Combinatorial Stratifications

Instances of Impact in Geometry and Topology

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Jedes Suchen hat sein vorgängiges Geleit
aus dem Gesuchten her.
Fragen ist erkennendes Suchen.

M. Heidegger, Sein und Zeit, §*2*

PROLOGUE

Central objects in geometry and topology often come equipped with intrinsic combinatorial structure, for instance a natural combinatorial stratification. Implications of distinguishing such combinatorial core data are manifold. At times it proves useful for determining invariants of the spaces involved; alternatively, it might serve as a starting point for linking geometric situations in seemingly distant contexts.

One of the most beautiful and most characteristic examples of intrinsic combinatorial data determining invariants of a space is the topology of arrangements of hyperplanes. The complements of complex hyperplane arrangements, for instance, have an intricate topology. For describing their cohomology algebras, however, knowing the lattices of intersections of hyperplanes is fully sufficient. A presentation for the integral cohomology in terms of generators and relations, the famous *Orlik-Solomon algebra*, can be read from there.

Discrete structures that emerge from geometry are often of interest to combinatorialists in their own right. They call for abstraction that, if considered with open eyes and mind, bears the potential to disclose unexpected and far-reaching connections.

The five chapters of the present thesis are intertwined by a common geometric object and by a common theme. *De Concini-Procesi wonderful arrangement models* are ubiquitous (compare [DP3] as a main reference); we analyze their combinatorial core data and present results in accordance with the general spirit outlined above.

Combinatorial Resolutions: In the first chapter (joint work with Dmitry Kozlov [FK1]) we provide a combinatorial framework that is designed to describe the incidence change in stratifications throughout a resolution process. Though inspired by the construction of De Concini-Procesi models, our setting of combinatorial resolutions, including notions of building sets, nested sets and combinatorial blowups, is purely order theoretic.

Returning to the source of our notions, we find that they in fact describe the incidence combinatorics of stratifications through every step of the De Concini-Procesi model construction. They also serve in other contexts, e.g., resolutions of toric varieties. Since a preprint version of this chapter has circulated, our combinatorial framework for resolutions has been taken up in the context of model constructions for real subspace and halfspace arrangements and for real stratified manifolds [Ga2]. Moreover, it provided the outset for my joint work with Sergey Yuzvinsky, which appears as Chapter 2 in this thesis.

An algebra defined by atomic lattices: In the second chapter (joint work with Sergey Yuzvinsky [FY]) we study a graded algebra $D = D(\mathcal{L}, \mathcal{G})$ over $\mathbb{Z}$ defined by a finite lattice $\mathcal{L}$ and a subset $\mathcal{G}$ in $\mathcal{L}$, where $\mathcal{G}$ is a combinatorial building set as it appears in the first chapter of this thesis. For intersection lattices of hyperplane arrangements this algebra specializes to the cohomology algebra of hyperplane arrangement compactifications of De Concini and Procesi [DP4].

Our main result is a representation of D, for an arbitrary atomic lattice $\mathcal{L}$, as the Chow ring of a smooth toric variety that we construct from $\mathcal{L}$ and $\mathcal{G}$. Stepping aside from the original geometric context of model constructions, but holding on to the combinatorial structure at the core of the matter, we find a new and seemingly unrelated geometric interpretation of the graded algebra D. We describe the toric variety both by its fan and geometrically by a series of blowups and orbit removal. Also we find a Gröbner basis of the relation ideal of D and a monomial basis of D thereby generalizing earlier results by Yuzvinsky [Y] on a linear basis for arrangement model cohomology.

Nested set complexes: In the third chapter (joint work with my student Irene Müller [FM]) we look at the abstract simplicial complex of nested sets from the point of view of topological combinatorics. We show that nested set complexes are homotopy equivalent to the order complexes of the underlying meet-semilattices without their minimal elements. For atomic semilattices, we consider the realization of nested set complexes by simplicial fans proposed in the second chapter of this thesis, and we strengthen our previous result showing that in this case nested set complexes in fact are homeomorphic to the mentioned order complexes.

Desingularizing finite group actions: In the two remaining chapters (both joint work with Dmitry Kozlov [FK2, FK3]) we study abelianizations of real linear, respectively diffeomorphic, actions of finite groups.

The fourth chapter is concerned with one of the most natural real linear actions, the permutation action of the symmetric group $\mathcal{S}_n$ on $\mathbb{R}^n$. We study its abelianization provided by the maximal De Concini-Procesi wonderful model for the braid arrangement. We show that stabilizers of points in the arrangement model are not merely abelian but, in fact, elementary abelian 2-groups. To prove that, we develop a combinatorial framework for explicitly describing the stabilizers in terms of automorphism groups of set diagrams over families of cubes. We observe that the natural nested set stratification on the arrangement model is not stabilizer distinguishing with respect to the $\mathcal{S}_n$-action, i.e., stabilizers of points are not in general isomorphic on open strata. Motivated by this structural deficiency, we furnish a new stratification of the De Concini-Procesi arrangement model that distinguishes stabilizers.

In the fifth and last chapter of this thesis we provide abelianizations of diffeomorphic actions of finite groups on smooth real manifolds. De Concini-Procesi wonderful models for *local* subspace arrangements and a careful analysis of linear actions on real vector spaces are at the core of our construction. Again, we can show that stabilizers are elementary abelian 2-groups, a setting for which we suggest the term *digitalization*. As our main examples, we discuss the resulting digitalizations of the permutation actions of the symmetric group on $\mathbb{R}^n$, and on real projective space.

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My perception and my acquaintance with mathematical research was formed at the various locations that I was fortunate enough to visit or to work at. Among them foremost the Mathematics Department at M.I.T., the School of Mathematics at the Institute for Advanced Study in Princeton, and during the last four years the Mathematics Department and the Forschungsinstitut für Mathematik at ETH Zürich. Yet, it is people who create the atmosphere, the spirit, and the challenge at all these places – I am grateful for meeting many wonderful mathematicians, for learning from them, and for getting their help and advice.

In particular, I want to mention the longstanding and successful collaboration with Dmitry Kozlov and with Sergey Yuzvinsky. It is my pleasure to thank them and my student, Irene Müller, for their permission to include our joint works into this thesis.

CONTENTS

Chapter 1

Incidence Combinatorics of Resolutions

1.1 Introduction

For an arbitrary meet-semilattice we introduce notions of *combinatorial blowups*, *building sets*, and *nested sets*. The definitions are given on a purely order-theoretic level without any reference to geometry. This provides a common abstract framework for the incidence combinatorics occurring in at least two different situations in algebraic geometry: the construction of De Concini-Procesi models of subspace arrangements [DP1], and the resolution of singularities in toric varieties.

The various parts of this abstract framework have received different emphasis within different situations: while the notion of combinatorial blowups clearly specializes to stellar subdivisions of defining fans in the context of toric varieties, building sets and nested sets were introduced in the context of model constructions by De Concini & Procesi [DP3] (earlier and in a more special setting by Fulton & MacPherson [FuM]), from where we adopt our terminology. This correspondence however is not complete: the building sets in [DP3, FuM] are not canonical, they depend on the geometry, while ours do not. See Section 1.4.1 for further details.

It was proved in [DP3] that a sequence of blowups within an arrangement of complex linear subspaces leads from the intersection stratification of complex space given by the maximal subspaces of the arrangement to an arrangement model stratified by divisors with normal crossings. In the context of toric varieties, there exist many different procedures for stellar subdivisions of a defining fan that result in a simplicial fan, so-called simplicial resolutions.

The purpose of our Main Theorem 1.3.4 is to unify these two situations on the combinatorial level: a sequence of combinatorial blowups, performed on a (combinatorial) building set in linear extension compatible order, transforms the initial semilattice to a semilattice where all intervals are boolean algebras, more precisely to the face poset of the corresponding simplicial complex of nested sets. In particular, the structure of the resulting semilattice can be fully described by the initial data of nested sets. Both the formulation and the proof of our main theorem are purely combinatorial.

We sketch the content of this chapter: In Section 1.2 we define building sets and nested sets for meet-semilattices in purely order-theoretic terms and develop general structure theory for these notions. We present the definition of combinatorial blowups for meet-semilattices in Section 1.3 and study their effect on building sets and nested sets. The section contains our Main Theorem 1.3.4 which describes the result of blowing up the elements of a building set in terms of the initial nested set complex.

Section 1.4 is devoted to relating our abstract framework to two different contexts in algebraic geometry: In 1.4.1 we briefly review the construction of De Concini-Procesi models for subspace arrangements. We show that the change of the incidence combinatorics of the stratification in a single construction step is described by a combinatorial blowup of the semilattice of strata. In 1.4.2 we draw the connection to simplicial resolutions of toric varieties: we recognize stellar subdivisions as combinatorial blowups of the face posets of defining fans and discuss the notions of building and nested sets in this context.

In Section 1.5 we report on some developments that took place after a first preprint version of this chapter has been written and has circulated in the fall of 2000. Our combinatorial framework for the incidence combinatorics of resolutions has been taken up in various contexts. We outline the model construction for real subspace and halfspace arrangements and for real stratified manifolds by G. Gaiffi [Ga2]. Moreover, we give a short account of the study of a graded algebra associated with any finite lattice in [FY], which appears as Chapter 2 of this thesis. In this work, our combinatorial generalization of originally geometric notions leads to the construction of an, at first sight, unrelated geometric counterpart for wonderful models of hyperplane arrangements.

1.2 BUILDING SETS AND NESTED SETS OF MEET-SEMILATTICES

1.2.1 Poset terminology

We recall some notions from the theory of partially ordered sets, and refer to [St, Ch. 3] for further details.

All posets discussed in this chapter will be finite. A poset $\mathcal{L}$ is called a *meet-semilattice* if any two elements $x, y \in \mathcal{L}$ have a greatest lower bound, i.e., the set $\{z \in \mathcal{L} \mid z \leq x, z \leq y\}$ has a maximal element, called the *meet*, $x \wedge y$, of x and y. Greatest lower bounds of subsets $A = \{a_1, \ldots, a_t\}$ in $\mathcal{L}$ we denote with $\bigwedge A = a_1 \wedge \ldots \wedge a_t$. In particular, meet-semilattices have a unique minimal element denoted $\hat{0}$. Minimal elements in $\mathcal{L} \setminus \{\hat{0}\}$ are called the *atoms* in $\mathcal{L}$. Meet-semilattices share the following property: for any subset $A = \{a_1, \ldots, a_t\} \subseteq \mathcal{L}$ the set $\{x \in \mathcal{L} \mid x \geq a$ for all $a \in A\}$ is either empty or it has a unique minimal element, called the *join*, $\bigvee A = a_1 \vee \ldots \vee a_t$, of A. If the meet-semilattice needs to be specified, we write $(\bigvee A)_{\mathcal{L}} = (a_1 \vee \ldots \vee a_t)_{\mathcal{L}}$ for the join of A in $\mathcal{L}$.

For brevity, we talk about semilattices throughout the chapter, meaning meet-semilattices.

Let P be an arbitrary poset. For $x \in P$ set: $P_{\leq x} = \{y \in P \mid y \leq x\}$; $P_{<x}$, and $P_{\geq x}$, $P_{>x}$ are defined analogously. For subsets $\mathcal{G} \subseteq P$ with the induced order, we define $\mathcal{G}_{\leq x} = \{y \in P \mid y \in \mathcal{G}, y \leq x\}$, and $\mathcal{G}_{<x}$ again analogously. For intervals in P we use the standard notations $[x, y] := \{z \in P \mid x \leq z \leq y\}$, $[x, y) := \{z \in P \mid x \leq z < y\}$, etc.

A poset is called *irreducible* if it is not a direct product of two other posets, both consisting of at least two elements. For a poset P with a unique minimal element $\hat{0}$, we call $I(P) = \{x \in P \mid [\hat{0}, x] \text{ is irreducible}\}$ the *set of irreducible elements* in P. In particular, the minimal element $\hat{0}$ and all atoms of P are irreducible elements in P. For $x \in P$, we call $D(x) := \max(I(P)_{\leq x})$ the *set of elementary divisors* of x – a term which is explained by the following proposition:

Proposition 1.2.1 *Let P be a poset with a unique minimal element $\hat{0}$. For $x \in P$ there exists a unique finest decomposition of the interval $[\hat{0}, x]$ in P as a direct product, which is given by an isomorphism $\varphi_x^{\mathrm{el}} : \prod_{j=1}^{l} [\hat{0}, y_j] \xrightarrow{\cong} [\hat{0}, x]$, with $\varphi_x^{\mathrm{el}}(\hat{0}, \ldots, y_j, \ldots, \hat{0}) = y_j$ for $j = 1, \ldots, l$. The factors of this decomposition are the intervals below the elementary divisors of x: $\{y_1, \ldots, y_l\} = D(x)$.*

Proof. Whenever a poset with a minimal element $\hat{0}$ is represented as a direct product, all elements which have more than one coordinate different from $\hat{0}$ are reducible. Hence, if $\prod_{j=1}^{l}[\hat{0}, y_j] \cong [\hat{0}, x]$, and the y_j are irreducible for $j = 1, \ldots, l$, then $\{y_1, \ldots, y_l\} = D(x)$. □

1.2.2 Building sets

In this subsection we define the notion of building sets of a semilattice and develop their structure theory.

Definition 1.2.2 Let $\mathcal{L}$ be a semilattice. A subset $\mathcal{G}$ in $\mathcal{L} \setminus \{\hat{0}\}$ is called a *building set* of $\mathcal{L}$ if for any $x \in \mathcal{L} \setminus \{\hat{0}\}$ and $\max \mathcal{G}_{\leq x} = \{x_1, \ldots, x_k\}$ there is an isomorphism of posets

$$\varphi_x : \prod_{j=1}^{k} [\hat{0}, x_j] \xrightarrow{\cong} [\hat{0}, x] \tag{1.2.1}$$

with $\varphi_x(\hat{0}, \ldots, x_j, \ldots, \hat{0}) = x_j$ for $j = 1, \ldots, k$. We call $F(x) := \max \mathcal{G}_{\leq x}$ the *set of factors* of x in $\mathcal{G}$.

The next proposition provides several equivalent conditions for a subset of $\mathcal{L} \setminus \{\hat{0}\}$ to be a building set.

Proposition 1.2.3 *For a semilattice $\mathcal{L}$ and a subset $\mathcal{G}$ of $\mathcal{L}\setminus\{\hat{0}\}$ the following are equivalent:*

(1) $\mathcal{G}$ *is a building set of* $\mathcal{L}$*;*

(2) $\mathcal{G}\supseteq I(\mathcal{L})\setminus\{\hat{0}\}$*, and for every* $x\in\mathcal{L}\setminus\{\hat{0}\}$ *with* $D(x)=\{y_1,\ldots,y_l\}$ *the elementary divisors of* x*, there exists a partition* $\pi_x=\pi_1|\ldots|\pi_k$ *of* $[l]$ *with blocks* $\pi_t=\{i_1,\ldots,i_{|\pi_t|}\}$ *for* $t=1,\ldots,k$*, such that the elements in* $\max\mathcal{G}_{\leq x}=\{x_1,\ldots,x_k\}$ *are of the form* $x_t = \varphi_x^{\mathrm{el}}(\hat{0},\ldots,\hat{0},y_{i_1},\hat{0},\ldots,\hat{0}, y_{i_2},\hat{0},\ldots,\hat{0},y_{i_{|\pi_t|}},\hat{0})$.

Informally speaking, the factors of x *in* $\mathcal{G}$ *are products of disjoint sets of elementary divisors of* x*.*

(3) $\mathcal{G}$ *generates* $\mathcal{L}\setminus\{\hat{0}\}$ *by* $\vee$*, and for any* $x\in\mathcal{L}$*, any* $\{y,y_1,\ldots,y_t\}\subseteq\max\mathcal{G}_{\leq x}$*, and* $z\in\mathcal{L}$ *with* $z<y$*, we have* $\mathcal{G}_{\leq y}\cap\mathcal{G}_{\leq z\vee y_1\vee\ldots\vee y_t}=\mathcal{G}_{\leq z}$.

(4) $\mathcal{G}$ *generates* $\mathcal{L}\setminus\{\hat{0}\}$ *by* $\vee$*, and for any* $x\in\mathcal{L}$*, any* $\{y,y_1,\ldots,y_t\}\subseteq\max\mathcal{G}_{\leq x}$*, and* $z\in\mathcal{L}$ *with* $z<y$*, the following two conditions are satisfied:*

$i)$	$\mathcal{G}_{\leq y}\cap\mathcal{G}_{\leq y_1\vee\ldots\vee y_t}=\emptyset$	*"disjointness,"*
$ii)$	$z\vee y_1\vee\ldots\vee y_t<y\vee y_1\vee\ldots\vee y_t$	*"necessity."*

Proof. (1)⇒(2): That $\mathcal{G}$ contains $I(\mathcal{L})\setminus\{\hat{0}\}$ follows directly from the definition of building sets. We have the following isomorphisms: $\varphi_x:\prod_{j=1}^k[\hat{0},x_j]\longrightarrow[\hat{0},x]$ by the building set property, and $\varphi_{x_j}^{\mathrm{el}}:\prod_{y\in D(x_j)}[\hat{0},y]\longrightarrow[\hat{0},x_j]$ for $j=1,\ldots,k$ by Proposition 1.2.1. The composition $\varphi_x\circ(\prod_{j=1}^k\varphi_{x_j}^{\mathrm{el}})$ yields the finest decomposition φ_x^{el} of $[\hat{0},x]$. Thus, $D(x)=\biguplus_{j=1}^k D(x_j)$, which gives the partition described in (2).

(2)⇒(1): The decomposition of $[\hat{0},x]$ into intervals below the elements in $\max\mathcal{G}_{\leq x}$ follows from Proposition 1.2.1 by assembling factors $[\hat{0},y_j]$ with maximal elements indexed by elements from the same block of the partition π_x into one factor.

(1)⇒(3): (3) is a direct consequence of $[\hat{0},x]$ decomposing into a direct product of the form described in the definition of building sets.

