,\rangle\,,$$

the linear subspace in T_xX spanned by lines that are invariant under the action of H. Denote the arrangement of proper subspaces $L(x,H)$ in T_xX, $\mathcal{A}(\operatorname{stab} x \circlearrowright T_xX)$, by $\mathcal{A}_x$.

For any subgroup H in $\operatorname{stab} x$, we take up the homomorphism that occurred in the proof of Theorem 3.1, and define

$$\alpha_{x,H}:\; H \;\longrightarrow\; \mathbb{Z}_2^{\dim L(x,H)}$$

by choosing a basis $v_1,\dots,v_t$, $t := \dim L(x,H)$, for $L(x,H)$, and setting

$$\alpha_{x,H}(h) \;=\; (\epsilon_1,\dots,\epsilon_t)\,,$$

for $h\in H$, with $\epsilon_i \in \mathbb{Z}_2$ determined by $h(v_i)=\epsilon_i v_i$ for $i=1,\dots,t$.

Moreover, we define

$$F(x,H) \;:=\; \ker \alpha_{x,H}\,.$$

Note that $F(x,H)$ is the normal subgroup of elements in H that fix all of $L(x,H)$ pointwise. We denote by $\mathcal{L}(x,H)$ the connected component of $\operatorname{Fix}(F(x,H) \circlearrowright X)$ in X that contains x.

Consider the stratification of X by the collection of submanifolds $\mathcal{L}(x,H)$ for $x\in X$, $H \leq \operatorname{stab} x$,

$$\mathcal{L} \;:=\; \{\mathcal{L}(x,H)\}_{x\in X,\, H\leq \operatorname{stab} x}\,.$$

We will refer to this stratification as the *$\mathcal{L}$-stratification* of X. Observe that $\mathcal{L}$ is a locally finite stratification.

We recall the following fact from the theory of group actions on smooth manifolds:

Proposition 3.2. *Let G be a compact Lie group acting differentiably on a smooth manifold X, and let $x_0 \in X$. Then there exists a* $\operatorname{stab} x_0$*-equivariant diffeomorphism Φ_{x_0} from an open neighborhood U of x_0 in X to the tangent space $T_{x_0}X$ of X in x_0.*

This is a special case of the so-called *slice theorem* [A, tD] that originally appeared in work of Bochner [Bo].

We return to our setting of G being a finite group.

Proposition 3.3. *The diffeomorphism Φ_{x_0} maps the $\mathcal{L}$-stratification of X to the arrangement stratification on $T_{x_0}X$ given by $\mathcal{A}_{x_0}$, i.e.,*

$$\Phi_{x_0}(\mathcal{L}(x_0,H)) \;=\; L(x_0,H) \qquad \textit{for any } H \leq \operatorname{stab} x_0\,.$$

Proof. By definition, $\mathcal{L}(x_0,H) = \operatorname{Fix}(F(x_0,H) \circlearrowright X)$, which, using the $\operatorname{stab} x_0$-equivariance of Φ_{x_0}, implies that $\Phi_{x_0}(\mathcal{L}(x_0,H)) = \operatorname{Fix}(F(x_0,H) \circlearrowright T_{x_0}X)$. We are left to show that

$$\operatorname{Fix}(F(x_0,H) \circlearrowright T_{x_0}X) \;=\; L(x_0,H)\,.$$

Obviously, $L(x_0,H) \subseteq \operatorname{Fix}(F(x_0,H) \circlearrowright T_{x_0}X)$, and we need to see that $\operatorname{Fix}(F(x_0,H) \circlearrowright T_{x_0}X)$ does not exceed $L(x_0,H)$.

Note that H acts on $L(x_0,H)$. By definition, $F(x_0,H)$ is a normal subgroup of H with quotient $H/F(x_0,H) \cong \mathbb{Z}_2^d$ for some $d \leq t = \dim L(x_0,H)$, and we find that H acts on $\mathrm{Fix}\,(F(x_0,H) \circlearrowleft T_{x_0}X)$: For $x\in \mathrm{Fix}\,(F(x_0,H) \circlearrowleft T_{x_0}X)$, $h\in H$, and $h_1\in F(x_0,H)$, we have $h_1hx=h\tilde{h}_1x$ for some $\tilde{h}_1\in F(x_0,H)$, thus $h_1hx=hx$, i.e., $hx\in \mathrm{Fix}\,(F(x_0,H) \circlearrowleft T_{x_0}X)$.

Instead of considering the action of H on $\mathrm{Fix}\,(F(x_0,H) \circlearrowleft T_{x_0}X)$, we consider the induced action of $H/F(x_0,H)$ on $\mathrm{Fix}\,(F(x_0,H) \circlearrowleft T_{x_0}X)$. Since $H/F(x_0,H)\cong \mathbb{Z}_2^d$ for some $d\leq t$, $\mathrm{Fix}\,(F(x_0,H) \circlearrowleft T_{x_0}X)$ decomposes into 1-dimensional representation spaces, which, as lines that are invariant under the action of H, must be contained in $L(x_0,H)$ by definition. This shows that $\mathrm{Fix}\,(F(x_0,H) \circlearrowleft T_{x_0}X)$ does not exceed $L(x_0,H)$, and thus completes our proof. □

Remark 3.4. Applying Proposition 3.3 to a linear action $G\circlearrowleft X=\mathbb{R}^n$ for $x_0=0$, we see that the linear subspaces $L(H)$, for $H\leq G$, are fixed point sets of subgroups of H, namely $L(H)=\mathrm{Fix}(F(H))$, where $F(H)$ is the subgroup of elements in H that fix $L(H)$ point-wise, $F(H)=\{h\in H\mid hx=x \text{ for all } x\in L(H)\}$.

In particular, Proposition 3.3 shows that the submanifolds $\mathcal{L}(x,H)$ in the $\mathcal{L}$-stratification form a local subspace arrangement in X. Moreover, the $\mathcal{L}$-stratification is invariant under the action of G since $g(\mathcal{L}(x,H))=\mathcal{L}(g(x),gHg^{-1})$ for any $x\in X$, $H\leq \mathrm{stab}\,x$, and any $g\in G$. Hence, we have at hand the maximal G-equivariant wonderful model $Y_{\mathcal{L}}=Y_{X,\mathcal{L}}$ of the local subspace arrangement $\mathcal{L}$ in X as outlined in 2.3.

3.2.2. *Digitalizing manifolds.* We propose the maximal wonderful model of X with respect to the $\mathcal{L}$-stratification as a digitalization of the manifold X.

Theorem 3.5. *Let G be a finite group acting differentiably and effectively on a smooth manifold X. Then the maximal wonderful model of X with respect to the $\mathcal{L}$-stratification $Y_{X,\mathcal{L}}$ is a digitalization of the given action.*

Proof. Let x be a point in $Y_{X,\mathcal{L}}$, $x_0=p(x)$ its image under the blowdown map $p: Y_{X,\mathcal{L}}\longrightarrow X$. Since p is G-equivariant, $\mathrm{stab}\,x\subseteq \mathrm{stab}\,x_0$, hence we can restrict our attention to $\mathrm{stab}\,x_0$ when determining the stabilizer of x in G.

Consider the $\mathrm{stab}\,x_0$-equivariant diffeomorphism Φ_{x_0} as discussed above (Proposition 3.2),

$$\Phi_{x_0}:\quad U\longrightarrow T_{x_0}X\,,$$

where U is an open neighborhood of x_0 in X, such that Φ_{x_0} maps the $\mathcal{L}$-stratification on U to the arrangement stratification on the tangent space at x_0. Since the De Concini-Procesi model is defined locally, the diffeomorphism Φ_{x_0} induces a $\mathrm{stab}\,x_0$-equivariant diffeomorphism between the inverse image of U under the blowdown map, $p^{-1}U=Y_{U,\mathcal{L}}$, and the De Concini-Procesi model for the arrangement $\mathcal{A}_{x_0}$ in the tangent space $T_{x_0}X$,

$$\widetilde{\Phi}_{x_0}:\quad Y_{U,\mathcal{L}}\longrightarrow Y_{T_{x_0}X,\mathcal{A}_{x_0}}\,.$$

In particular,

$$\mathrm{stab}\,x\;\cong\;\mathrm{stab}\,\widetilde{\Phi}_{x_0}(x)\,,$$

which, by our analysis of the linear setting, is an elementary abelian 2-group, provided we can see that $\mathrm{stab}\,x_0$ acts effectively on $T_{x_0}X$. To settle this remaining point, assume that there exists a group element $g\neq e$ in $\mathrm{stab}\,x_0$ that fixes all of $T_{x_0}X$. By Proposition 3.2, g

then fixes an open neighborhood of x_0 in X, which implies that g fixes all of X, contrary to our assumption of the action being effective. □