(3)⇒(4): $i)$ follows by setting $z=\hat{0}$ in (3). Equality in $ii)$ implies with (3) that $\mathcal{G}_{\leq y}=\mathcal{G}_{\leq z}$, in particular, $y\in\mathcal{G}_{\leq z}$ – a contradiction to $z<y$.

(4)⇒(1): For $x\in\mathcal{L}\setminus\{\hat{0}\}$ and $\max\mathcal{G}_{\leq x}=\{x_1,\ldots,x_k\}$ consider the poset map

$$\phi:\ \prod_{j=1}^k[\hat{0},x_j]\ \longrightarrow\ [\hat{0},x]\,,\quad (\alpha_1,\ldots,\alpha_k)\ \longmapsto\ \alpha_1\vee\ldots\vee\alpha_k\,.$$

i) ϕ *is surjective*: For $\hat{0} \neq y \leq x$, let $\max \mathcal{G}_{\leq y} = \{y_1, \ldots, y_t\}$. First, observe that $\bigvee_{i=1}^{t} y_i = y$, since $\mathcal{G}$ generates $\mathcal{L}$ by $\vee$. Second, define $\gamma_j := \bigvee_{y_i \in S_j} y_i$ with $S_j := (\max \mathcal{G}_{\leq y}) \cap \mathcal{G}_{\leq x_j}$ for $j = 1, \ldots, k$. Clearly, $\gamma_j \in [\hat{0}, x_j]$, and $\cup_{j=1}^{k} S_j = \max \mathcal{G}_{\leq y}$, since $\mathcal{G}_{\leq y} \subseteq \mathcal{G}_{\leq x}$. Hence, $\phi(\gamma_1, \ldots, \gamma_k) = \bigvee_{i=1}^{t} y_i = y$.
ii) ϕ *is injective*: a) Assume $\phi(\alpha_1, \ldots, \alpha_k) = \phi(\beta_1, \ldots, \beta_k) = y \neq x$. Let $\max \mathcal{G}_{\leq y} = \{y_1, \ldots, y_t\}$. By induction on the number of elements in $[\hat{0}, x]$ we can assume that $[\hat{0}, y]$ decomposes as a direct product $[\hat{0}, y] \cong \prod_{i=1}^{t} [\hat{0}, y_i]$. Moreover, the subsets S_j of $\max \mathcal{G}_{\leq y}$ defined in i) actually partition $\max \mathcal{G}_{\leq y}$ as follows from the disjointness property applied to pairwise intersections of the $\mathcal{G}_{\leq x_j}$. Thus, $[\hat{0}, y] \cong \prod_{j=1}^{k} [\hat{0}, \gamma_j]$, with elements $\gamma_j \in [\hat{0}, x_j]$ as above, and it follows that $\alpha_j = \beta_j = \gamma_j$ for $j = 1, \ldots, k$.
b) Assume that $\phi(\alpha_1, \ldots, \alpha_k) = \phi(\beta_1, \ldots, \beta_k) = x$. By the necessity property it follows that $\alpha_j = \beta_j = x_j$ for $j = 1, \ldots, k$. □

Remark 1.2.4 The definition of building sets and of irreducible elements, as well as the characterization of building sets in Proposition 1.2.3 (2), are independent of the existence of a join operation and can be formulated for any poset with a unique minimal element.

We gather a few important properties of building sets.

Proposition 1.2.5 *For a building set $\mathcal{G}$ of $\mathcal{L}$, the following holds:*

(1) Let $x \in \mathcal{L}$, $F(x) = \{x_1, \ldots, x_k\}$ the set of factors of x in $\mathcal{G}$, and $\hat{0} \neq y \in \mathcal{G}$ with $y \leq x$. Then there exists a unique $j \in \{1, \ldots, k\}$ such that $y \leq x_j$; i.e., $F(x) = \max \mathcal{G}_{\leq x}$ induces a partition of $\mathcal{G}_{\leq x}$.

(2) For $x \in \mathcal{L}$ and $x_0 \in F(x)$,

$$\bigvee (F(x) \setminus \{x_0\}) \; < \; \bigvee F(x) \; = x \, ,$$

i.e., each factor of x in $\mathcal{G}$ is needed to generate x.

(3) If $h_1, \ldots, h_k$ in $\mathcal{G}$ are such that $(h_i, \bigvee_{j=1}^{k} h_j] \cap \mathcal{G} = \emptyset$ for $i = 1, \ldots, k$, then $F(\bigvee_{j=1}^{k} h_j) = \{h_1, \ldots, h_k\}$.

Proof. (1) is a consequence of Proposition 1.2.3 (4)*i*), as was noted already in the proof of (4)⇒(1), part ii) a), in the previous proposition. Taking the full set of factors and setting $z = \hat{0}$ in Proposition 1.2.3 (4)*ii*), yields (2). For (3) note that $\{h_1, \ldots, h_k\} \subseteq F(\bigvee_{j=1}^{k} h_j)$ by assumption. If $\{h_1, \ldots, h_k\}$ were not the complete set of factors, we would obtain a contradiction to (2). □

Example 1.2.6 (1) For the boolean lattice $\mathcal{B}_n$ of rank n, its atoms form the minimal building set. As with any other semilattice, the full poset without its minimal element gives the maximal building set.

In the smallest interesting example, the rank 3 boolean lattice $\mathcal{B}_3$, we see that there are other building sets between these extremal choices: The atoms can be combined with any other rank 2 element to form a building set. Moreover, atoms can be combined with the top element to form a building set, and any other subset of $\mathcal{B}_3$ containing the latter is in fact a building set.

(2) For the partition lattice Π_n, the minimal building set is given by the 1-block partitions. Again, the maximal building set is given by the full lattice without its minimal element. Looking at Π_4, we see that we can add any 2-block partition to the minimal building set, e.g., (12)(34), to obtain building sets other than the extreme ones.

(3) The lattice D_n of positive integral divisors of a natural number $n > 0$ ordered by division relation has the prime powers dividing n as its minimal building set. Note that this example includes the boolean lattice for any n having no square divisors, hence there are ample building sets between the extreme choices.

1.2.3 Nested sets

In this subsection we define the notion of nested subsets of a building set of a semilattice and prove some of their properties.

Definition 1.2.7 Let $\mathcal{L}$ be a semilattice and $\mathcal{G}$ a building set of $\mathcal{L}$. A subset N in $\mathcal{G}$ is called *nested* if, for any set of incomparable elements $x_1, \ldots, x_t$ in N of cardinality at least two, the join $x_1 \vee \ldots \vee x_t$ exists and does not belong to $\mathcal{G}$. The nested sets in $\mathcal{G}$ form an abstract simplicial complex, denoted $\mathcal{N}(\mathcal{G})$.

Note that the elements of $\mathcal{G}$ are the vertices of the complex of nested sets $\mathcal{N}(\mathcal{G})$. Moreover, the order complex of $\mathcal{G}$ is a subcomplex of $\mathcal{N}(\mathcal{G})$, since linearly ordered subsets of $\mathcal{G}$ are nested.

Proposition 1.2.8 *For a given semilattice $\mathcal{L}$ and a subset N of a building set $\mathcal{G}$ of $\mathcal{L}$, the following are equivalent:*

(1) N is nested.

(2) Whenever $x_1, \ldots, x_t$ are noncomparable elements in N, the join $x_1 \vee \ldots \vee x_t$ exists, and $F(x_1 \vee \ldots \vee x_t) = \{x_1, \ldots, x_t\}$.

(3) There exists a chain $C \subseteq \mathcal{L}$, such that $N = \bigcup_{x \in C} F(x)$.

(4) *$N \in \Lambda$, where Λ is the maximal subset of $2^{\mathcal{G}}$, for which the following three conditions are satisfied:*

(o) *$\emptyset \in \Lambda$, and $\{g\} \in \Lambda$, for $g \in \mathcal{G}$;*

(i) *if $N \in \Lambda$ and $x \in \max N$, then $N_{<x} \in \Lambda$;*

(ii) *if $N \in \Lambda$, then $\max N = F(\bigvee \max N)$.*

Proof.
(1)⇒(2): Let N be a nested set, and $M = \{x_1, \dots, x_t\} \subseteq N$ a set of incomparable elements with $\bigvee_{i=1}^t x_i \notin \mathcal{G}$. We can assume that for some x_j: $(x_j, \bigvee_{i=1}^t x_i] \cap \mathcal{G} \neq \emptyset$, otherwise the claim follows by Proposition 1.2.5 (3). Moreover, we can assume that there exists an element $y \in (x_1, \bigvee_{i=1}^t x_i] \cap \mathcal{G}$ and that $y \in \max \mathcal{G}_{\leq \bigvee M}$. Define $M' := \{x_1, \dots, x_t\} \cap \mathcal{G}_{\leq y} = \{x_1 = x_{j_0}, x_{j_1}, \dots, x_{j_k}\}$ and $z := \bigvee_{l=0}^k x_{j_l}$. Since $M' = \{x_{j_0}, x_{j_1}, \dots, x_{j_k}\}$ is nested (it is a subset of N), we have the strict inequality $z < y$. Furthermore,

$$\bigvee_{i=1}^{t} x_i \;=\; z \vee \bigvee (M \setminus M') \;\leq\; z \vee \bigvee (\max G_{\leq \bigvee M} \setminus \{y\}) \;<\; \bigvee_{i=1}^{t} x_i \,,$$

where the first inequality follows from Proposition 1.2.5 (1) and the second inequality from Proposition 1.2.5 (2). We thus arrive to a contradiction, which finishes the proof.
(2)⇒(1): Obvious.
(2)⇒(3): Let N be a set satisfying condition (2). Fix a particular linear extension $\{x_1, \dots, x_k\}$ on the partial order of N, and define $\alpha_j := x_1 \vee \dots \vee x_j$, for $j = 1, \dots, k$. By (2) we have $F(\alpha_j) = \max\{x_1, \dots, x_j\}$, and therefore $x_j \in F(\alpha_j)$ and $x_{j+1} \notin F(\alpha_j)$ for $j = 1, \dots, k$. Hence, the α_j's are different and form a chain $C = \alpha_1 < \alpha_2 < \dots < \alpha_k$. By construction, $N = \bigcup_{x \in C} F(x)$.
(1),(2)⇒(4): Let N be a nested set, we shall prove that $N \in \Lambda$ by induction on the size of N:

1. if $|N| = 0$, then $N \in \Lambda$ by condition (o);

2. if $|N| \geq 1$, then $\max N = F(\bigvee \max N)$ by condition (2). Furthermore, since $|N_{<x}| < |N|$, and $N_{<x}$ is nested (it is a subset of N), $N_{<x} \in \Lambda$ by induction. Hence $N \in \Lambda$.

(3)⇒(1): Let $C = (\alpha_1 < \dots < \alpha_k)$ be a chain in $\mathcal{L}$ and $N = \bigcup_{x \in C} F(x)$. Let $N' = \{x_1, \dots, x_t\} \subseteq N$, $t \geq 2$, be an antichain in N, and s the maximal index in C such that $N' \cap F(\alpha_s) \neq \emptyset$. In particular, $N' \cap F(\alpha_s) \neq \{\alpha_s\}$ due to $|N'| > 1$ and N' being an antichain.

Let $y \in N' \cap F(\alpha_s)$. If $|N' \cap F(\alpha_s)| > 1$,

$$y < \bigvee (N' \cap F(\alpha_s)) \leq \bigvee N' \leq \alpha_s ,$$

where the strict inequality is a consequence of the necessity property for building sets. Thus, $\bigvee N' \notin \mathcal{G}$. If $|N' \cap F(\alpha_s)| = 1$, we have $y < \bigvee N' \leq \alpha_s$, due to N' being an antichain with $|N'| > 1$, and again $\bigvee N' \notin \mathcal{G}$.
(4)⇒(3): We need the following fact:

Fact. *If there are elements $x_1, \dots, x_t$ and $y_1, \dots, y_k$ in $\mathcal{L}$, such that $x_t > y_j$ for $j = 1, \dots, k$, and $F(\bigvee_{i=1}^t x_i) = \{x_1, \dots, x_t\}$, and $F(\bigvee_{j=1}^k y_j) = \{y_1, \dots, y_k\}$, then $F(x_1 \vee \dots \vee x_{t-1} \vee y_1 \vee \dots \vee y_k) = \{x_1, \dots, x_{t-1}, y_1, \dots, y_k\}$.*

Once the fact above is proved, one can derive (3) as follows: For any $N \in \Lambda$ we shall form a chain $C = (\alpha_1 < \dots < \alpha_{|N|})$ such that $N = \bigcup_{i=1}^{|N|} F(\alpha_i)$. Choose a linear extension $\{x_1, \dots, x_t\}$ of N. Set $\alpha_t = \bigvee \max N$, moreover, $\alpha_{t-1} = \bigvee \max(N \setminus \{x_t\})$, $\alpha_{t-2} = \bigvee \max(N \setminus \{x_t, x_{t-1}\})$, and so on. By (4)(ii), $F(\alpha_t) = \max N$. Applying (4)(i) to $x_t \in \max N$, and (4)(ii) to $N_{<x_t}$, we obtain $F(\bigvee \max N_{<x_t}) = \max N_{<x_t}$. With the fact above, we conclude that $F(\alpha_{t-1}) = \max(N \setminus \{x_t\})$, and, using the same argument iteratively, we arrive to $N = \bigcup_{i=1}^t F(\alpha_i)$.

Proof of the fact. Set $\alpha := x_1 \vee \dots \vee x_{t-1} \vee y_1 \vee \dots \vee y_k$. Since $\alpha \leq \bigvee_{i=1}^t x_i$, the factors of α can be partitioned into groups of elements below the x_i for $i = 1, \dots, t$, by Proposition 1.2.5 (1). Since $x_i \leq \alpha$ for $i = 1, \dots, t-1$, we obtain $F(\alpha) = \{x_1, \dots, x_{t-1}, \gamma_1, \dots, \gamma_m\}$ with $\gamma_j \leq x_t$ for $j = 1, \dots, m$.

Again using Proposition 1.2.5 (1), the $y_1, \dots, y_k$ can be partitioned into groups below the factors γ_j for $j = 1, \dots, m$. The occurrence of one strict inequality $\bigvee \{y_l \mid y_l \leq \gamma_j\} < \gamma_j$ for some $j \in \{1, \dots, m\}$ yields a contradiction to $\alpha = \bigvee_{i=1}^{t-1} x_i \vee \bigvee_{j=1}^k y_j = \bigvee_{i=1}^{t-1} x_i \vee \bigvee_{j=1}^m \gamma_j$, due to the necessity property of building sets. Moreover, since the y_i are factors themselves, joins of more than two of the y_i's are not elements of $\mathcal{G}$. Thus, $y_i = \gamma_i$, for $i = 1, \dots, k{=}m$, as claimed. □

Example 1.2.9 (1) For the boolean lattice B_n with its minimal building set, any subset of atoms is nested. The nested set complex hence is a simplex on n vertices. As for any other semilattice with maximal building set, the nested sets are the totally ordered subsets of the poset, hence the nested set complex is the order complex of the poset. In the particular case of $\mathcal{B}_n$ it is the barycentric subdivision of a simplex on n vertices. For $\mathcal{B}_3$ with building set $\mathcal{G} = \{1, 2, 3, 23\}$ the nested set complex consists of two triangles, namely $\{1, 2, 23\}$ and $\{1, 3, 23\}$.

(2) For the partition lattice Π_n with its minimal building set of 1-block partitions, a subset of such partitions is nested if and only if any two non-trivial blocks

are either contained one in another or disjoint. This is the example which has suggested the terminology of *nested* sets in the first place, it appeared as the central combinatorial structure in the paper of Fulton & MacPherson [FuM] on models for configuration spaces of smooth complex varieties.

1.3 Sequences of combinatorial blowups

We introduce the notion of a combinatorial blowup of an element in a semilattice and prove that the set of semilattices is closed under this operation.

1.3.1 Combinatorial blowups

Definition 1.3.1 For a semilattice $\mathcal{L}$ and an element $\alpha \in \mathcal{L}$ we define a poset $\mathrm{Bl}_{\alpha}\mathcal{L}$, the *combinatorial blowup of $\mathcal{L}$ at α*, as follows:

- elements of $\mathrm{Bl}_{\alpha}\mathcal{L}$:
  - (1) $y \in \mathcal{L}$, such that $y \not\geq \alpha$;
  - (2) $[\alpha, y]$, for $y \in \mathcal{L}$, such that $y \not\geq \alpha$ and $(y \vee \alpha)_{\mathcal{L}}$ exists (in particular, $[\alpha, \hat{0}]$ can be thought of as the result of blowing up α);
- order relations in $\mathrm{Bl}_{\alpha}\mathcal{L}$:
  - (1) $y > z$ in $\mathrm{Bl}_{\alpha}\mathcal{L}$ if $y > z$ in $\mathcal{L}$;
  - (2) $[\alpha, y] > [\alpha, z]$ in $\mathrm{Bl}_{\alpha}\mathcal{L}$ if $y > z$ in $\mathcal{L}$;
  - (3) $[\alpha, y] > z$ in $\mathrm{Bl}_{\alpha}\mathcal{L}$ if $y \geq z$ in $\mathcal{L}$;

  where in all three cases $y, z \not\geq \alpha$.

Note that the atoms in $\mathrm{Bl}_{\alpha}\mathcal{L}$ are the atoms of $\mathcal{L}$ together with the element $[\alpha, \hat{0}]$. It is easy, albeit tedious, to check that the class of (meet-)semilattices is closed under combinatorial blowups.