4. Permutation actions on linear and on projective spaces

One of the most natural linear actions of a finite group is the action of the symmetric group $\mathcal{S}_n$ permuting the coordinates of a real n-dimensional vector space. This action induces a differentiable action of $\mathcal{S}_n$ on $(n-1)-$dimensional real projective space $\mathbb{RP}^{n-1}$. Our goal in this section is to give explicit descriptions of the $\mathcal{L}$-stratifications and the resulting digitalizations in both cases. The *braid arrangement* $\mathcal{A}_{n-1}$ will play a central role for both stratifications; we thus recall that $\mathcal{A}_{n-1}=\{H_{ij}\colon x_j-x_i=0 \mid 1\leq i,j\leq n\}$. Its intersection lattice is the lattice Π_n of set partitions of $[n] := \{1,\dots,n\}$ ordered by reversed refinement.

We will show that, in the case of the real permutation action, the arrangement $\mathcal{A}(\mathcal{S}_n)$ coincides with the rank 2 truncation of the braid arrangement, $\mathcal{A}_{n-1}^{\mathrm{rk}\geq 2}$, i.e., the braid arrangement $\mathcal{A}_{n-1}$ without its hyperplanes. The abelianization construction proposed in the present article hence specializes to the maximal model of the braid arrangement discussed in [FK2].

For the permutation action on $\mathbb{RP}^{n-1}$, we show that the $\mathcal{L}$-stratification coincides with the rank 2 truncation of the projectivized braid arrangement, $\mathbb{P}\mathcal{A}_{n-1}^{\mathrm{rk}\geq 2}$, thus the digitalization proposed in 3.2 coincides with the maximal projective arrangement model for $\mathcal{A}_{n-1}$ (cf. [DP1, §4]).

Let us fix some notation that will come in handy when dealing with the permutation action on $\mathbb{R}^n$, and the induced action on $\mathbb{RP}^{n-1}$, respectively. Any permutation $\sigma\in\mathcal{S}_n$ determines a set partition of $[n]$, $\rho(\sigma)\vdash[n]$, by its cycle decomposition. Moreover, any set partition $\pi=(B_1|\dots|B_k)\vdash[n]$ gives rise to an intersection of hyperplanes U_π in the braid arrangement $\mathcal{A}_{n-1}$, namely

$$U_\pi := \bigcap_{r=1}^{k}\ \bigcap_{i,j\in B_r} H_{ij}\,.$$

We call U_π the *braid space* associated to π.

4.1. Digitalizing the real permutation action. As outlined above, we will now recover the rank 2 truncation of the braid arrangement as the arrangement $\mathcal{A}(\mathcal{S}_n)$ arising from the real permutation action.

Theorem 4.1. *The arrangement $\mathcal{A}(\mathcal{S}_n)$ associated with the real permutation action as described in* 3.1 *coincides with the rank* 2 *truncation of the braid arrangement. In particular, the digitalization $Y_{\mathcal{A}(\mathcal{S}_n)}$ of* Theorem 3.1 *coincides with the maximal wonderful model of the braid arrangement as discussed in* [FK2].

Proof. We first show that any proper subspace $L(H)$ in $\mathbb{R}^n$, for H a subgroup of $\mathcal{S}_n$, is a braid space of codimension at least 2:
Recall from Remark 3.4 that any subspace $L(H)$ is a fixed point set, namely, $L(H)=\mathrm{Fix}(F(H))$ where $F(H)$ is the subgroup of elements in H that fix $L(H)$ pointwise. Obviously, $\mathrm{Fix}(F(H)) = \cap_{\sigma\in F(H)}\mathrm{Fix}(\sigma)$. Therefore, it is enough to prove that

$\mathrm{Fix}(\sigma)$ is a braid space for any permutation $\sigma \in \mathcal{S}_n$. However, it is obvious that $\mathrm{Fix}(\sigma) = U_{\rho(\sigma)}$, where $\rho(\sigma)$ is the cycle decomposition of σ as defined above. Observe here that proper subspaces of type $L(H)$ are never of codimension 1; if $L(H)$ were that, then H would also leave the orthogonal line $L(H)^\perp$ invariant, in contradiction to $L(H)$ being proper.

We will now prove that all braid spaces other than hyperplanes occur in the arrangement $\mathcal{A}(\mathcal{S}_n)$. To this end, we show that we can realize braid spaces U_π, $\pi \vdash n$, with $\mathrm{type}(\pi) = (3, 1^{n-3})$ and $\mathrm{type}(\pi) = (2^2, 1^{n-4})$ as subspaces $L(H)$ for some subgroups H of $\mathcal{S}_n$. Any braid space of higher codimension will be obtained as an intersection of those.

Without loss of generality, set $\pi = 123|4|\dots|n$ to cover elements of the first type. Let H be the Young subgroup $\mathcal{S}_{\{1,2,3\}} \times \mathcal{S}_{\{4\}} \times \cdots \times \mathcal{S}_{\{n\}}$. Clearly, if a line is invariant under the action of H, then it must be fixed point-wise. This is the case if and only if the first 3 coordinates of a generating vector are equal. We conclude that $L(\mathcal{S}_{\{1,2,3\}}) = U_{123}$.

Now, set $\pi = 12|34|5|\dots|n$. Let H be the subgroup of $\mathcal{S}_n$ generated by the transpositions (12) and (34), and the involution $(13)(24)$. We see that H is isomorphic to a wreath product $\mathcal{S}_2 \wr \mathcal{S}_2$. The subspace of points fixed by H is the braid space U_{1234}. Furthermore, the line spanned by the vector $(1, 1, -1, -1, 0, \dots, 0)$ belongs to $L(H)$, it is flipped by H. It is not difficult to see that U_{1234} and the line $\langle(1, 1, -1, -1, 0, \dots, 0)\rangle$ span the entire $L(H)$; on the other hand, they span $U_{12|34}$, so we conclude that $L(\mathcal{S}_2 \wr \mathcal{S}_2) = U_{12|34}$. □

4.2. **Digitalizing the permutation action on real projective space.** We will consider the $\mathcal{L}$-stratification on $\mathbb{RP}^{n-1}$ induced by the permutation action of $\mathcal{S}_n$, and we will give a description of the digitalization proposed in 3.2.

Let us first prove a lemma that describes the fixed point set of a single permutation $\sigma \in \mathcal{S}_n$ on $\mathbb{RP}^{n-1}$. The fact that $\mathcal{L}$-strata are defined as connected components of fixed point sets, points to the significance of this lemma for the following considerations.

Lemma 4.2. *Let σ be a permutation in $\mathcal{S}_n$. The fixed point set of σ on $\mathbb{RP}^{n-1}$ decomposes as a disjoint union of submanifolds*

$$\mathrm{Fix}(\sigma \circlearrowright \mathbb{RP}^{n-1}) \;=\; \mathbb{P}U_{\rho(\sigma)} \,\cup\, \mathbb{P}V_\sigma\,, \tag{4.1}$$

where V_σ is the span of lines generated by vectors v_B, one for each even length block B in the partition $\rho(\sigma)$ associated with σ. The vector v_B has sign-alternating entries ± 1 in the coordinates contained in B in the linear order determined by the iteration of σ, and has entries 0 otherwise.

Proof. Let $L^+(\sigma)$ denote the span of lines in $\mathbb{R}^n$ that are point-wise fixed by $\sigma \in \mathcal{S}_n$ with respect to the permutation action on $\mathbb{R}^n$, and let $L^-(\sigma)$ denote the span of lines in $\mathbb{R}^n$ that are flipped by σ. Obviously,

$$\mathrm{Fix}(\sigma \circlearrowright \mathbb{RP}^{n-1}) \;=\; \mathbb{P}(L^+(\sigma) \,\cup\, L^-(\sigma))\,.$$