Lemma 1.3.2 *Let $\mathcal{L}$ be a semilattice and $\alpha \in \mathcal{L}$, then $\mathrm{Bl}_{\alpha}\mathcal{L}$ is a semilattice.*

Proof. The joins in $\mathrm{Bl}_{\alpha}\mathcal{L}$ are defined by the rule

$$\begin{aligned}([\alpha, y] \vee [\alpha, z])_{\mathrm{Bl}_{\alpha}\mathcal{L}} &= [\alpha, (y \vee z)_{\mathcal{L}}],\\ ([\alpha, y] \vee z)_{\mathrm{Bl}_{\alpha}\mathcal{L}} &= [\alpha, (y \vee z)_{\mathcal{L}}],\\ (y \vee z)_{\mathrm{Bl}_{\alpha}\mathcal{L}} &= (y \vee z)_{\mathcal{L}},\end{aligned}$$

which is applicable only if $(y \vee z)_{\mathcal{L}}$ exists, otherwise the corresponding joins in $\mathrm{Bl}_{\alpha}\mathcal{L}$ do not exist. Also, the first and second formulae are applicable only in the case $(y \vee z)_{\mathcal{L}} \not\geq \alpha$, otherwise the corresponding joins do not exist. The check of this is straightforward and is left to the reader. □

Observe that it is possible that $(x \vee y)_{\mathcal{L}}$ exists, while $(x \vee y)_{\mathrm{Bl}_{\alpha}\mathcal{L}}$ does not.

1.3.2 Blowing up building sets

In this subsection we prove that if one combinatorially blows up a building set of a semilattice in any chosen linear extension order, then one ends up with the face poset of the simplicial complex of nested sets of this building set. The following proposition provides the essential step for the proof.

Proposition 1.3.3 *Let $\mathcal{L}$ be a semilattice, $\mathcal{G}$ a building set of $\mathcal{L}$, and $\alpha \in \max \mathcal{G}$. Then, $\widetilde{\mathcal{G}} = (\mathcal{G} \setminus \{\alpha\}) \cup \{[\alpha, \hat{0}]\}$ is a building set of $\mathrm{Bl}_{\alpha}\mathcal{L}$. Furthermore, the nested subsets of $\widetilde{\mathcal{G}}$ are precisely the nested subsets of $\mathcal{G}$ with α replaced by $[\alpha, \hat{0}]$.*

Proof. It is easy to see that $\widetilde{\mathcal{G}}$ is a building set of $\mathrm{Bl}_{\alpha}\mathcal{L}$. Indeed, given $x \in \mathcal{L} \setminus \mathcal{L}_{\geq \alpha}$, (1.2.1) is obvious for $x \in \mathrm{Bl}_{\alpha}\mathcal{L}$, and, if $(x \vee \alpha)_{\mathcal{L}}$ exists, it follows for $[\alpha, x] \in \mathrm{Bl}_{\alpha}\mathcal{L}$ from the identity

$$[\hat{0}, [\alpha, x]]_{\mathrm{Bl}_{\alpha}\mathcal{L}} \;=\; [\hat{0}, x]_{\mathrm{Bl}_{\alpha}\mathcal{L}} \times B_1\,,$$

where B_1 is the subposet consisting of the two comparable elements $\hat{0} < [\alpha, \hat{0}]$.

Let us now see that the sets of nested subsets of $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ are the same when replacing α by $[\alpha, \hat{0}]$:

Let N be a nested set in $\mathcal{G}$, not containing α. For incomparable elements $x_1, \ldots, x_t$ in N, $\bigvee_{i=1}^{t} x_i \not\geq \alpha$, since otherwise we had $\alpha \in \max \mathcal{G}_{\leq \bigvee x_i} = F(\bigvee_{i=1}^{t} x_i) = \{x_1, \ldots, x_t\}$ by Proposition 1.2.8(2). Thus, $\bigvee_{i=1}^{t} x_i$ exists in $\mathrm{Bl}_{\alpha}\mathcal{L}$ and $\bigvee_{i=1}^{t} x_i \notin \widetilde{\mathcal{G}}$. Hence, N is nested in $\widetilde{\mathcal{G}}$. A nested subset in $\widetilde{\mathcal{G}}$ not containing $[\alpha, \hat{0}]$ is obviously nested in $\mathcal{G}$.

Let now N be nested in $\mathcal{G}$ containing α, and set $\widetilde{N} = (N \setminus \{\alpha\}) \cup \{[\alpha, \hat{0}]\}$. Subsets of incomparable elements in $\widetilde{N}$ not containing $[\alpha, \hat{0}]$ can be dealt with as above. Thus assume that $[\alpha, \hat{0}], x_1, \ldots, x_t$ are incomparable in $\widetilde{N}$. Then, $x_1, \ldots, x_t$ are incomparable in the nested set N, and, as above, we conclude that $\bigvee_{i=1}^{t} x_i$ exists and $\bigvee_{i=1}^{t} x_i \not\geq \alpha$. Moreover, $\alpha \vee \bigvee_{i=1}^{t} x_i$ exists in $\mathcal{L}$ (joins of nested sets always exist!), thus, $[\alpha, \bigvee_{i=1}^{t} x_i] = [\alpha, \hat{0}] \vee \bigvee_{i=1}^{t} x_i$ exists in $\mathrm{Bl}_{\alpha}\mathcal{L}$ and is obviously not contained in $\widetilde{\mathcal{G}}$. We conclude that $\widetilde{N}$ is nested in $\widetilde{\mathcal{G}}$.

Vice versa, let $\widetilde{N}$ be nested in $\widetilde{\mathcal{G}}$ containing $[\alpha, \hat{0}]$, and set $N = (\widetilde{N} \setminus \{[\alpha, \hat{0}]\}) \cup \{\alpha\}$. Again it suffices to consider subsets of incomparable elements $\alpha, x_1, \ldots, x_t$

in N. With $[\alpha, \hat{0}], x_1, \ldots, x_t$ incomparable in $\widetilde{N}$, $[\alpha, \hat{0}] \vee \bigvee_{i=1}^{t} x_i = [\alpha, \bigvee_{i=1}^{t} x_i]$ exists in $\mathrm{Bl}_{\alpha}\mathcal{L}$, thus $\alpha \vee \bigvee_{i=1}^{t} x_i$ exists in $\mathcal{L}$. Incomparability implies that $\alpha \vee \bigvee_{i=1}^{t} x_i > \alpha$, and thus $\alpha \vee \bigvee_{i=1}^{t} x_i \notin \mathcal{G}$. We conclude that N is nested in $\mathcal{G}$. □

By iterating the combinatorial blowup described in Proposition 1.3.3 through all of $\mathcal{G}$, we obtain the following theorem, which serves as a motivation for the entire development.

Theorem 1.3.4 *Let $\mathcal{L}$ be a semilattice and $\mathcal{G}$ a building set of $\mathcal{L}$ with some chosen linear extension: $\mathcal{G} = \{G_1, \ldots, G_t\}$, with $G_i > G_j$ implying $i < j$. Let $\mathrm{Bl}_k\mathcal{L}$ denote the result of subsequent blowups $\mathrm{Bl}_{G_k}(\mathrm{Bl}_{G_{k-1}}(\ldots \mathrm{Bl}_{G_1}\mathcal{L}))$. Then the final semilattice $\mathrm{Bl}_t\mathcal{L}$ is equal to the face poset of the simplicial complex $\mathcal{N}(\mathcal{G})$.*

Proof. The building set $\mathcal{G}_t$ of $\mathrm{Bl}_t\mathcal{L}$ that results from iterated application of Proposition 1.3.3 obviously is the set of atoms $\mathfrak{A}$ in $\mathrm{Bl}_t\mathcal{L}$. Every element $x \in \mathrm{Bl}_t\mathcal{L}$ is the join of atoms below it: $x = \bigvee \mathfrak{A}_{\leq \mathfrak{x}}$. The subset $\mathfrak{A}_{\leq \mathfrak{x}}$ of $\mathcal{G}_t$ is nested, in particular, it is the set of factors of x in $\mathrm{Bl}_t\mathcal{L}$ with respect to $\mathcal{G}_t$ (Proposition 1.2.8(2)). Proposition 1.2.5(2) implies that the interval $[\hat{0}, x]$ in $\mathrm{Bl}_t\mathcal{L}$ is boolean. We conclude that $\mathrm{Bl}_t\mathcal{L}$ is the face poset of a simplicial complex with faces in one-to-one correspondence with the nested sets in $\mathcal{G}_t$, which in turn correspond to the nested sets in $\mathcal{G}$ by Proposition 1.3.3. □

1.4 Instances of combinatorial blowups

1.4.1 De Concini-Procesi models of subspace arrangements

Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be an arrangement of linear subspaces in complex space $\mathbb{C}^d$. Much effort has been spent on describing the cohomology of the complement $\mathcal{M}(\mathcal{A}) = \mathbb{C}^d \setminus \bigcup \mathcal{A}$ of such an arrangement and, in particular, on answering the question whether the cohomology algebra is completely determined by the combinatorial data of the arrangement. Here, combinatorial data is understood as the lattice $\mathcal{L}(\mathcal{A})$ of intersections of subspaces of $\mathcal{A}$ ordered by reverse inclusion together with the complex codimensions of the intersections. A major step towards the solution of this problem (for a complete answer see [DGM, dLS]) was the construction of smooth models for the complement $\mathcal{M}(\mathcal{A})$ by De Concini & Procesi [DP3] that allowed for an explicit description of rational models for $\mathcal{M}(\mathcal{A})$ following [M]. The De Concini-Procesi models for arrangements in turn are one instance in a sequence of model constructions reaching from compactifications of symmetric spaces [DP1, DP2], over the Fulton-MacPherson compactifications of configuration spaces [FuM] to the general framework of wonderful conical compactifications proposed by MacPherson & Procesi [MP].

Given a complex subspace arrangement $\mathcal{A}$ in $\mathbb{C}^d$, De Concini & Procesi describe a smooth irreducible variety Y together with a proper map $\pi : Y \longrightarrow \mathbb{C}^d$ such that π is isomorphism over $\mathcal{M}(\mathcal{A})$, and the complement of the preimage of $\mathcal{M}(\mathcal{A})$ is a union of irreducible divisors with normal crossings in Y. The model Y can be constructed by a sequence of blowups of smooth subvarieties that is prescribed by the stratification of complex space induced by the arrangement.

Building sets for subspace arrangements

In order to enumerate the strata in the intersection stratification of Y given by the irreducible divisors, De Concini & Procesi introduced the notions of building sets, nested sets and irreducible elements as follows:

Definition 1.4.1 ([DP3, §2]) Let $\mathcal{L}(\mathcal{A})$ be the intersection lattice of an arrangement $\mathcal{A}$ of linear subspaces in a finite dimensional complex vector space. Consider the lattice $\mathcal{L}(\mathcal{A})^*$ formed by the orthogonal complements of intersections ordered by inclusion.

(1) For $U \in \mathcal{L}(\mathcal{A})^*$, $U = \oplus_{i=1}^k U_i$ with $U_i \in \mathcal{L}(\mathcal{A})^*$, is called a *decomposition* of U if for any $V \subseteq U$, $V \in \mathcal{L}(\mathcal{A})^*$, $V = \oplus_{i=1}^k (U_i \cap V)$ and $U_i \cap V \in \mathcal{L}(\mathcal{A})^*$ for $i = 1, \ldots, k$.

(2) Call $U \in \mathcal{L}(\mathcal{A})^* \setminus \{\hat{0}\}$ *irreducible* if it does not admit a non-trivial decomposition.

(3) $\mathcal{G} \subseteq \mathcal{L}(\mathcal{A})^* \setminus \{\hat{0}\}$ is called a *building set* for $\mathcal{A}$ if for any $U \in \mathcal{L}(\mathcal{A})^*$ and $G_1, \ldots, G_k$ maximal in $\mathcal{G}$ below U, $U = \oplus_{i=1}^k G_i$ is a decomposition (the $\mathcal{G}$-decomposition) of U.

(4) A subset $\mathcal{S} \subseteq \mathcal{G}$ is called *nested* if for any set of non-comparable elements $U_1, \ldots, U_k$ in $\mathcal{S}$, $U = \oplus_{i=1}^k U_i$ is the $\mathcal{G}$-decomposition of U.

Note that $\mathcal{L}(\mathcal{A})^*$ coincides with $\mathcal{L}(\mathcal{A})$ as abstract lattices. We will therefore talk about irreducible elements, building sets and nested sets in $\mathcal{L}(\mathcal{A})$ without explicitly referring to the dual setting of the preceding definition.

The notions of Definition 1.4.1 are in part based on the earlier notions introduced by Fulton & MacPherson in [FuM] to study compactifications of configuration spaces. Our terminology is naturally adopted from [FM, DP3]. Building sets and nested sets in the sense of De Concini & Procesi are building and nested sets for the intersection lattices of subspace arrangements in our combinatorial sense (see Proposition 1.4.5 (1) below). However, there are differences. The opposite is not true: A combinatorial building set for the intersection lattice of a subspace arrangement is not necessarily a building set for this arrangement in the sense of De Concini & Procesi, neither are irreducible elements in the sense of De Concini & Procesi irreducible in our sense.

Example 1.4.2 (*Combinatorial versus De Concini-Procesi building sets*)
Consider the following arrangement $\mathcal{A}$ of 3 subspaces in $\mathbb{C}^4$:

$$A_1 : \; z_4 = 0 \,, \quad A_2 : \; z_1 = z_2 = 0 \,, \quad A_3 : \; z_1 = z_3 = 0 \,.$$

The intersection lattice $\mathcal{L}(\mathcal{A})$ is a boolean algebra on 3 elements. Combinatorial building sets of this lattice have been discussed in Example 1.2.6, in particular, the set of atoms $\{A_1, A_2, A_3\} \subseteq \mathcal{L}(\mathcal{A})$ is the minimal combinatorial building set. However, any building set for $\mathcal{A}$ in the sense of De Concini & Procesi necessarily includes the intersection $A_2 \cap A_3$, since its orthogonal complement does not decompose in $\mathcal{L}(\mathcal{A})^*$. The minimal building set for $\mathcal{A}$, i.e., the set of irreducibles for $\mathcal{A}$, in the sense of De Concini & Procesi is $\{A_1, A_2, A_3, A_2 \cap A_3\}$. Any other building set contains this minimal building set and the total intersection $\bigcap \mathcal{A} = 0$.

The main difference between our combinatorial set-up and the original context of De Concini-Procesi model constructions can be formulated in the following way: our constructions are order-theoretically canonical for a given semilattice. The set of combinatorial building sets, in particular the set of irreducible elements, depends only on the semilattice itself and not on the geometry of the subspace arrangement which it encodes. See Proposition 1.4.5 for a complete explanation.

Local subspace arrangements

In order to trace the De Concini-Procesi construction step by step we need the more general notion of a local subspace arrangement.

Definition 1.4.3 Let M be a smooth complex d-dimensional manifold and $\mathcal{A}$ a union of finitely many smooth complex submanifolds of M such that all non-empty intersections of submanifolds in $\mathcal{A}$ are connected smooth complex submanifolds. $\mathcal{A}$ is called a *local subspace arrangement* if for any $x \in \mathcal{A}$ there exists an open set N in M with $x \in N$, a subspace arrangement $\widetilde{\mathcal{A}}$ in $\mathbb{C}^d$, and a biholomorphic map $\phi : N \to \mathbb{C}^d$, such that $\phi(N \cap \mathcal{A}) = \widetilde{\mathcal{A}}$.

Given a subspace arrangement $\mathcal{A}$, the initial ambient space $\mathbb{C}^d$ of $\mathcal{M}(\mathcal{A})$ carries a natural stratification by the subspaces of $\mathcal{A}$ and their intersections, the poset of strata being the intersection lattice $\mathcal{L}(\mathcal{A})$ of the arrangement. For a local subspace arrangement $\mathcal{A} = \{A_1, \ldots, A_n\}$ in M we again consider the stratification of M by all possible intersections of the A_i's, just like in the global case. The poset of strata is also denoted by $\mathcal{L}(\mathcal{A})$ and is called the intersection semilattice (it is a lattice if the intersection of all maximal strata is nonempty).

Definition 1.4.4 Let $\mathcal{A}$ be a local subspace arrangement and $\mathcal{L}(\mathcal{A})$ its intersection semilattice. For $U \in \mathcal{L}(\mathcal{A})$, $U_1, \ldots, U_k \in \mathcal{L}(\mathcal{A})$ are said to form a *decomposition* of U if for any $x \in U$ there exists an open set N with $x \in N$ and a biholomorphic map $\phi : N \to \mathbb{C}^d$, such that $\phi(N \cap U_1), \ldots, \phi(N \cap U_k)$ form a decomposition of $\phi(N \cap U)$ in the sense of Definition 1.4.1(1).

As in the global case, $\mathcal{G} \subseteq \mathcal{L}(\mathcal{A})$ is a *building set* for $\mathcal{A}$ if for any $U \in \mathcal{L}(\mathcal{A})$, the set of strata $\max \mathcal{G}_{\leq U}$ gives a decomposition of U.

We shall refer to these building sets as *geometric* building sets. The difference between combinatorial building sets and geometric ones is contained in the dimension function as is explained in the following proposition.

Proposition 1.4.5 *Let $\mathcal{A}$ be a local subspace arrangement with intersection semilattice $\mathcal{L}(\mathcal{A})$.*

(1) If $\mathcal{G} \subseteq \mathcal{L}(\mathcal{A})$ is a geometric building set of $\mathcal{A}$, then it is a combinatorial building set.

(2) If $\mathcal{G} \subseteq \mathcal{L}(\mathcal{A})$ is a combinatorial building set of $\mathcal{L}(\mathcal{A})$, and for any $x \in \mathcal{L}(\mathcal{A})$ the sum of codimensions of its factors is equal to the codimension of x, then $\mathcal{G}$ is a geometric building set.

Proof. In both cases it is enough to consider the case when $\mathcal{A}$ is a subspace arrangement.

(1) Consider $\mathcal{G}$ as a subset of $\mathcal{L}(\mathcal{A})^*$, then, for $U \in \mathcal{G}$, the isomorphism φ_U requested in Definition 1.2.2 is given by taking direct sums:

$$\varphi_U : \prod_{j=1}^{k} [\hat{0}, G_j] \xrightarrow{\oplus_{j=1}^{k}} [\hat{0}, U],$$

where $G_1, \ldots, G_k$ are maximal in $\mathcal{G}$ below U.