First we note that $L^+(\sigma) = U_{\rho(\sigma)}$. To describe $L^-(\sigma)$ we proceed as follows: In order for a line $l = \langle v\rangle$, $v \in \mathbb{R}^n$, to be flipped by the action of σ, the non-zero coordinates of v must have the same absolute value within each cycle of σ, i.e., each block of $\rho(\sigma)$, with actual entries alternating in sign in the order prescribed by iteration of σ. This implies that the coordinates of v, which are contained in the odd length blocks of $\rho(\sigma)$, must be 0. Moreover, we see that $L^-(\sigma)$ is generated by vectors v_B for the even length blocks B in $\rho(\sigma)$, with v_B having alternating entries ± 1 in B and 0 otherwise. Thus, $L^-(\sigma) = V_\sigma$ as described in the statement of the Lemma. The subspaces $L^+(\sigma)$ and $L^-(\sigma)$ are orthogonal in $\mathbb{R}^n$, hence

$$\mathrm{Fix}(\sigma \circlearrowright \mathbb{RP}^{n-1}) \;=\; \mathbb{P}(U_{\rho(\sigma)} \,\cup\, V_\sigma) \;=\; \mathbb{P}U_{\rho(\sigma)} \,\cup\, \mathbb{P}V_\sigma\,,$$

and the union is disjoint. □

Theorem 4.3. *The $\mathcal{L}$-stratification on $\mathbb{RP}^{n-1}$ induced by the permutation action of $\mathcal{S}_n$ coincides with the rank 2 truncation of the projectivized braid arrangement. In particular, the digitalization $Y_{\mathbb{RP}^{n-1},\mathcal{L}}$ coincides with the maximal projective arrangement model for $\mathbb{P}\mathcal{A}_{n-1}$.*

Proof. We first show that any projectivized braid space of codimension at least 2 occurs as an $\mathcal{L}$-stratum on $\mathbb{RP}^{n-1}$. As in the proof of Theorem 4.1, it will be enough to show that we can realize projectivizations of braid spaces $\mathbb{P}U_\pi$ with type(π)=$(3,1^{n-3})$ and type(π)=$(2^2,1^{n-4})$ as $\mathcal{L}$-strata $\mathcal{L}(\ell,H)$ for some points $\ell \in \mathbb{RP}^{n-1}$ and subgroups H of $\mathcal{S}_n$.

Again, we start by setting $\pi = 123|4|\ldots|n$ without loss of generality. We choose $\ell = \langle(1,\ldots,1)\rangle$ as the reference point in $\mathbb{RP}^{n-1}$, and $H = \langle(123)\rangle \cong \mathbb{Z}_3$ as subgroup of $\mathcal{S}_n$. Observe that, due to our special choice of ℓ, H acts on the tangent space $T_\ell\mathbb{RP}^{n-1}$ by permuting coordinates. We find that $L(\ell,\mathbb{Z}_3) = U_{123}$, and $F(\ell,\mathbb{Z}_3) = \mathbb{Z}_3$. Referring to Lemma 4.2, we see that $\mathrm{Fix}(\mathbb{Z}_3 \circlearrowright \mathbb{RP}^{n-1}) = \mathrm{Fix}((123) \circlearrowright \mathbb{RP}^{n-1})$ has only one connected component $\mathbb{P}U_{123}$. The second submanifold in (4.1) does not come into play, since there are no even length block sizes occurring in the partition $123|4|\ldots|n$. We conclude that $\mathcal{L}(\ell,H) \;=\; \mathbb{P}U_{123}$.

We now set $\pi = 12|34|5|\ldots|n$, and we choose $\ell = \langle(1,\ldots,1)\rangle$ in $\mathbb{RP}^{n-1}$ as before, and $H = \langle(1234)\rangle \cong \mathbb{Z}_4$ as subgroup of $\mathcal{S}_n$. Again, H acts on $T_\ell\mathbb{RP}^{n-1}$ by permuting coordinates and we see that $L(\ell,\mathbb{Z}_3) = U_{12|34}$. The latter space is *spanned* by the lines that are fixed by (1234), i.e., by U_{1234}, *and* the line $\langle(1,-1,1,-1,0,\ldots,0)\rangle$ which is flipped by (1234). We find that $F(\ell,\mathbb{Z}_4) = \langle(13)(24)\rangle \cong \mathbb{Z}_2$, and $\mathcal{L}(\ell,\mathbb{Z}_4)$ is the connected component of $\mathrm{Fix}((12)(34) \circlearrowright \mathbb{RP}^{n-1}) \;=\; \mathbb{P}U_{12|34} \cup \mathbb{P}V_{(12)(34)}$ that contains ℓ. The first submanifold contains our chosen point ℓ and is disjoint from the second, hence $\mathcal{L}(\ell,\mathbb{Z}_4) = \mathbb{P}(U_{12|34})$.