(2) For $U \in \mathcal{L}(\mathcal{A})^*$, the set $\{U_1, \ldots, U_k\} = \max \mathcal{G}_{\leq U}$ gives a decomposition of U because:

a) By the definition of $\mathcal{L}(\mathcal{A})^*$ and the definition of combinatorial building sets, we have $U = \text{span}(U_1, \ldots, U_k)$, and, since $\sum_{i=1}^{k} \dim U_i = \dim U$, we have $U = \bigoplus_{i=1}^{k} U_i$;

b) for any $V \subseteq U$, $\bigoplus_{i=1}^{k}(U_i \cap V) \subseteq V = \text{span}(U_1 \wedge V, \ldots, U_k \wedge V) \subseteq \bigoplus_{i=1}^{k}(U_i \cap V)$, where "$\wedge$" denotes the meet operation in $\mathcal{L}(\mathcal{A})^*$, hence $V = \bigoplus_{i=1}^{k}(U_i \cap V)$.

□

Intersection stratification of local arrangements after blowup

Let a space X be given with an intersection stratification induced by a local subspace arrangement, and let G be a stratum in X. In the blowup of X at G, $\mathrm{Bl}_G X$, we find the following maximal strata:

- maximal strata in X that do not intersect with G,
- blowups of maximal strata V at $G \cap V$, $\mathrm{Bl}_{G\cap V} V$, where V is maximal in X and intersects G,
- the exceptional divisor $\widetilde{G}$ replacing G.

We consider the intersection stratification of $\mathrm{Bl}_G X$ induced by these maximal strata. We will later see (proof of Proposition 1.4.7) that in case G is maximal in a building set for the local arrangement in X, then the union of maximal strata in $\mathrm{Bl}_G X$ is again a local arrangement with induced intersection stratification. In general, this is not the case, see Example 1.4.6

For ease of notation, let us agree here that formally blowing up an empty (non-existing) stratum has no effect on the space. We think about a stratum Y in X, intersection of all maximal strata $V_1, \dots, V_t$ that contain Y, as being replacing by the intersection of corresponding maximal strata in $\mathrm{Bl}_G X$:

$$\mathrm{Bl}_{G\cap V_1} V_1 \;\cap\; \dots \;\cap\; \mathrm{Bl}_{G\cap V_t} V_t \,, \tag{1.4.1}$$

(recall that $\mathrm{Bl}_{G\cap V_j} V_j = V_j$ for $G \cap V_j = \emptyset$). The intersection (1.4.1) being empty means that the stratum Y vanishes under blowup of G. For notational convenience, we most often retain names of strata under blowups, thereby referring to the replacement of strata described above.

Example 1.4.6 (*Local subspace arrangements are not closed under blowup*)
We give an example which shows that blowing up a stratum in a local subspace arrangement does not necessarily result in a local subspace arrangement again. Consider the following arrangement of 2 planes and 1 line in $\mathbb{C}^3$:

$$A_1 : \; y - z = 0\,, \quad A_2 : \; y + z = 0\,, \quad L : \; x = y = 0\,.$$

After blowing up L, the planes A_1 and A_2 are replaced by complex line bundles over $\mathbb{C}\mathrm{P}^1$, which have in common their zero section Z and a complex line Y; L is replaced by a direct product of $\mathbb{C}$ and $\mathbb{C}\mathrm{P}^1$, which intersects both line bundles in Z. The new maximal strata fail to form a local subspace arrangement in the point $Z \cap Y$.

Tracing incidence structure during arrangement model construction

We now give a more detailed description of the model construction by De Concini & Procesi via successive blowups, and then proceed with linking our notion of combinatorial blowups to the context of arrangement models.

Let $\mathcal{A}$ be a complex subspace arrangement, $\mathcal{G} \subseteq \mathcal{L}(\mathcal{A})$ a geometric building set for $\mathcal{A}$, and $\{G_1, \ldots, G_t\}$ some linear extension of the partial containment order on associated strata in $\mathbb{C}^d$ such that $G_k \supset G_l$ implies $l < k$. The De Concini-Procesi model $Y = Y_{\mathcal{G}}$ of $\mathcal{M}(\mathcal{A})$ is the result of blowing up the strata indexed by elements of $\mathcal{G}$ in the given order. Note that the linear order was chosen so that at each step the stratum which is to be blown up does *not* contain any other stratum indexed by an element of $\mathcal{G}$. At each step we consider intersection stratifications as described above, and we denote the poset of strata after blowup of G_i with $\mathcal{L}_i^{\mathcal{G}}(\mathcal{A})$. For the case of a stratum G_i being empty after previous blowups remember our agreement of considering blowups of $\emptyset$ as having no effect on a space. The later Proposition 1.4.7 however shows that strata indexed by elements in $\mathcal{G}$ do not disappear during the sequence of blowups.

Let us remark that the combinatorial data of the initial stratification, i.e., of the arrangement, prescribes much of the geometry of $Y_{\mathcal{G}}$: the complement $Y_{\mathcal{G}} \setminus \mathcal{M}(\mathcal{A})$ is a union of smooth irreducible divisors indexed by elements of $\mathcal{G}$, and these divisors intersect if and only if the set of indices is nested in $\mathcal{G}$ [DP3, Thm 3.2].

Proposition 1.4.7 *Let $\mathcal{A}$ be an arrangement of complex subspaces, $\mathcal{G}$ a building set for $\mathcal{A}$ in the sense of De Concini & Procesi, and $\{G_1, \ldots, G_t\}$ some linear extension of the partial containment order on associated strata as described above. Let $\mathrm{Bl}_i^{\mathcal{G}}(\mathcal{A})$ denote the geometric result of successively blowing up strata $G_1, \ldots, G_i$, for $1 \leq i \leq t$. Then,*

(1) The poset of strata $\mathcal{L}_i^{\mathcal{G}}(\mathcal{A})$ of $\mathrm{Bl}_i^{\mathcal{G}}(\mathcal{A})$ can be described as the result of a sequence of combinatorial blowups of the intersection lattice $\mathcal{L} = \mathcal{L}(\mathcal{A})$:

$$\mathcal{L}_i^{\mathcal{G}}(\mathcal{A}) \; = \; \mathrm{Bl}_i(\mathcal{L}), \qquad \textit{for } 1 \leq i \leq t.$$

(*Recall that* $\mathrm{Bl}_i(\mathcal{L}) = \mathrm{Bl}_{G_i}(\mathrm{Bl}_{G_{i-1}}(\ldots \mathrm{Bl}_{G_1}\mathcal{L}))$ *for* $1 \leq i \leq t$.)

(2) The union of maximal strata $\mathcal{A}_i^{\mathcal{G}}$ in $\mathrm{Bl}_i^{\mathcal{G}}(\mathcal{A})$ is a local subspace arrangement, with $\mathcal{G}$ in $\mathcal{L}_i^{\mathcal{G}}(\mathcal{A})$ being a building set for $\mathcal{A}_i^{\mathcal{G}}$ in the sense of Definition 1.4.4. (*Recall that $\mathcal{G}$ here refers to the preimages of the original strata in $\mathcal{G} \subseteq \mathcal{L}(\mathcal{A})$ under the sequence of blowups.*)

Proof. We proceed by induction on the number of blowups. The induction start is obvious, since the lattice of strata $\mathcal{L}_0^{\mathcal{G}}(\mathcal{A})$ of the initial stratification of $\mathbb{C}^d$ coincides with the intersection lattice $\mathcal{L}(\mathcal{A}) = \mathrm{Bl}_0(\mathcal{L})$ of the arrangement $\mathcal{A}$. The union of maximal strata is the arrangement $\mathcal{A}$ itself with its given building set $\mathcal{G}$.

Assume that $\mathcal{L}^{\mathcal{G}}_{i-1}(\mathcal{A}) = \mathrm{Bl}_{i-1}(\mathcal{L})$ for some $1 \leq i \leq t$, the union of maximal strata $\mathcal{A}^{\mathcal{G}}_{i-1}$ in $\mathrm{Bl}^{\mathcal{G}}_{i-1}(\mathcal{A})$ being a local arrangement, and $\mathcal{G}$ a building set for $\mathcal{L}^{\mathcal{G}}_{i-1}(\mathcal{A})$. Let $G = G_i$ be the next stratum to be blown up. First, we proceed in 4 steps to show that $\mathcal{L}^{\mathcal{G}}_i(\mathcal{A}) = \mathrm{Bl}_i(\mathcal{L})$. In 2 further steps we then verify the claims in (2).

Step 1: *Assign strata of* $\mathrm{Bl}^{\mathcal{G}}_i(\mathcal{A})$ *to elements in* $\mathrm{Bl}_i(\mathcal{L})$.

We distinguish two types of elements in $\mathrm{Bl}_i(\mathcal{L})$:

$$\begin{array}{lrl} \text{Type I}: & Y & \text{with } Y \in \mathrm{Bl}_{i-1}(\mathcal{L}) \text{ and } Y \not\geq G\,, \\ \text{Type II}: & [G,Y] & \text{with } Y \in \mathrm{Bl}_{i-1}(\mathcal{L})\,,\ Y \not\geq G\,, \\ & & \text{and } Y \vee G \text{ exists in } \mathrm{Bl}_{i-1}(\mathcal{L})\,. \end{array}$$

To $Y \in \mathrm{Bl}_i(\mathcal{L})$ of type I, assign $\mathrm{Bl}_{G\cap Y} Y$ (recall that blowing up an empty stratum does not change the space). Note that $\dim \mathrm{Bl}_{G\cap Y} Y = \dim Y$.

To $[G,Y] \in \mathrm{Bl}_i(\mathcal{L})$ of type II, assign $(\mathrm{Bl}_{G\cap Y} Y) \cap \widetilde{G}$, where $\widetilde{G}$ denotes the exceptional divisor that replaces G in $\mathrm{Bl}^{\mathcal{G}}_i(\mathcal{A})$. This description comprises $\widetilde{G}$ being assigned to $[G, \hat{0}]$. Note that $\dim(\mathrm{Bl}_{G\cap Y} Y) \cap \widetilde{G} = \dim Y - 1$.

Step 2: *Reverse inclusion order on the assigned spaces coincides with the partial order on* $\mathrm{Bl}_i(\mathcal{L})$.

(1) $X, Y \in \mathrm{Bl}_i(\mathcal{L})$, both of type I:

$$X \leq_{\mathrm{Bl}_i(\mathcal{L})} Y \Leftrightarrow X \leq_{\mathrm{Bl}_{i-1}(\mathcal{L})} Y \Leftrightarrow X \supseteq_{\mathrm{Bl}^{\mathcal{G}}_{i-1}(\mathcal{A})} Y \Leftrightarrow \mathrm{Bl}_{G\cap X} X \supseteq \mathrm{Bl}_{G\cap Y} Y,$$

where "$\Leftarrow$" in the last equivalence can be seen by first noting that $Y \setminus (G\cap Y) \subseteq X \setminus (G \cap X)$, and then comparing points in the exceptional divisors.

(2) $X, [G,Y] \in \mathrm{Bl}_i(\mathcal{L})$, X of type I, $[G,Y]$ of type II:

As above we conclude

$$\begin{aligned} X \leq_{\mathrm{Bl}_i(\mathcal{L})} [G,Y] \;&\Leftrightarrow\; X \leq_{\mathrm{Bl}_{i-1}(\mathcal{L})} Y \\ &\Leftrightarrow\; X \supseteq_{\mathrm{Bl}^{\mathcal{G}}_{i-1}(\mathcal{A})} Y \;\Rightarrow\; \mathrm{Bl}_{G\cap X} X \supseteq \mathrm{Bl}_{G\cap Y} Y \cap \widetilde{G}\,. \end{aligned}$$

To prove the converse is rather subtle. Note first that $G \cap Y \subseteq G \cap X$. Assume that G strictly contains $G \cap X$, then both $G \cap X$ and $G \cap Y$ are not in the building set due to the linear order chosen on $\mathcal{G}$, and G is a factor of both $G\cap X$ and $G \cap Y$. Let $F(G\cap X) = \{G, G_1, \ldots, G_k\}$, $F(G \cap Y) = \{G, H_1, \ldots, H_t\}$. X written as a join of elements in $\mathrm{Bl}_{i-1}(\mathcal{L})$ below the factors of $G \cap X$ reads

$$X \;=\; g_X \vee Z_1 \vee \ldots \vee Z_k$$

for some $g_X \in [\hat{0}, G]$, $Z_i \in [\hat{0}, G_i]$ for $i = 1, \ldots, k$.

If $Z_i < G_i$ for some $i \in \{1, \ldots, k\}$, we have

$$\begin{aligned} G \vee X &= G \vee (g_X \vee Z_1 \vee \ldots \vee Z_i \vee \ldots \vee Z_k) \\ &\leq G \vee (g_X \vee G_1 \vee \ldots \vee Z_i \vee \ldots \vee G_k) \\ &< G \vee G_1 \vee \ldots \vee G_k = G \vee X \,, \end{aligned}$$

by the “necessity” property of Proposition 1.2.3(4), yielding a contradiction. Hence,

$$X = g_X \vee G_1 \vee \ldots \vee G_k \,,$$

and similarly, $Y = g_Y \vee H_1 \vee \ldots \vee H_t$ for some $g_Y \in [\hat{0}, G]$.

For each $j \in \{1, \ldots, k\}$ there exists a unique $i_j \in \{1, \ldots, t\}$ such that $G_j \leq H_{i_j}$ by Proposition 1.2.5(1). Thus, $\bigvee G_i < \bigvee H_j$, and, for showing that $X \leq Y$, it is enough to see that $g_X \leq g_Y$.

We show that in an open neighborhood of any point $y \in G \cap Y$, $g_Y \subseteq g_X$. This yields our claim since strata in $\mathrm{Bl}^{\mathcal{G}}_{i-1}(\mathcal{A})$ have pairwise transversal intersections: if they coincide locally, they must coincide globally. With $\mathcal{A}^{\mathcal{G}}_{i-1}$ being a local arrangement, there exists an open neighborhood of $y \in G \cap Y$ where the stratification is biholomorphic to a stratification induced by a subspace arrangement. We tacitly work in the arrangement setting, using that $(\mathrm{Bl}_{\,i-1}(\mathcal{L}))_{\leq G \vee Y}$ is the intersection lattice of a product arrangement. The $\mathcal{G}$-decomposition of $(G \vee Y)^{\perp}$ described in Definition 1.4.4 yields (when transferred to the primal setting):

$$g_Y = \mathrm{span}(G, Y) \,.$$

Analogously, $g_X = \mathrm{span}(G, X)$.

In the linear setting we are concerned with, we interpret points in the exceptional divisor of a blowup as follows:

$$\mathrm{Bl}_{\,G \cap Y} Y \cap \widetilde{G} = \{(a, \mathrm{span}(p, G \cap Y)) \mid a \in G \cap Y,\ p \in Y \setminus (G \cap Y)\} \,. \quad (1.4.2)$$

In terms of this description, the inclusion map $\mathrm{Bl}_{\,G \cap Y} Y \cap \widetilde{G} \hookrightarrow \mathrm{Bl}_{\,G}(\mathrm{Bl}^{\mathcal{G}}_{i-1}(\mathcal{A}))$ reads

$$(a, \mathrm{span}(p, G \cap Y)) \longmapsto (a, \mathrm{span}(p, G)) \,.$$

Therefore, $\mathrm{Bl}_{\,G \cap Y} Y \cap \widetilde{G}$ being contained in $\mathrm{Bl}_{\,G \cap X} X \subseteq \mathrm{Bl}_{\,G}(\mathrm{Bl}^{\mathcal{G}}_{i-1}(\mathcal{A}))$ means that for $(a, \mathrm{span}(p, G \cap Y)) \in \mathrm{Bl}_{\,G \cap Y} Y \cap \widetilde{G}$ there exists $q \in X \setminus (G \cap X)$ such that $\mathrm{span}(p, G) = \mathrm{span}(q, G)$. In particular, $\mathrm{span}(Y, G) \subseteq \mathrm{span}(X, G)$, which by our previous arguments implies that $Y \subseteq X$.

We assumed above that $G \supset G \cap X$. If $G \cap X$ coincides with G, i.e., X contains G, then $g_X = X$ and a similar reasoning applies to see that $Y \subseteq X$. Similarly for $G \cap X = G \cap Y = G$.

(3) $[G, X]$, $[G, Y] \in \mathrm{Bl}_i(\mathcal{L})$, both of type II:

$$[G,X] \leq_{\mathrm{Bl}_i(\mathcal{L})} [G,Y] \;\Leftrightarrow\; X \leq_{\mathrm{Bl}_{i-1}(\mathcal{L})} Y$$
$$\Leftrightarrow\; X \supseteq_{\mathrm{Bl}^{\mathcal{G}}_{i-1}(\mathcal{A})} Y \;\Leftrightarrow\; \mathrm{Bl}_{G\cap X} X \cap \widetilde{G} \supseteq \mathrm{Bl}_{G\cap Y} Y \cap \widetilde{G}\,,$$

where "$\Leftarrow$" follows from (2) and $\mathrm{Bl}_{G\cap X} X \supseteq \mathrm{Bl}_{G\cap X} X \cap \widetilde{G} \supseteq \mathrm{Bl}_{G\cap Y} Y \cap \widetilde{G}$.

Step 3: *Each of the assigned spaces is the intersection of maximal strata in* $\mathrm{Bl}^{\mathcal{L}}_i(\mathcal{A})$.

It is enough to show that spaces assigned to elements of type I in $\mathrm{Bl}_i(\mathcal{L})$ are intersections of new maximal strata. Those associated to elements of type II then are intersections as well by definition.

Let $Y \in \mathrm{Bl}_i(\mathcal{L})$, $Y \not\geq G$, and $Y = \cap_{i=1}^t V_i$ with $V_1, \ldots, V_t$ the maximal strata in $\mathrm{Bl}^{\mathcal{G}}_{i-1}(\mathcal{A})$ containing Y. We claim that

$$\mathrm{Bl}_{G\cap Y} Y \;=\; \bigcap_{i=1}^{t} \mathrm{Bl}_{G\cap V_i} V_i\,. \tag{1.4.3}$$

For the inclusion "$\subseteq$" note that $\mathrm{Bl}_{G\cap Y} Y \subseteq \mathrm{Bl}_{G\cap V_i} V_i$ is a direct consequence of $Y \subseteq V_i$ as discussed in Step 2 (1).

For the reverse inclusion we need the following identity:

$$\bigvee_{i=1}^{t} (G \wedge V_i) \;=\; G \wedge Y\,. \tag{1.4.4}$$

This identity holds in any semilattice without referring to G being an element of the building set.

Let $\alpha \in \cap_{i=1}^t \mathrm{Bl}_{G\cap V_i} V_i$. In case $\alpha \in \cap_{i=1}^t V_i \setminus (G \cap V_i)$, we conclude that $\alpha \in Y \setminus (G \cap Y)$. We thus assume that α is contained in the intersection of exceptional divisors $\widetilde{G \cap V_i}$, $i = 1, \ldots, t$. We again switch to local considerations in the neighborhood of a point $y \in G \cap Y$, using that it carries a stratification biholomorphic to an arrangement stratification.

Using the description (1.4.2) of points in exceptional divisors that are created by blowups in the arrangement setting, $\alpha \in \cap_{i=1}^t \widetilde{G \cap V_i} \subseteq \cap_{i=1}^t \mathrm{Bl}_{G\cap V_i} V_i$ means that there exist $a \in \cap_{i=1}^t (G \cap V_i)$, and $p_i \in V_i \setminus (G \cap V_i)$ for $i = 1, \ldots, t$, with

$$\alpha \;=\; (a, \mathrm{span}(p_i, G \cap V_i)) \;\in\; \mathrm{Bl}_{G\cap V_i} V_i\,.$$

In particular, $\mathrm{span}(p_i, G) = \mathrm{span}(p_j, G)$ for $1 \leq i, j \leq t$. Thus,

$$\mathrm{span}(p_j, G) \;\subseteq \bigcap_{i=1}^{t} \mathrm{span}(V_i, G) \;=\; \mathrm{span}(Y, G)$$

using the identity (1.4.4). We conclude that there exists $y \in Y \setminus (G \cap Y)$ such that $\mathrm{span}(y, G) = \mathrm{span}(p_j, G)$ for all $j \in \{1, \ldots, k\}$, hence

$$\alpha \;=\; (a, \mathrm{span}(y, G \cap Y)) \;\in\; \mathrm{Bl}_{G\cap Y} Y\,.$$

Though we are for the moment not concerned with the case of $Y \subseteq G$, we note for later reference that (1.4.3) remains true, with $\mathrm{Bl}_Y Y = \emptyset$ meaning that the intersection on the right-hand side is empty. Following the proof of the inclusion "$\supseteq$" in (1.4.3) for $G \cap Y = Y$, we first find that the intersection of blowups can only contain points in the exceptional divisors. Assuming $\alpha \in \bigcap_{i=1}^t \widetilde{G \cap V_i}$ we arrive to a contradiction when concluding that $\mathrm{span}(p_j, G) \subseteq \bigcap_{i=1}^t \mathrm{span}(V_i, G) = \mathrm{span}(Y, G) = G$ for $j = 1, \ldots, t$.

Step 4: *Any intersection of maximal strata in* $\mathrm{Bl}_i^{\mathcal{G}}(\mathcal{A})$ *occurs as an assigned space.*

Every intersection involving the exceptional divisor $\widetilde{G}$ occurs if we can show that all other intersections occur (intersections that additionally involve $\widetilde{G}$ then are assigned to corresponding elements of type II).

Consider $W = \bigcap_{i=1}^t \mathrm{Bl}_{G\cap V_i} V_i$, where the V_i are maximal strata in $\mathrm{Bl}_{i-1}^{\mathcal{G}}(\mathcal{A})$; recall here that a blowup in an empty stratum does not alter the space. We can assume that $\bigcap_{i=1}^t V_i \neq \emptyset$, otherwise the intersection W were empty. With the identity (1.4.3) in Step 3 we conclude that either $W = \emptyset$ (in case $\bigcap_{i=1}^t V_i \subseteq G$) or $W = \mathrm{Bl}_{G \cap \bigcap_{i=1}^t V_i} \bigcap_{i=1}^t V_i$, in which case it is assigned to the element $\bigcap_{i=1}^t V_i$ in $\mathrm{Bl}_i(\mathcal{L})$.

Step 5: $\mathcal{A}_i^{\mathcal{G}}$ *is a local subspace arrangement in* $\mathrm{Bl}_i^{\mathcal{G}}(\mathcal{A})$.

It follows from the description (1.4.3) of strata in $\mathrm{Bl}_i^{\mathcal{G}}(\mathcal{A})$ that all intersections of maximal strata are connected and smooth. It remains to show that $\mathcal{A}_i^{\mathcal{G}}$ locally looks like a subspace arrangement. Let $y \in \mathcal{A}_i^{\mathcal{G}}$. We can assume that y lies in the exceptional divisor $\widetilde{G}$. Let $x \in G \subseteq \mathcal{A}_{i-1}^{\mathcal{G}}$ be the image of y under the blowdown map.

We first give a local description around x in $\mathcal{A}_{i-1}^{\mathcal{G}}$. By induction hypothesis, there exists a neighborhood N of x, and an arrangement of linear subspaces $\mathcal{B}$ in $\mathbb{C}^n$ such that the pair $(N, \mathcal{A}_{i-1}^{\mathcal{G}} \cap N)$ is biholomorphic to the pair $(\mathbb{C}^n, \mathcal{B})$. We can assume that under this biholomorphic map, x is mapped to the origin. Let $T = \bigcap_{B\in\mathcal{B}} B$ and note that $G \cap N$ is mapped to some subspace Γ in $\mathcal{B}$.

With G being maximal in the building set for $\mathcal{A}_{i-1}^{\mathcal{G}}$, $\mathcal{B}/T$ is a product arrangement with one of the factors being an arrangement in Γ/T. More precisely, there exists a subspace $\Gamma' \subseteq \mathbb{C}^n$, and two subspace arrangements, $\mathcal{C}$ in Γ/T and $\mathcal{C}'$ in Γ'/T, such that

(1) $\Gamma/T \,\oplus\, \Gamma'/T \,\oplus\, T \;=\; \mathbb{C}^n$,

(2) $\mathcal{B} \;=\; \{A \,\oplus\, \Gamma'/T \,\oplus\, T \mid A \in \mathcal{C}\} \,\cup\, \{\Gamma/T \,\oplus\, A' \,\oplus\, T \mid A' \in \mathcal{C}'\}$.

Blowing up G in $\mathrm{Bl}\,^{\mathcal{G}}_{i-1}(\mathcal{A})$ locally corresponds to blowing up Γ in $\mathbb{C}^n$. Let t be the point on the special divisor $\widetilde{\Gamma}$ corresponding to $y \in \widetilde{G}$, thus t maps to the origin in $\mathbb{C}^n$ under the blowdown map. A neighborhood of t in $\mathrm{Bl}\,_{\Gamma}\mathbb{C}^n$ is an n-dimensional open ball which can be parameterized as a direct sum

$$M \,\oplus\, M' \,\oplus\, I \,\oplus\, T\,.$$

Here, M is an open ball around 0 in Γ/T, M' is an open ball on the unit sphere in Γ'/T around the point of intersection with the line $\langle p\rangle$ in Γ'/T that defines t as a point in the exceptional divisor, $t = (0, \mathrm{span}(p, \Gamma)) \in \widetilde{\Gamma}$ (compare (1.4.2)), and I an open unit ball in $\mathbb{C}$.

The maximal strata in this neighborhood are the following:

- ∘ the hyperplane $M \,\oplus\, M' \,\oplus\, \{0\} \,\oplus\, T$, as the exceptional divisor,
- ∘ $(M \cap A) \,\oplus\, M' \,\oplus\, I \,\oplus\, T$, replacing $A \,\oplus\, \Gamma'/T \,\oplus\, T$ after blowup,
- ∘ $M \,\oplus\, (M' \cap A') \,\oplus\, I \,\oplus\, T$, replacing $\Gamma/T \,\oplus\, A' \,\oplus\, T$ after blowup for $A' \neq 0$.

This proves that around t in $\mathrm{Bl}\,_{\Gamma}\mathbb{C}^n$ we have the structure of a local subspace arrangement, which in turn shows the local arrangement property around y in $\mathcal{A}_i^{\mathcal{G}}$.
Step 6: *$\mathcal{G}$ is a building set for $\mathcal{A}_i^{\mathcal{G}}$ in the sense of* Definition 1.4.4.
$\mathcal{G}$ is a combinatorial building set by Proposition 1.3.3. Complementing this with the dimension information about the strata, we conclude, by Proposition 1.4.5(2), that $\mathcal{G}$ is a geometric building set. □

1.4.2 Simplicial resolutions of toric varieties

The study of toric varieties has proved to be a field of fruitful interplay between algebraic and convex geometry: toric varieties are determined by rational polyhedral fans, and many of their algebraic geometric properties are reflected by combinatorial properties of their defining fans.

We recall one such correspondence – between subdivisions of fans and special toric morphisms – and show that so-called stellar subdivisions are instances of combinatorial blowups. This allows us to apply our Main Theorem in the present context: Given a polyhedral fan, we specify a class of *simplicial* subdivisions, and, interpreting our notions of building sets and nested sets, we describe the incidence combinatorics of the subdivisions in terms of the combinatorics of the initial fan. For background material on toric varieties we refer to the standard sources [Da, Od, Fu, Ew].

Let X_Σ be a toric variety defined by a rational polyhedral fan Σ. Any subdivision of Σ gives rise to a proper, birational toric morphism between the associated toric varieties (cf [Da, 5.5.1]). In particular, simplicial subdivisions yield toric morphisms from quasi-smooth toric varieties to the initial variety – so-called *simplicial resolutions*. Quasi-smooth toric varieties being rational homology manifolds, such morphisms can replace smooth resolutions for (co)homological considerations.

We define a particular, elementary, type of subdivisions:

Definition 1.4.8 Let $\Sigma = \{\sigma\}_{\sigma\in\Sigma} \subseteq \mathbb{R}^d$ be a polyhedral fan, i.e., a collection of closed polyhedral cones σ in $\mathbb{R}^d$ such that $\sigma \cap \tau$ is a cone in Σ for any $\sigma, \tau \in \Sigma$. Let $\mathrm{cone}(x)$ be a ray in $\mathbb{R}^d$ generated by $x \in \mathrm{relint}\,\tau$ for some $\tau \in \Sigma$. The *stellar subdivision* $\mathrm{sd}(\Sigma, x)$ of Σ in x is given by the collection of cones

$$(\,\Sigma \setminus \mathrm{star}(\tau,\Sigma)\,) \;\cup\; \{\,\mathrm{cone}(x,\rho) \mid \rho \subseteq \sigma \;\text{ for some }\; \sigma \in \mathrm{star}(\tau,\Sigma)\,\}\,,$$

where $\mathrm{star}(\tau,\Sigma) = \{\sigma \in \Sigma \mid \tau \subseteq \sigma\}$, and $\mathrm{cone}(x,\rho)$ the closed polyhedral cone spanned by ρ and x. If only concerned with the combinatorics of the subdivided fan, we also talk about stellar subdivision of Σ in τ, $\mathrm{sd}(\Sigma,\tau)$, meaning any stellar subdivision in x for $x \in \mathrm{relint}\,\tau$.

Proposition 1.4.9 *Let $\mathcal{F}(\Sigma)$ be the face poset of a polyhedral fan Σ, i.e., the set of closed cones in Σ ordered by inclusion, together with the zero cone $\{0\}$ as a minimal element. For $\tau \in \Sigma$, the face poset of the stellar subdivision of Σ in τ can be described as the combinatorial blowup of $\mathcal{F}(\Sigma)$ at τ:*

$$\mathcal{F}(\mathrm{sd}(\Sigma,\tau)) \;=\; \mathrm{Bl}_{\,\tau}\mathcal{F}(\Sigma)\,.$$

Proof. Removing $\mathrm{star}(\tau,\Sigma)$ from Σ corresponds to removing $\mathcal{F}(\Sigma)_{\geq\tau}$ from $\mathcal{F}(\Sigma)$, adding cones as described in Definition 1.4.8 corresponds to extending $\mathcal{F}(\Sigma) \setminus \mathcal{F}(\Sigma)_{\geq\tau}$ by elements $[\tau,\rho]$ for $\rho \in \mathcal{F}(\Sigma)$, $\rho \subseteq \sigma$ for some $\sigma \in \mathrm{star}(\tau,\Sigma)$. The comparison of order relations is straightforward. $\square$

We apply our Main Theorem to the present context.

Theorem 1.4.10 *Let Σ be a polyhedral fan in $\mathbb{R}^d$ with face poset $\mathcal{F}(\Sigma)$. Let $\mathcal{G} \subseteq \mathcal{F}(\Sigma)$ be a building set of $\mathcal{F}(\Sigma)$ in the sense of Definition 1.2.2, $\mathcal{N}(\mathcal{G})$ the complex of nested sets in $\mathcal{G}$ (cf. Definition 1.2.7). Then, the consecutive application of stellar subdivisions in every cone $G \in \mathcal{G}$ in a non-increasing order yields a simplicial subdivision of Σ with face poset equal to the face poset of $\mathcal{N}(\mathcal{G})$.*

As examples of building sets for face lattices of polyhedral fans let us mention:

(1) the full set of faces, with the corresponding complex of nested sets being the order complex of $\mathcal{F}(\Sigma)$ (stellar subdivision in all cones results in the barycentric subdivision of the fan);

(2) the set of rays together with the non-simplicial faces of Σ;

(3) the set of irreducible elements in $\mathcal{F}(\Sigma)$: the set of rays together with all faces of Σ that are not products of some of their proper faces.

Remark 1.4.11 For a smooth toric variety X_Σ, the union of closed codimension 1 torus orbits is a local subspace arrangement, in particular, the codimension 1 orbits form a divisor with normal crossings, [Fu, p. 100]. The intersection stratification of this local arrangement coincides with the torus orbit stratification of the toric variety. For any face τ in the defining fan Σ, the torus orbit $\mathcal{O}_\tau$ together with all orbits corresponding to rays in Σ form a geometric building set. Our proof in 1.4.1 applies in this context with $\mathcal{O}_\tau$ playing the role of G. We conclude that under blowup of X_Σ in the closed torus orbit $\mathcal{O}_\tau$, the incidence combinatorics of torus orbits changes exactly in the way described by a stellar subdivision of Σ in τ. This is the combinatorial part of the well-known fact that in the smooth case, the blowup of X_Σ in a torus orbit $\mathcal{O}_\tau$ corresponds to a regular stellar subdivision of the fan Σ in τ [MO].

1.5 An outlook

1.5.1 Models for real subspace arrangements and stratified manifolds

In the spirit of the De Concini-Procesi wonderful model construction for subspace arrangements, Gaiffi [Ga2] presents a model construction for the complement of arrangements of real linear subspaces modulo $\mathbb{R}^+$: Given a central subspace arrangement $\mathcal{A}$ in some Euclidean vector space V, denote by $\widehat{\mathcal{M}}(\mathcal{A})$ the quotient of its complement by $\mathbb{R}^+$. Denote the unit sphere in V by $S(V)$, and consider, for a given (geometric) building set $\mathcal{G}$ in $\mathcal{L}(\mathcal{A})$, the embedding

$$\rho \,:\, \widehat{\mathcal{M}}(\mathcal{A}) \;\longrightarrow\; S(V) \times \prod_{G \in \mathcal{G}} G \cap S(V)\,.$$

The map is obtained by composing the natural section $\widehat{\mathcal{M}}(\mathcal{A}) \to \mathcal{M}(\mathcal{A})$, $[x] \mapsto \frac{x}{|x|}$, with a projection onto each factor of the right-hand side product. Denote the closure of this map by $Y_\mathcal{G}$. $Y_\mathcal{G}$ is shown to be a manifold with corners, which enjoys much of the properties familiar from the projective setting: the boundary of $Y_\mathcal{G}$ is stratified by codimension 1 manifolds with corners indexed with building set elements and having non-empty intersection whenever the index set is nested with respect to $\mathcal{G}$. The set-up allows for a straightforward generalization to mixed subspace and halfspace arrangements motivated by compactifications of configuration spaces in work of Kontsevich [Ko]. A step aside from classical (linear) arrangements, our combinatorial framework still applies is this context.

In a second part of his paper, Gaiffi extends the previous construction to conically stratified manifolds with corners. Replacing the explicit construction of taking the closure of an embedding into a product of spheres as above, he describes a sequence of "real blowups" in the sense of Kuperberg & Thurston [KT]. The sequence is prescribed by the choice of a subset of strata in the original manifold that is a combinatorial building set in our sense. The resulting space is a manifold with corners with its boundary stratified by codimension 1 manifolds with corners that are indexed by the building set elements, and intersections being non-empty if and only if the corresponding index sets are nested.

1.5.2 A graded algebra associated with a finite lattice

In joint work with Sergey Yuzvinsky, which appears as second chapter in this thesis [FY], we start out from the combinatorial notions of building sets and nested sets given in the present chapter and define a commutative graded algebra in purely combinatorial terms:

Definition 1.5.1 For a finite lattice $\mathcal{L}$, $\mathfrak{A}$ its set of atoms, and $\mathcal{G}$ a combinatorial building set in $\mathcal{L}$, define the algebra $D(\mathcal{L},\mathcal{G})$ as the quotient of a polynomial algebra over $\mathbb{Z}$ with generators in 1-1 correspondence with the elements of $\mathcal{G}$:

$$D(\mathcal{L},\mathcal{G}) \;:=\; \mathbb{Z}\left[\{x_G\}_{G\in\mathcal{G}}\right] \Big/ \mathcal{I}\,,$$

where the ideal of relations $\mathcal{I}$ is generated by

$$\prod_{i=1}^{t} x_{G_i}\,, \qquad \text{for } \{G_1,\ldots,G_t\} \text{ not nested}\,,$$

$$\sum_{G\geq H} x_G\,, \qquad \text{for } H\in\mathfrak{A}\,.$$

For $\mathcal{L}$ the intersection lattice of an arrangement of complex hyperplanes $\mathcal{A}$ and $\mathcal{G}$ its minimal building set, this algebra was shown to be isomorphic to the integer cohomology algebra of the compact wonderful arrangement model in [DP4, 1.1]. We show that the algebra in fact is isomorphic to the cohomology algebra of the arrangement model for *any* choice of a building set in the intersection lattice.