Summarizing up to this point, we see that the rank 2 truncation of the projectivized braid arrangement is part of the $\mathcal{L}$-stratification induced by the permutation action on real projective space. It remains to show that all $\mathcal{L}$-strata indeed are projectivized braid spaces.

To this end, let ℓ be a point in $\mathbb{RP}^{n-1}$, i.e., $\ell = \langle v\rangle$ is a line in $\mathbb{R}^n$ with generating vector v of unit length, and let H be a subgroup of $\mathrm{stab}\,\ell$.

We interpret the tangent space $T_\ell\mathbb{RP}^{n-1}$ as the linear hyperplane orthogonal to $\langle v\rangle$ in $\mathbb{R}^n$, $T_\ell\mathbb{RP}^{n-1} = \langle v\rangle^\perp := T$. Instead of the linear $\mathrm{stab}\,\ell$-action on the tangent space T induced by $\mathcal{S}_n$ acting on $\mathbb{RP}^{n-1}$, we can consider the permutation action of $\mathrm{stab}\,\ell$ on T. These two actions differ by at most a sign, which in particular implies that the construction of $L(H)$ yields the same subspaces with respect to both actions.

We moreover observe that we can consider $\operatorname{stab}\ell$ acting by permutation of coordinates on $T \oplus \langle v\rangle \cong \mathbb{R}^n$ and we obtain the space $L(H \circlearrowleft T)$ by restricting $L(H \circlearrowleft \mathbb{R}^n)$ to T. The latter space *is* a braid space of codimension at least 2 by Theorem 4.1,

$$L(H \circlearrowleft T) \oplus \langle v\rangle \;=\; U_\pi\,,$$

for some partition $\pi \vdash n$, $\operatorname{rk}\pi \geq 2$.

Translating T along ℓ into $T+v$, we can give an explicit description of the Bochner map Φ_ℓ (cf. Proposition 3.2) that maps a neighborhood U of ℓ in $\mathbb{RP}^{n-1}$ differentiably and $\operatorname{stab}\ell$-equivariantly to the tangent space $T_\ell\mathbb{RP}^{n-1}$,

$$\begin{array}{rccc} \Phi_\ell\,: & U & \longrightarrow & T_\ell\mathbb{RP}^{n-1} \\ & u & \longmapsto & u \cap (\ell^\perp + v)\,. \end{array}$$

We see that $L(H)+v$ in $T_\ell\mathbb{RP}^{n-1}$ has the projectivization $\mathbb{P}U_\pi$ as its inverse image under Φ_ℓ. By Proposition 3.3 we conclude that $\mathcal{L}(\ell, H)$ indeed is the projectivization of a braid space, which completes our proof. □

Example 4.4. To illustrate our theorem on the $\mathcal{L}$-stratification of real projective space induced by the permutation action and the resulting digitalization, we look at $\mathcal{S}_3$ acting on $\mathbb{RP}^2$ in some detail.

We depict $\mathbb{RP}^2$ using the upper hemisphere model in Figure 1, where we place $\mathbb{P}\Delta^\perp$, for $\Delta = \langle(1,1,1)\rangle$, on the equator.

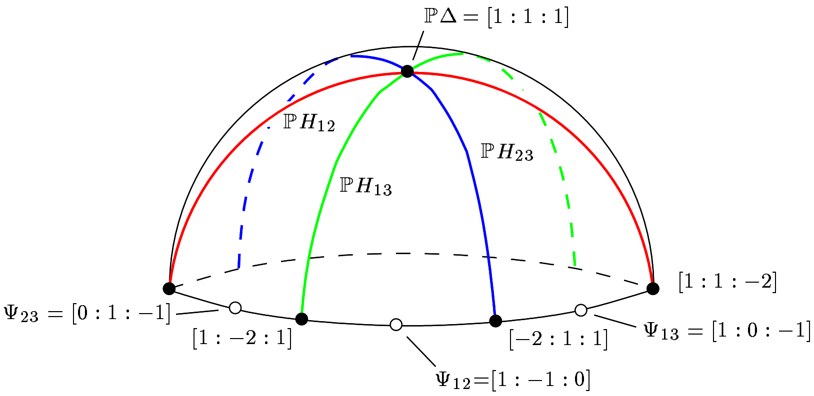

FIGURE 1. $\mathbb{RP}^2$ stratified by loci of non-trivial stabilizers.