Going beyond the arrangement context, we can provide yet another geometric interpretation of the algebras $D(\mathcal{L},\mathcal{G})$: For an arbitrary atomic lattice and a given combinatorial building set we construct a smooth, non-compact toric variety $X_{\Sigma(\mathcal{L},\mathcal{G})}$ and show that its Chow ring is isomorphic to the algebra $D(\mathcal{L},\mathcal{G})$.

In a sense, this is a prototype result of what we had hoped for when working on our combinatorial framework: to provide the outset for going beyond the geometric context of resolutions and yet get back to it in a different, elucidating, and, other than via the abstract combinatorial detour, seemingly unrelated way.

CHAPTER 2

CHOW RINGS OF TORIC VARIETIES DEFINED BY ATOMIC LATTICES

2.1 INTRODUCTION

In this chapter we study a graded algebra $D = D(\mathcal{L},\mathcal{G})$ over $\mathbb{Z}$ that is defined by a finite lattice $\mathcal{L}$ and a special subset, a so-called building set, $\mathcal{G}$ in $\mathcal{L}$. The definition of this algebra is inspired by a presentation for the cohomology of arrangement compactifications as it appears in work of De Concini and Procesi [DP4].

In [DP3, DP4] the authors studied a compactification of the complement of subspaces in a projective space defined by a building set in the intersection lattice $\mathcal{L}$ of the subspaces. In particular they gave a description of the cohomology algebra H^* of this compactification in terms of generators and relations. In general, the set of defining relations for H^* is much larger than the one we propose for D. However, in the case of all subspaces being of codimension 1 and $\mathcal{G}$ the set of irreducibles in $\mathcal{L}$, the former can be reduced to the latter [DP4, Prop. 1.1]. We show that this reduction holds for arbitrary building sets in $\mathcal{L}$, thus giving a first geometric interpretation of the algebra $D(\mathcal{L},\mathcal{G})$ (compare Corollary 2.4.3).

Our first result about D is that for an arbitrary atomic lattice $\mathcal{L}$ a larger set of relations, similar to the defining relations of H^*, holds in D. To define the new relations for arbitrary lattices beyond the geometric context of arrangements, we need to introduce a special metric on the chains of $\mathcal{L}$. In fact, this new set of relations forms a Gröbner basis of the relation ideal which allows us to define a basis of D over $\mathbb{Z}$ generalizing the basis defined in [Y] and [Ga1].

Our main result about D motivating its definition is Theorem 2.5.4 which asserts that D is naturally isomorphic to the Chow ring of a smooth toric variety $X = X_{\Sigma(\mathcal{L},\mathcal{G})}$ constructed from an atomic lattice $\mathcal{L}$ and a building set $\mathcal{G}$ in $\mathcal{L}$. This result gives a second geometric interpretation of D, this time for arbitrary atomic lattices. We introduce the toric variety X by means of its polyhedral fan $\Sigma(\mathcal{L},\mathcal{G})$ that we build directly from $\mathcal{L}$ and $\mathcal{G}$. Then we give a more geometric construction of X as the result of several toric blowups of an affine complex space and subsequent removal of certain open torus orbits.

The chapter is organized as follows. In section 2, we recall the necessary combinatorial definitions and define the algebra $D = D(\mathcal{L},\mathcal{G})$. In section 3, we extend the set of relations for D to a Gröbner basis of the relation ideal and exhibit a basis of the algebra. In section 4, we review the De Concini-Procesi compactifications of arrangement complements and relate D to their cohomology algebras. Also we give some examples of the Poincaré series of these compactifications using our basis. Section 5 is devoted to the definition of the toric variety X from a pair $(\mathcal{L}, \mathcal{G})$. We prove our main theorem asserting that D is naturally isomorphic to the Chow ring of X. In section 6, we give another construction of X as the result of a series of toric blowups and subsequent removal of some open orbits. Finally, in section 7, we consider a couple of simple examples.

2.2 THE ALGEBRA $D(\mathcal{L}, \mathcal{G})$

We start with defining some lattice-theoretic notions, *building sets* and *nested sets*, that provide the combinatorial essence for our algebra definition below. These notions, in the special case of intersection lattices of subspace arrangements, are crucial for the arrangement model construction of De Concini and Procesi [DP3]. For our purpose, we choose to present purely order-theoretic generalizations of their notions that previously appeared in [FK1].

By a lattice, in this chapter, we mean a finite partially ordered set all of whose subsets have a least upper bound (join, $\vee$) and a greatest lower bound (meet, $\wedge$). The least element of any lattice is denoted by $\hat{0}$. For any subset $\mathcal{G}$ of a lattice $\mathcal{L}$ we denote by $\max \mathcal{G}$ the set of maximal elements of $\mathcal{G}$. Also, for any $X \in \mathcal{L}$ we put $\mathcal{G}_{\leq X} = \{G \in \mathcal{G} \mid G \leq X\}$, similarly for $\mathcal{G}_{\geq}$. To denote intervals in $\mathcal{L}$ we use the notation $[X, Y] := \{Z \in \mathcal{L} \mid X \leq Z \leq Y\}$ for $X, Y \in \mathcal{L}$.

Definition 2.2.1 Let $\mathcal{L}$ be a finite lattice. A subset $\mathcal{G}$ in $\mathcal{L} \setminus \{\hat{0}\}$ is called a *building set* in $\mathcal{L}$ if for any $X \in \mathcal{L} \setminus \{\hat{0}\}$ and $\max \mathcal{G}_{\leq X} = \{G_1, \ldots, G_k\}$ there is an isomorphism of posets

$$\varphi_X : \prod_{i=1}^{k} [\hat{0}, G_i] \xrightarrow{\cong} [\hat{0}, X]$$

with $\varphi_X(\hat{0}, \ldots, G_i, \ldots, \hat{0}) = G_i$ for $i = 1, \ldots, k$. We call $\max \mathcal{G}_{\leq X}$ the *set of factors* of X in $\mathcal{G}$.

As a first easy example one can take the maximal building set $\mathcal{L} \setminus \{\hat{0}\}$. Looking at the other extreme, the elements $X \in \mathcal{L} \setminus \{\hat{0}\}$ for which $[\hat{0}, X]$ does *not* decompose as a direct product, so-called *irreducibles* in $\mathcal{L}$, form the minimal building set in a given lattice $\mathcal{L}$.

The choice of a building set $\mathcal{G}$ in $\mathcal{L}$ gives rise to a family of *nested sets*. Roughly speaking these are the subsets of $\mathcal{L}$ whose antichains are sets of factors with respect to the building set $\mathcal{G}$. The precise definition is as follows.

Definition 2.2.2 Let $\mathcal{L}$ be a finite lattice and $\mathcal{G}$ a building set in $\mathcal{L}$. A subset $\mathcal{S}$ in $\mathcal{G}$ is called *nested* if, for any set of pairwise incomparable elements $G_1,\dots,G_t$ in $\mathcal{S}$ of cardinality at least two, the join $G_1 \vee \dots \vee G_t$ does not belong to $\mathcal{G}$. The nested sets in $\mathcal{G}$ form an abstract simplicial complex, the *simplicial complex of nested sets* $\mathcal{N}(\mathcal{L},\mathcal{G})$.

For the maximal building set $\mathcal{G}=\mathcal{L}\backslash\{\hat{0}\}$ the nested set complex coincides with the order complex of $\mathcal{L}\backslash\{\hat{0}\}$. Smaller building sets yield nested set complexes with fewer vertices, but allow for more dense collections of simplexes.

An important property of a nested set is that for any two distinct maximal elements X and Y we have $X\wedge Y=\hat{0}$ (see [FK1, Prop. 2.5(1), 2.8(2)]).

We now have all notions at hand to define the main character of this chapter.

Definition 2.2.3 Let $\mathcal{L}$ be a finite lattice, $\mathfrak{A}(\mathcal{L})$ its set of atoms, and $\mathcal{G}$ a building set in $\mathcal{L}$. We define the algebra $D(\mathcal{L},\mathcal{G})$ of $\mathcal{L}$ with respect to $\mathcal{G}$ as

$$D(\mathcal{L},\mathcal{G}) \;:=\; \mathbb{Z}\,[\{x_G\}_{G\in\mathcal{G}}]\Big/\mathcal{I}\,,$$

where the ideal $\mathcal{I}$ of relations is generated by

$$\prod_{i=1}^{t} x_{G_i} \qquad\qquad \text{for } \{G_1,\dots,G_t\} \notin \mathcal{N}(\mathcal{L},\mathcal{G})\,, \tag{2.2.1}$$

and

$$\sum_{G\geq H} x_G \qquad\qquad \text{for } H\in\mathfrak{A}(\mathcal{L})\,. \tag{2.2.2}$$

Note that the algebra $D(\mathcal{L},\mathcal{G})$ is a quotient of the face algebra of the simplicial complex $\mathcal{N}(\mathcal{L},\mathcal{G})$. Although D is defined for an arbitrary lattice our main constructions and results make sense only for atomic lattices, i.e., lattices in which any element is the join of some atoms. Thus we will restrict our considerations to this case.

In the special case of $\mathcal{L}$ being the intersection lattice of an arrangement of complex linear hyperplanes and $\mathcal{G}$ being the minimal building set in $\mathcal{L}$, this algebra appears in work of De Concini and Procesi [DP4]. It is the cohomology algebra of a compactification of the projectivized arrangement complement; for details we refer to section 2.4.

2.3 GRÖBNER BASIS

The set of generators of the ideal $\mathcal{I}$ in Definition 2.2.3, while being elegant, is too small for being a Gröbner basis of this ideal. In this section, we extend this set to a Gröbner basis. In particular, we will obtain a $\mathbb{Z}$-basis of $D(\mathcal{L},\mathcal{G})$.

To define the larger set of relations we need to introduce a metric on chains in $\mathcal{L}$.

Definition 2.3.1 Let $\mathcal{L}$ be an atomic lattice and $X,Y\in\mathcal{L}$ with $X\leq Y$. We denote by $d(X,Y)$ the minimal number of atoms $H_1,\dots,H_d$ in $\mathcal{L}$ such that $Y=X\vee\bigvee_{i=1}^{d}H_i$.

The following four properties of the function d are immediate:

(i) $d(X,Z)\geq d(Y,Z)$ for $X,Y,Z\in\mathcal{L}$ with $X\leq Y\leq Z$. Notice that equality is possible even if all three X,Y, and Z are distinct. Also it is not necessarily true that $d(X,Y)\leq d(X,Z)$.

(ii) $d(X,Y)+d(Y,Z)\geq d(X,Z)$ for $X,Y,Z\in\mathcal{L}$ with $X\leq Y\leq Z$.

(iii) $d(X\vee Z,Y\vee Z)\leq d(X,Y)$ for $X\leq Y\in\mathcal{L}$ and $Z\in\mathcal{L}$ arbitrary.

(iv) $d(A,A\vee B)\leq d(A\wedge B,B)$ for $A,B\in\mathcal{L}$.

For example, (iv) follows from the fact that if $(A\wedge B)\vee\bigvee_i H_i=B$ for some atoms $H_1,\dots,H_d$ then $A\vee\bigvee_i H_i=A\vee B$. If $\mathcal{L}$ is geometric (for instance, the intersection lattice of a hyperplane arrangement) then $d(X,Y)=\mathrm{rk}Y-\mathrm{rk}X$ whence in (ii) equality holds and (iv) is the semimodular inequality.

Now we can introduce the new set of generators for $\mathcal{I}$. The new relations are analogous to the defining relations for the cohomology algebra of the compactification of the complement of an arrangement of projective subspaces described in [DP3].

Theorem 2.3.2 *The ideal of relations $\mathcal{I}$ in Definition 2.2.3 is generated by polynomials of the following type:*

$$h_{\mathcal{S}}=\prod_{G\in\mathcal{S}}x_G \qquad \text{for } \mathcal{S}\notin\mathcal{N}(\mathcal{L},\mathcal{G}), \tag{2.3.1}$$

$$g_{\mathcal{H},B}=\prod_{i=1}^{k}x_{A_i}\Big(\sum_{G\geq B}x_G\Big)^{d}, \tag{2.3.2}$$

where $A_1,\dots,A_k$ are maximal elements in a nested set $\mathcal{H}\in\mathcal{N}(\mathcal{L},\mathcal{G})$, $B\in\mathcal{G}$ with $B>A=\bigvee_{i=1}^{k}A_i$, and $d=d(A,B)$.

Proof. First notice that polynomials (2.2.1) and (2.2.2) are among polynomials $h_{\mathcal{S}}$ and $g_{\mathcal{H},B}$. (To see that polynomials (2.2.2) are among $g_{\mathcal{H},B}$ choose $\mathcal{H} = \emptyset$, and $B = H \in \mathfrak{A}(\mathcal{L})$. Here and everywhere we use the usual agreement that the join of the empty set is $\hat{0}$.) Hence it is left to show that any $g_{\mathcal{H},B}$ is in $\mathcal{I}$, i.e., it is a combination of polynomials (2.2.1) and (2.2.2).

We prove our claim by induction on d.

<u>$d = 1$.</u> Choose an atom H of $\mathcal{L}$ with $H \vee A = B$. Then using (2.2.2) we have

$$\prod_{i=1}^{k} x_{A_i} \Big(\sum_{G \geq H} x_G \Big) \in \mathcal{I}. \tag{2.3.3}$$

We want to show that for any $G \geq H$, $\{G, A_1, \ldots, A_k\} \in \mathcal{N} = \mathcal{N}(\mathcal{L}, \mathcal{G})$ implies that $G \geq B$. Then, any summand with $G \not\geq B$ can be omitted from (2.3.3) using polynomials (2.2.1), and we obtain $g_{\mathcal{H},B} \in \mathcal{I}$ for $d = 1$.

First note that G cannot be smaller than or equal to any of the A_i, $i = 1, \ldots, k$, since $G \leq A_i$ would imply $H \leq A_i$ contradicting the choice of H.

Assume that G is incomparable with $A_1, \ldots, A_s$ for some $s \geq 1$, and $G \geq A_i$ for $i = s{+}1, \ldots, k$. Since $\{G, A_1, \ldots, A_k\} \in \mathcal{N}$ these elements are the factors of the $\mathcal{G}$-decomposition in

$$\widetilde{G} \; := \; G \vee \bigvee_{i=1}^{s} A_i \; = \; G \vee \bigvee_{i=1}^{k} A_i \; \geq \; H \vee \bigvee_{i=1}^{k} A_i \; = \; B\,.$$

Since $B \in \mathcal{G}$, the elements A_i, $i = 1, \ldots, s$, are not maximal in $\mathcal{G}$ below $\widetilde{G}$, which contradicts the A_i being factors of $\widetilde{G}$.

We conclude that G is comparable with, i.e., larger than all A_i whence $G \geq \bigvee_{i=1}^{k} A_i \vee H = B$.

<u>$d > 1$.</u> Choose an atom H of $\mathcal{L}$ from the set of atoms in the definition of $d(A, B)$. Then $A < A \vee H < B$. Using (2.2.2) we have

$$\prod_{i=1}^{k} x_{A_i} \Big(\sum_{G \geq H} x_G \Big) \Big(\sum_{G \geq B} x_G \Big)^{d-1} \in \mathcal{I}. \tag{2.3.4}$$

We show, using polynomials (2.2.1) and (2.2.2) and the induction hypothesis, that any G with $G \not\geq B$ can be omitted from the first sum modulo $\mathcal{I}$.

Let $G_0 \in \mathcal{G}$, $G_0 \geq H$ but $G_0 \not\geq B$. Using polynomials (2.2.1) we can assume that $\{G_0, A_1, \ldots, A_k\} \in \mathcal{N}$. Due to the choice of H, G_0 cannot be smaller than any of the A_i. Further note that if G_0 is incomparable with say $A_1, \ldots A_s$, $s \leq k$, then it is incomparable also with all $A_1, \ldots, A_k$. Indeed the join $G_0 \vee A_1 \vee \ldots \vee A_s = G_0 \vee A_1 \vee \ldots \vee A_k$ is a $\mathcal{G}$-decomposition. Hence the two following cases remain to be considered.

Case 1. *G_0 is comparable with all A_i, $i = 1, \ldots, k$, hence $G_0 \geq A$.*
Our goal is to rewrite

$$x_{G_0} \Big(\sum_{G \geq B} x_G \Big)^{d-1} \tag{2.3.5}$$

modulo $\mathcal{I}$ so that it contains an expression of the form (2.3.2) with exponent $< d$ as a factor. First observe that $G_0 \vee B \in \mathcal{G}$ since $G_0, B \in \mathcal{G}$ but $H < G_0 \wedge B$ [DP3, Thm. 2.3, 3b']. The building set element $G_0 \vee B$ is to take the role of B in (2.3.2).

Let $G \in \mathcal{G}$ with $G \geq B$. We want to show that any G with $G \not\geq G_0 \vee B$ can be omitted from (2.3.5) modulo $\mathcal{I}$. We can assume that $\{G, G_0\} \in \mathcal{N}$. If $G \leq G_0$ then $B \leq G_0$, contradicting the choice of G_0. If G and G_0 were incomparable then $G \vee G_0 \notin \mathcal{G}$ contradicting the fact that they both are greater than H. Hence $G \geq G_0$ and thus $G \geq G_0 \vee B$.