The locus of points in $\mathbb{RP}^2$ with non-trivial stabilizer groups consists of the three lines $\mathbb{P}H_{ij}$, $1 \leq i,j \leq 3$, which are projectivizations of the hyperplanes in $\mathcal{A}_2$, intersecting in $\mathbb{P}\Delta = [1{:}1{:}1]$, and points Ψ_{ij} on $\mathbb{P}\Delta^\perp$, where Ψ_{ij} is the line orthogonal to H_{ij} in $\mathbb{R}^3$ for $1 \leq i,j \leq 3$.

Observe that the transposition $(i,j) \in \mathcal{S}_3$ acts on $\mathbb{RP}^2$ as a central symmetry in $\Psi_{i,j}$, respectively, as a reflection in $\mathbb{P}H_{i,j}$, which we illustrate in Figure 2.

We find that the arrangements $\mathcal{A}_\ell(\operatorname{stab}\ell \circlearrowleft T_\ell\mathbb{RP}^2)$ associated with the induced linear actions of the stabilizers on tangent spaces for $\ell \in \mathbb{RP}^2$ are empty unless $\ell = [1{:}1{:}1]$. In this case, we see that $\mathcal{S}_3 \circlearrowleft T_{[1:1:1]}\mathbb{RP}^2$ coincides with the standard action of $\mathcal{S}_3$ on $\mathbb{R}^3/\Delta$,

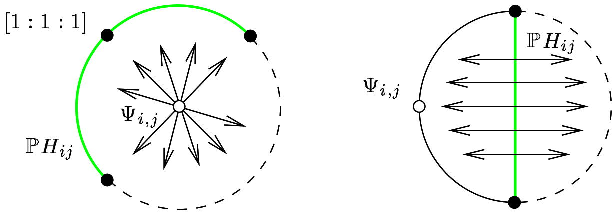

FIGURE 2. $(i,j) \in \mathcal{S}_3$ acting on $\mathbb{RP}^2$.

since transpositions, as we observed above, act as reflections in the hyperplanes of the projectivized braid arrangement. Thus, $\mathcal{A}_{[1:1:1]}(\mathcal{S}_3 \circlearrowleft T_{[1:1:1]}\mathbb{RP}^2)$ coincides with the rank 2 truncation of the braid arrangement consisting of the origin in the tangent space.

We conclude that the $\mathcal{L}$-stratification is given by the single point [1:1:1] in $\mathbb{RP}^2$, hence the digitalization we propose is the blowup of $\mathbb{RP}^2$ in this point,

$$Y_{\mathbb{RP}^2,\mathcal{L}} \;=\; \mathrm{Bl}_{[1:1:1]}(\mathbb{RP}^2)\,.$$

Topologically, this means to glue a Möbius band into a pointed $\mathbb{RP}^2$, equivalently, to glue two Möbius bands along their boundaries, resulting in a Klein bottle (cf. Figure 3).

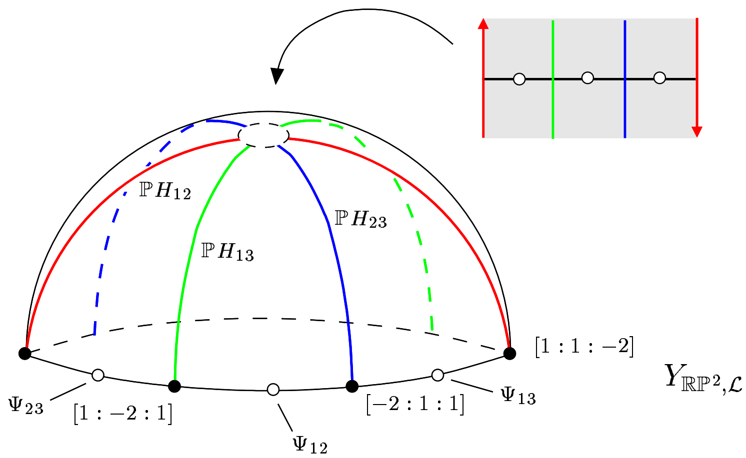

FIGURE 3. The wonderful model $Y_{\mathbb{RP}^2,\mathcal{L}}$.

Remark 4.5. As already the low-dimensional Example 4.4 shows, the $\mathcal{L}$-stratification associated with the permutation action of $\mathcal{S}_n$ on $\mathbb{RP}^{n-1}$ is different from the codimension 2 truncation of the stabilizer stratification.

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