Thus (2.3.5) reduces to

$$x_{G_0} \Big(\sum_{G \geq G_0 \vee B} x_G \Big)^{d-1} . \tag{2.3.6}$$

Using properties (iv) and (i) of our metric d we obtain

$$d(G_0, G_0 \vee B) \leq d(G_0 \wedge B, B) \leq d(A \vee H, B) < d\,. \tag{2.3.7}$$

Hence (2.3.6) contains a polynomial of the form (2.3.2) with exponent $< d$ as a factor whence it lies in $\mathcal{I}$ by induction hypothesis.
Case 2. *G_0 is incomparable with $A_1, \ldots, A_k$.*
Since $\{G_0, A_1, \ldots A_k\} \in \mathcal{N}$ we have $\widetilde{G}_0 := G_0 \vee A_1 \vee \ldots \vee A_k \notin \mathcal{G}$. We want to rewrite

$$\Big(\prod_{i=1}^{k} x_{A_i} \Big) \, x_{G_0} \Big(\sum_{G \geq B} x_G \Big)^{d-1} \tag{2.3.8}$$

modulo $\mathcal{I}$ so that it contains a polynomial of the form (2.3.2) with exponent $< d$ as a factor.

Observe that $\widetilde{G}_0 \vee B = G_0 \vee B$, and, as in Case 1, $G_0 \vee B \in \mathcal{G}$. This time, $\widetilde{G}_0 \vee B = G_0 \vee B$ is to take the role of B, and $\widetilde{G}_0$ the role of A in (2.3.2).

As in Case 1, we see that

$$\Big(\prod_{i=1}^{k} x_{A_i} \Big) \, x_{G_0} \Big(\sum_{G \geq B} x_G \Big)^{d-1} \equiv \Big(\prod_{i=1}^{k} x_{A_i} \Big) \, x_{G_0} \Big(\sum_{G \geq G_0 \vee B} x_G \Big)^{d-1} \text{modulo}\, \mathcal{I},$$

arguing as before for nested pairs $\{G, G_0\}$.

Now the right hand side has a factor of the form (2.3.2) with exponent $< d$ because again

$$d(\widetilde{G}_0, \widetilde{G}_0 \vee B) \leq d(\widetilde{G}_0 \wedge B, B) \leq d(B, A \vee H) < d\,.$$

This implies that the right hand side lies in $\mathcal{I}$ by induction hypothesis which completes the proof. □

The main feature of the new generating set is that it is a Gröbner basis of $\mathcal{I}$. As the main reference for Gröbner bases we use [Ei]. Fix a linear order on $\mathcal{G}$ that refines the reverse of the partial order on $\mathcal{L}$. It defines a lexicographic order on the monomials which we use in the following theorem.

Theorem 2.3.3 *The generating system (2.3.1) and (2.3.2) is a Gröbner basis of* $\mathcal{I}$.

Proof. To prove that a set of monic polynomials is a Gröbner basis for the ideal it generates it suffices to consider all pairs of their initial monomials with a common indeterminate, compute their syzygies, and show that these syzygies have standard expressions in generators (without remainders). We will prove this by a straightforward calculation. To make the calculation easier to follow we will use several agreements. For any polynomial $p \in \mathcal{I}$ we will be dealing with, we will exhibit a generator g whose initial monomial $in(g)$ divides a monomial μ of p and call $p - c(\mu)\frac{\mu}{in(g)} g$ the reduction of p by g (here $c(\mu)$ is the coefficient of μ in p). Reducing a polynomial all the way to 0 gives a standard expression for it. Also since reduction by monomial generators is very simple we will not name specific generators of the form $h_{\mathcal{S}}$ but just call this reduction h-equivalence.

We use certain new notation in the proof. For each $\mathcal{S} \subset \mathcal{G}$ put $\pi_{\mathcal{S}} = \prod_{A \in \mathcal{S}} x_A$ and for any $B \in \mathcal{G}$ put $y_B = \sum_{Y \in \mathcal{G}_{>B}} x_Y$.

Now we consider pairs (g_1, g_2) of generators of $\mathcal{I}$ of several types.

1. At least one of the generators is $h_{\mathcal{S}}$. If they both are of this type then the syzygy is 0. If the other one is $g_{\mathcal{H},B}$ with $B \notin \mathcal{S}$ then the syzygy is divisible by $h_{\mathcal{S}}$ whence h-equivalent to 0. Finally if $B \in \mathcal{S}$ then the only nontrivial case is where $T = (\mathcal{S} \cup \mathcal{H}) \setminus \{B\} \in \mathcal{N} = \mathcal{N}(\mathcal{G}, \mathcal{L})$. Notice that then $\mathcal{S} \cup \mathcal{H} \notin \mathcal{N}$. The syzygy is h-equivalent to $\pi_T y_B^{d(A,B)}$ where $A = \bigvee_{X \in \mathcal{H}} X$ as usual. Put $\bar{A} = \bigvee_{X \in T} X$. If $X \in \mathcal{G}_{>B}$ and $X \leq \bar{A} \vee B$ then X cannot form a nested set with T. Indeed, if it did then $\bar{A} \vee B = X \vee \bar{A} \notin \mathcal{G}$ contradicting $T \cup \{X\} \notin \mathcal{N}$. Similarly, if $X \in \mathcal{G}_{>B}$ and X is incomparable with $\bar{A}$ then X cannot form a nested set with T. Indeed if they did then $X \vee (\bar{A} \vee B) = X \vee \bar{A} \notin \mathcal{N}$ implying that X forms a nested set with $\mathcal{S} \cup \mathcal{H}$. This would contradict $X > B$.

Now using property (i) of the metric d we can reduce the syzygy to 0 by $g_{T, \bar{A} \vee B}$.

For the rest of the proof we need to consider only pairs with $g_i = g_{\mathcal{H}_i,B_i}$ $(i=1,2)$. We denote the exponent of $x_{B_i}+y_{B_i}$ in g_i by d_i.

2. Suppose $B_1 \neq B_2$ and $B_i \notin \mathcal{H}_j$. In this case the syzygy is

$$\pi_{\mathcal{H}_2\setminus\mathcal{H}_1} g_1(g_1 - in(g_1)) - \pi_{\mathcal{H}_1\setminus\mathcal{H}_2} g_2(g_2 - in(g_2))$$

and this is in fact a standard expression for it. (Here and to the end of the proof we use $\pi_{\mathcal{S}}$ for arbitrary subsets $\mathcal{S}$ of $\mathcal{L}$ meaning that if $\mathcal{S}$ is not nested the product is h-equivalent to 0.)

3. Suppose $B_1 = B_2 = B$ and $d = d_2 - d_2 \geq 0$. Then the syzygy is

$$\pi_{\mathcal{H}_1\cup\mathcal{H}_2}\,[\,x_B^d(x_B+y_B)^{d_1} - (x_B+y_B)^{d_2}\,]$$

and it reduces to 0 by g_1.

4. At last, suppose $B_1 \in \mathcal{H}_2$. Put $\mathcal{H} = (\mathcal{H}_1 \cup \mathcal{H}_2)\setminus\{B_1\}$ and $x_{B_i} = x_i$, $y_{B_i} = y_i$. Then the syzygy is

$$s = \pi_{\mathcal{H}}\,[\,(x_1+y_1)^{d_1}x_2^{d_2} - x_1^{d_1}(x_2+y_2)^{d_2}\,]\,.$$

Adding to s the polynomial $f = \pi_{\mathcal{H}}(x_1+y_1)^{d_1}[(x_2+y_2)^{d_2} - x_2^{d_2}]$ we obtain

$$s' = s + f = \pi_{\mathcal{H}}\,[\,(x_1+y_1)^{d_1} - x_1^{d_1}\,]\,(x_2+y_2)^{d_2}\,.$$

Notice that f is divisible by g_1 and $in(f) \leq in(s)$. Thus it suffices to reduce s' to 0. Also by g_2 we can immediately reduce s' to

$$s'' = \pi_{\mathcal{H}}\,y_1^{d_1}(x_2+y_2)^{d_2}\,.$$

For the next steps we sort out summands of y_1. Using property (i) of the metric d we can delete the summands x_Y with $B_1 < Y < B_2$ reducing by $g_{\mathcal{H}\cup\{Y\},B_2}$. The sum of all summands x_Y with $Y \geq B_2$ forms $\pi_{\mathcal{H}}(x_2+y_2)^{d_1+d_2}$ that reduces to 0 by $g_{\mathcal{H},B_2}$. Indeed, denote the join of $\mathcal{H}_i$ by C_i and the join of $\mathcal{H}_2\setminus\{B_1\}$ by C_2'. This gives the join of $\mathcal{H}$ as $C_1 \vee C_2'$. Then, using properties (ii) and (iii) of the metric d, we have

$$\begin{aligned} d(C_1\vee C_2', B_2) &\leq d(C_1\vee C_2', B_1\vee C_2') + d(B_1\vee C_2', B_2) \\ &\leq d(C_1,B_1) + d(C_2,B_2) = d_1 + d_2\,. \end{aligned}$$

After the reductions in the previous paragraph we are left with a sum each summand of which is divisible by a polynomial

$$t_Z = \pi_{\mathcal{H}} x_Z (x_2+y_2)^{d_2}\,,$$

where $Z \in \mathcal{G}_{>B_1}$, Z is incomparable with B_2, and $\mathcal{H} \cup \{Z\} \in \mathcal{N}$. To reduce this polynomial we sort out the summands in the second sum. If $Y \in \mathcal{G}_{\geq B_2}$ is not greater than or equal to $Z \vee B_2$ then it is incomparable with Z whence $\{Z, Y\} \notin \mathcal{N}$ since $B_1 < Z, Y$. This implies that t_Z is h-equivalent to

$$t'_Z = \pi_{\mathcal{H}} x_Z (\sum_{Y \geq B_2 \vee Z} x_Y)^{d_2}.$$

Finally t'_Z reduces to 0 by $g_{\mathcal{H} \cup \{Z\}, B_2 \vee Z}$ since, by property (iii) of the metric d, we have

$$d(C'_2 \vee Z, B_2 \vee Z) = d(C_2 \vee Z, B_2 \vee Z) \leq d(C_2, B_2) = d_2.$$

This reduction completes the proof. □

Corollary 2.3.4 *The following monomials form a $\mathbb{Z}$-basis of $D(\mathcal{L}, \mathcal{G})$:*

$$\prod_{A \in \mathcal{S}} x_A^{m(A)},$$

where $\mathcal{S}$ is running over all nested subsets of $\mathcal{G}$ and $m(A) < d(A', A)$, A' being the join of $\mathcal{S} \cap \mathcal{L}_{<A}$.

If $\mathcal{L}$ is the intersection lattice of a complex central hyperplane arrangement then this basis coincides with the basis exhibited in [Y]. In the next section we will give some examples of computing the Hilbert series of the algebra using this basis.

2.4 Arrangement compactifications

As we mentioned before, for a geometric lattice the metric d defined in section 2.3 coincides with the difference of ranks. This holds in particular for the intersection lattice of a hyperplane arrangement. In this setting and for $\mathcal{G}$ being the minimal building set, the algebra $D(\mathcal{L}, \mathcal{G})$ appeared in [DP4] as the cohomology algebra of a compactification of the projectivized arrangement complement. From our work in previous sections we can conclude that for *any* building set $\mathcal{G}$ in $\mathcal{L}$ the algebra $D(\mathcal{L}, \mathcal{G})$ can be interpreted geometrically as the cohomology algebra of the corresponding arrangement compactification.

We first review the construction of arrangement models due to De Concini and Procesi in the special case of complex hyperplane arrangements [DP3].

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an arrangement of complex linear hyperplanes in $\mathbb{C}^d$. Factoring by $\bigcap H_i$ if needed, we can assume $\mathcal{A}$ to be essential, i.e.,

$\bigcap H_i = \{0\}$. The combinatorial data of such an arrangement is customarily recorded by its intersection lattice $\mathcal{L}(\mathcal{A})$, i.e., the poset of intersections of all subsets of hyperplanes ordered by reverse inclusion. The greatest element of $\mathcal{L}(\mathcal{A})$ is 0 and the least element is $\mathbb{C}^d$. Let $\mathcal{G} \subseteq \mathcal{L}(\mathcal{A})$ be a building set in $\mathcal{L}(\mathcal{A})$, and let us assume here that $0 \in \mathcal{G}$.

We define a map on $\mathcal{M}(\mathcal{A}) := \mathbb{C}^d \setminus \bigcup \mathcal{A}$, the complement of the arrangement,

$$\Phi : \quad \mathcal{M}(\mathcal{A}) \longrightarrow \mathbb{C}^d \times \prod_{G \in \mathcal{G}} \mathbb{P}(\mathbb{C}^d/G)\,,$$

where Φ is the natural inclusion into the first factor and the natural projection to the other factors restricted to $\mathcal{M}(\mathcal{A})$. The map Φ defines an embedding of $\mathcal{M}(\mathcal{A})$ in the right hand side space and we let $Y_{\mathcal{G}}$ denote the closure of its image. The space $Y_{\mathcal{G}}$ is a smooth algebraic variety containing $\mathcal{M}(\mathcal{A})$ as an open set. The complement $Y_{\mathcal{G}} \setminus \mathcal{M}(\mathcal{A})$ is a divisor with normal crossings with irreducible components indexed by building set elements. An intersection of several components is non-empty (moreover, transversal and irreducible) if and only if the index set is nested as a subset of $\mathcal{G}$ [DP3, 3.1,3.2].

There is a projective analogue of $Y_{\mathcal{G}}$. Consider the projectivization $\mathbb{P}\mathcal{A}$ of $\mathcal{A}$, i.e., the family of codim 1 projective spaces $\mathbb{P}H$ in $\mathbb{CP}^{d-1}$ for H in $\mathcal{A}$. The following construction yields a compactification of the complement $\mathcal{M}(\mathbb{P}\mathcal{A}) := \mathbb{CP}^{d-1} \setminus \bigcup \mathbb{P}\mathcal{A}$. The map Φ described above is $\mathbb{C}^*$-equivariant, where $\mathbb{C}^*$ acts by scalar multiplication on $\mathcal{M}(\mathcal{A})$ and on $\mathbb{C}^d$, and trivially on $\prod_{G \in \mathcal{G}} \mathbb{P}(\mathbb{C}^d/G)$. We obtain a map

$$\overline{\Phi} : \quad \mathcal{M}(\mathbb{P}\mathcal{A}) \longrightarrow \mathbb{CP}^{d-1} \times \prod_{G \in \mathcal{G}} \mathbb{P}(\mathbb{C}^d/G)\,,$$

and again take the closure of its image to define a model $\overline{Y_{\mathcal{G}}}$ for $\mathcal{M}(\mathbb{P}\mathcal{A})$. The space $\overline{Y_{\mathcal{G}}}$ is a smooth projective variety and the complement $\overline{Y_{\mathcal{G}}} \setminus \mathcal{M}(\mathbb{P}\mathcal{A})$ is a divisor with normal crossings. Irreducible components are indexed by building set elements in $\mathcal{G}^0 := \mathcal{G} \setminus \{\{0\}\}$, and intersections of irreducible components are non-empty if and only if corresponding index sets are nested in $\mathcal{G}$.

Geometrically, the arrangement models $Y_{\mathcal{G}}$ and $\overline{Y_{\mathcal{G}}}$ are related as follows. The model $Y_{\mathcal{G}}$ is the total space of a line bundle over $\overline{Y_{\mathcal{G}}}$; in fact, it is the pullback of the tautological bundle on $\mathbb{CP}^{d-1}$ along the canonical map $\overline{Y_{\mathcal{G}}} \to \mathbb{CP}^{d-1}$. In particular, $\overline{Y_{\mathcal{G}}}$ is isomorphic to the divisor in $Y_{\mathcal{G}}$ associated to 0 [DP3, 4.1].

Example 2.4.1 Let $\mathcal{A}_{n-1}$ denote the rank $n-1$ complex *braid arrangement*, i.e., the family of partial diagonals, $H_{i,j} : z_j - z_i = 0$, $1 \le i < j \le n$, in $\mathbb{C}^n$. Its intersection lattice $\mathcal{L}(\mathcal{A}_{n-1})$ equals the lattice Π_n consisting of the set partitions of $[n] := \{1, \ldots, n\}$ ordered by reverse refinement. The set $\mathcal{F}$ of partitions with exactly one block of size $\geq\ 2$ forms the minimal building set in Π_n. The De Concini-Procesi arrangement compactification $\overline{Y_{\mathcal{F}}}$ is isomorphic to the Deligne-Knudson-Mumford compactification of the moduli space $M_{0,n+1}$ of $n{+}1$-punctured complex projective lines [DP3, 4.3].

In the more general setting of affine models for complex subspace arrangements, De Concini and Procesi provide explicit presentations for the cohomology algebras of irreducible components of divisors and of their intersections in terms of generators and relations [DP3, §5]. As mentioned above, the compactification of a complex hyperplane arrangement $\overline{Y_{\mathcal{G}}}$ is isomorphic to the divisor associated with the maximal building set element in the corresponding affine model. We recall a description of its integral cohomology algebra.

Proposition 2.4.2 ([DP3, Thm. 5.2]) *Let $\mathcal{A}$ be an essential arrangement of complex hyperplanes, $\mathcal{L}=\mathcal{L}(\mathcal{A})$ its intersection lattice, and $\mathcal{G}$ a building set in $\mathcal{L}$ containing $\{0\}$. Then the integral cohomology algebra of the arrangement compactification $\overline{Y_{\mathcal{G}}}$ can be described as*

$$H^*(\overline{Y_{\mathcal{G}}}) \ \cong\ \mathbb{Z}\,[\{c_G\}_{G\in\mathcal{G}}] \Big/ \mathcal{J}\,,$$

with generators c_G, $G\in\mathcal{G}$, corresponding to the cohomology classes of irreducible components of the normal crossing divisor, thus having degree 2.
The ideal of relations $\mathcal{J}$ is generated by polynomials of the following type:

$$\prod_{i=1}^{t} c_{G_i} \qquad \text{for}\ \ \{G_1,\ldots,G_t\}\notin\mathcal{N}(\mathcal{L},\mathcal{G})\,, \tag{2.4.1}$$

$$\prod_{i=1}^{k} c_{A_i}\Big(\sum_{G\geq B} c_G\Big)^{d}\,, \tag{2.4.2}$$

where $A_1,\ldots,A_k$ are maximal elements in a nested set $\mathcal{H}\in\mathcal{N}(\mathcal{L},\mathcal{G})$, $B\in\mathcal{G}$ with $B>\bigvee_{i=1}^{k}A_i$, and $d=\operatorname{codim}_{\mathbb{C}}B-\operatorname{codim}_{\mathbb{C}}\bigvee_{i=1}^{k}A_i$.

Comparing Proposition 2.4.2 with Theorem 2.3.2, we have a generalization of Proposition 1.1 from [DP4], where only the case of $\mathcal{G}$ being the minimal building set, i.e., the set of irreducibles, is considered.

Corollary 2.4.3 *Let $\mathcal{A}$ be an essential arrangement of complex hyperplanes, $\mathcal{L}=\mathcal{L}(\mathcal{A})$ its intersection lattice, and $\mathcal{G}$ a building set in $\mathcal{L}$ containing $\{0\}$. Then the cohomology algebra of the arrangement compactification $\overline{Y_{\mathcal{G}}}$ is isomorphic to the algebra $D(\mathcal{L},\mathcal{G})$ defined in section 2.2:*

$$H^*(\overline{Y_{\mathcal{G}}}) \ \cong\ D(\mathcal{L},\mathcal{G})\,.$$

In the rest of the section we will give several examples of the Poincaré series for compactifications of hyperplane arrangement complements. This means we compute the Hilbert series of $D(\mathcal{L},\mathcal{G})$. We restrict our computations to the compactifications with $\mathcal{G}$ being the maximal building set $\mathcal{L}\setminus\{\hat{0}\}$, although they can be easily generalized to arbitrary $\mathcal{G}$.

For these examples we use the basis of $D(\mathcal{L}) = D(\mathcal{L},\mathcal{L}\setminus\{\hat{0}\})$ from Corollary 2.3.4. In the considered case the basic monomials are parametrized by certain flags in $\mathcal{L}\setminus\{\hat{0}\}$ with multiplicity assigned to their elements. The upper bounds for multiplicities allow us to write the Hilbert series of $\mathrm{D}(\mathcal{L})$ in the following form. For each sequence r of natural numbers, $r = (0{=}r_0 < r_1 < \cdots < r_k \leq \mathrm{rk}\mathcal{L})$ denote by $f_{\mathcal{L}}(r)$ the number of flags in $\mathcal{L}$ whose sequence of ranks equals r. Set $k = k(r)$ and call it the length of r. Then we have

$$H(\mathrm{D}(\mathcal{L}),t) = 1 + \sum_r \left[\prod_{i=1}^{k(r)} \frac{t(1-t)^{r_i - r_{i-1}-1}}{1-t}\right] f_{\mathcal{L}}(r).$$

Here, r runs over all sequences as above and we use the agreement $\frac{t(t-1)^0}{t-1} = 1$.

In some important cases one can give more explicit descriptions of the numbers $f_{\mathcal{L}}(r)$ whence of the Hilbert series. We consider two such cases.

Generic arrangements. For arrangements from this class, the intersection lattice $\mathcal{L}$ is defined by the number n of atoms and the rank ℓ. We use both pieces of notation: $\mathcal{L}$ and $\mathcal{L}(n,\ell)$. The number of elements of $\mathcal{L}$ of rank $\ell' < \ell$ is $\binom{n}{\ell'}$ and for every $X \in \mathcal{L}$ of rank ℓ' the lattice $\{Y \in \mathcal{L} \mid Y \geq X\}$ is isomorphic to $\mathcal{L}(n{-}\ell', \ell{-}\ell')$. This immediately implies the following formula:

$$f_{\mathcal{L}}(r) = \prod_{i=1}^{k} \binom{n - r_{i-1}}{r_i - r_{i-1}},$$

where $k = k(r)$ if $r_{k(r)} < \ell$ and $k = k(r){-}1$ otherwise. This gives

$$\begin{aligned} H(\mathrm{D}(\mathcal{L}(n,\ell)),t) \;&= \\ 1 \;+\; \sum_r \Bigg\{ &\left[1 + \frac{t(1-t)^{\ell - r_k - 1}}{1-t}\right] \prod_{i=1}^{k(r)} \frac{t(1-t)^{r_i - r_{i-1}-1}}{1-t} \binom{n - r_{i-1}}{r_i - r_{i-1}} \Bigg\}, \end{aligned}$$

where the summation now is over all r with the extra condition $r_{k(r)} < \ell$ and we again use the agreement $\frac{t(t-1)^0}{t-1} = 1$.

Braid arrangements. For the rank $n{-}1$ complex braid arrangement (compare Example 2.4.1) the intersection lattice is given by the partition lattice Π_n of set partitions of $[n] := \{1,\ldots,n\}$ ordered by reverse refinement. Observe that the rank of a partition π coincides with $n-|\pi|$ where $|\pi|$ is the number of blocks of the partition. Thus the number of elements of Π_n of rank ℓ is $p_{n-\ell}(n)$ that is the number of partitions of $[n]$ in $n-\ell$ blocks. For every $X \in \Pi_n$ of rank ℓ the lattice $\{Y \in \Pi_n \mid Y \geq X\}$ is isomorphic to $\Pi_{n-\ell}$. This immediately implies the following formulas:

$$f_{\Pi_n}(r) = \prod_{i=1}^{k(r)} p_{n-r_i}(n - r_{i-1})$$

and

$$H(\mathrm{D}(\Pi_n), t) = 1 + \sum_r \left[\prod_{i=1}^{k(r)} \frac{t(1-t)^{r_i - r_{i-1} - 1}}{1-t} p_{n-r_i}(n - r_{i-1}) \right],$$

where the summation is over all r.

2.5 THE TORIC VARIETY $X_{\Sigma(\mathcal{L},\mathcal{G})}$

In this section we present another geometric interpretation of the algebra $D(\mathcal{L},\mathcal{G})$, this time for an arbitrary atomic lattice $\mathcal{L}$. For a given building set $\mathcal{G}$ in $\mathcal{L}$ we construct a toric variety $X_{\Sigma(\mathcal{L},\mathcal{G})}$ and show that its Chow ring is isomorphic to the algebra $D(\mathcal{L},\mathcal{G})$.

Given a finite lattice $\mathcal{L}$ with set of atoms $\mathfrak{A}(\mathcal{L}) = \{\mathfrak{A}_1, \ldots, \mathfrak{A}_n\}$, we will frequently use the following notation: For $X \in \mathcal{L}$, denote the set of atoms below X by $\lfloor X \rfloor := \{A \in \mathfrak{A}(\mathcal{L}) \mid \mathfrak{X} \geq \mathfrak{A}\}$. Define characteristic vectors v_X in $\mathbb{R}^n$ for $X \in \mathcal{L}$ with coordinates

$$(v_X)_i \ := \ \begin{cases} 1 & \text{if } A_i \in \lfloor X \rfloor, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for } i = 1, \ldots, n.$$

We will consider cones spanned by these characteristic vectors. We therefore agree to denote by $V(\mathcal{S})$ the cone spanned by the vectors v_X for $X \in \mathcal{S}$, $\mathcal{S} \subseteq \mathcal{L}$.

Let $\mathcal{L}$ be a finite atomic lattice and $\mathcal{G}$ a building set in $\mathcal{L}$. We define a rational, polyhedral fan $\Sigma(\mathcal{L},\mathcal{G})$ in $\mathbb{R}^n$ by taking cones $V(\mathcal{S})$ for any nested set $\mathcal{S}$ in $\mathcal{L}$,

$$\Sigma(\mathcal{L},\mathcal{G}) \ := \ \{ V(\mathcal{S}) \mid \mathcal{S} \in \mathcal{N}(\mathcal{L},\mathcal{G}) \}. \tag{2.5.1}$$

By definition, rays in $\Sigma(\mathcal{L},\mathcal{G})$ are in 1-1 correspondence with elements in $\mathcal{G}$; the face poset of $\Sigma(\mathcal{L},\mathcal{G})$ coincides with the face poset of $\mathcal{N}(\mathcal{L},\mathcal{G})$. To specify the set of cones in $\Sigma(\mathcal{L},\mathcal{G})$ of a fixed dimension k, or nested sets in $\mathcal{G}$ with k elements, we often use the notation $\Sigma(\mathcal{L},\mathcal{G})_k$ or $\mathcal{N}(\mathcal{L},\mathcal{G})_k$, respectively.

Proposition 2.5.1 *The polyhedral fan $\Sigma(\mathcal{L},\mathcal{G})$ is unimodular.*

Proof. We need to show that for any nested set $\mathcal{S} \in \mathcal{N}(\mathcal{L}, \mathcal{G})$ the set of generating vectors for $V(\mathcal{S})$, $\{ v_X \mid X \in \mathcal{S}\}$, can be extended to a lattice basis for $\mathbb{Z}^n$. To that end, fix a linear order $\prec$ on $\mathcal{S}$ that refines the given order on $\mathcal{L}$, and write the generating vectors v_X as rows of a matrix A following this linear order. Now transform A to a matrix $\widetilde{A}$, replacing each vector v_X by the characteristic vector $v_{\widetilde{X}}$ of $\widetilde{X}$, with

$$\widetilde{X} \;=\; \bigvee_{\substack{Y \in \mathcal{S}\\ Y \preceq X}} Y\,.$$

For each X this can be done by adding rows v_Z to v_X for elements

$$Z \;\in\; \max\nolimits_{\mathcal{L}}\{\, Y \in \mathcal{S} \mid Y \prec X,\, Y \text{ incomparable to } X \text{ in } \mathcal{L}\}\,,$$

the reason being that characteristic sets of atoms for incomparable elements of a nested set are disjoint [FK1, Prop. 2.5(1), 2.8]. The matrix $\widetilde{A}$ clearly has rows with strictly increasing support, hence can be easily extended to a square matrix with determinant ± 1. The same extra rows will complete the rows of the original matrix A to a lattice basis for $\mathbb{Z}^n$. □

Remark 2.5.2 In section 2.6 we will give a more constructive description of $\Sigma(\mathcal{L}, \mathcal{G})$, picturing the fan as the result of successive stellar subdivisions of faces of the n-dimensional cone spanned by the standard lattice basis for $\mathbb{Z}^n$ and subsequent removal of faces (compare Thm. 2.6.1). From this description, unimodality of the fan will follow immediately.

Let $X_{\Sigma(\mathcal{L},\mathcal{G})}$ denote the toric variety associated with $\Sigma(\mathcal{L}, \mathcal{G})$. If there is no risk of confusion, we will abbreviate notation by using X_Σ instead. X_Σ is a smooth, non-complete, complex algebraic variety. Crucial for us will be its stratification by torus orbits $\mathcal{O}_\mathcal{S}$, in one-to-one correspondence with cones $V(\mathcal{S})$ in $\Sigma(\mathcal{L}, \mathcal{G})$, thus with nested sets $\mathcal{S}$ in $\mathcal{G}$.

The orbit closures $[\mathcal{O}_\mathcal{S}]$, $\mathcal{S} \in \mathcal{N}(\mathcal{L}, \mathcal{G})_{n-k}$, generate the Chow groups $A_k(X_\Sigma)$, $k = 0, \ldots, n$. We describe generators for the groups of relations among the $[\mathcal{O}_\mathcal{S}]$, $\mathcal{S} \in \mathcal{N}(\mathcal{L}, \mathcal{G})_{n-k}$, in $A_k(X_\Sigma)$ for later reference. This description is due to Fulton and Sturmfels [FS]. We present here a slight adaptation to our present context.

Proposition 2.5.3 ([FS, 2.1]) *The group of relations among generators* $[\mathcal{O}_\mathcal{S}]$, $\mathcal{S} \in \mathcal{N}(\mathcal{L}, \mathcal{G})_{n-k}$, *for the k-th Chow group* $A_k(X_\Sigma)$, $k=0, \ldots, n$, *is generated by relations of the form*

$$r(\mathcal{T}, b) \quad = \sum_{\substack{\mathcal{S} \supset \mathcal{T}\\ \mathcal{S} \in \mathcal{N}(\mathcal{L},\mathcal{G})_{n-k}}} < b, z_{\mathcal{S},\mathcal{T}} > \; [\mathcal{O}_\mathcal{S}]\,, \qquad (2.5.2)$$

where $\mathcal{T}$ runs over all nested sets with $n-k-1$ elements and b over a generating set for the sublattice determined by $V(\mathcal{T})^{\perp}$ in the dual lattice Hom$(\mathbb{Z}^n,\mathbb{Z})$. Here, $z_{\mathcal{S},\mathcal{T}}$ is a lattice point in $V(\mathcal{S})$ generating the (1-dimensional) lattice $\mathrm{span}(V(\mathcal{S})\cap\mathbb{Z}^n)/\mathrm{span}(V(\mathcal{T})\cap\mathbb{Z}^n)$.

Since $X_{\Sigma(\mathcal{L},\mathcal{G})}$ is non-singular, the intersection product $\cdot$ makes $\mathrm{Ch}^*(X_\Sigma)=\oplus_{k=0}^n\mathrm{Ch}^k(X_\Sigma)$ with $\mathrm{Ch}^k(X_\Sigma)=A_{n-k}(X_\Sigma)$ into a commutative graded ring, the *Chow ring* of $X_{\Sigma(\mathcal{L},\mathcal{G})}$.

Theorem 2.5.4 *Let $X_{\Sigma(\mathcal{L},\mathcal{G})}$ be the toric variety associated with a finite atomic lattice $\mathcal{L}$ and a combinatorial building set $\mathcal{G}$ in $\mathcal{L}$ as described above. Then the assignment $x_G\mapsto[\mathcal{O}_{\{G\}}]$ for $G\in\mathcal{G}$, extends to an isomorphism*

$$D(\mathcal{L},\mathcal{G})\;\cong\;\mathrm{Ch}^*(X_{\Sigma(\mathcal{L},\mathcal{G})})\,.$$

Proof. Orbit closures $[\mathcal{O}_{\{G\}}]$ in X_Σ that correspond to the rays $V(\{G\})$ in $\Sigma(\mathcal{L},\mathcal{G})$ for $G\in\mathcal{G}$, generate $\mathrm{Ch}^*(X_\Sigma)$ multiplicatively, since

$$[\mathcal{O}_{\mathcal{S}}]\;=\;[\mathcal{O}_{\{G_1\}}]\cdot\ldots\cdot[\mathcal{O}_{\{G_k\}}]$$

for $\mathcal{S}=\{G_1,\ldots,G_k\}\in\mathcal{N}(\mathcal{L},\mathcal{G})$, $\cdot$ denoting the intersection product (see [Fu, p.100]).

Moreover, relations as in $D(\mathcal{L},\mathcal{G})$ hold. Indeed, the intersection products of orbit closures corresponding to rays that do *not* span a cone in $\Sigma(\mathcal{L},\mathcal{G})$ are 0 [Fu, p.100], which is exactly the monomial relations (2.2.1) for non-nested index sets in $D(\mathcal{L},\mathcal{G})$. Relations (2.5.2) in $\mathrm{Ch}^1(X_\Sigma)=A_{n-1}(X_\Sigma)$ as described above coincide with the linear relations (2.2.2) in $D(\mathcal{L},\mathcal{G})$

$$r(\emptyset,v_A)\;=\;\sum_{G\in\mathcal{G}}<v_A,v_G>[\mathcal{O}_{\{G\}}]\;=\;\sum_{G\geq A}[\mathcal{O}_{\{G\}}]\,,\tag{2.5.3}$$

the v_A, for $A\in\mathfrak{A}(\mathcal{L})$, forming a basis for the lattice orthogonal to $V(\emptyset)=0$ in $\mathbb{Z}^n$.

Thus, sending x_G to $[\mathcal{O}_{\{G\}}]$ for $G\in\mathcal{G}$, we have a surjective ring homomorphism from $D(\mathcal{L},\mathcal{G})$ to the Chow ring of X_Σ. It remains to show that the relations (2.5.2) in $\mathrm{Ch}^*(X_\Sigma)$ follow from relations (2.5.3) in $\mathrm{Ch}^1(X_\Sigma)$, and from monomials over non-nested index sets being zero.

Let us fix some notation. For $\mathcal{T}\in\mathcal{N}(\mathcal{L},\mathcal{G})$ and $X\in\mathcal{T}$ define

$$\Delta_{\mathcal{T}}(X)\;:=\;\lfloor X\rfloor\setminus\bigcup_{\substack{Y<X\\ Y\in\mathcal{T}}}\lfloor Y\rfloor\,,$$

the set of atoms that are below X, but not below the join of all Y in $\mathcal{T}$ that are smaller than X. Observe that $\Delta_{\mathcal{T}}(X) \neq \emptyset$ for any $X \in \mathcal{T}$, since $\mathcal{T}$ is nested, and

$$\lfloor \bigvee \mathcal{T} \rfloor \;=\; \bigcup_{X \in \mathcal{T}} \Delta_{\mathcal{T}}(X)\,.$$

For $\mathcal{T} \in \mathcal{N}(\mathcal{L},\mathcal{G})_{k-1}$, $k \geq 2$, the sublattice determined by $V(\mathcal{T})^{\perp}$ in the dual lattice is generated by vectors in $\mathcal{C}_1 \cup \mathcal{C}_2$, where

$$\begin{aligned} \mathcal{C}_1 &= \{\, v_{A_i} - v_{A_j} \mid A_i, A_j \in \Delta_{\mathcal{T}}(X) \text{ for some } X \in \mathcal{T} \}\,, \\ \mathcal{C}_2 &= \{\, v_A \mid A \in \mathfrak{A}(\mathcal{L}) \setminus \lfloor \bigvee \mathcal{T} \rfloor \,\}\,. \end{aligned}$$

Observe that $\mathcal{C}_1 \cup \mathcal{C}_2$ contains $\sum_{X \in \mathcal{T}}(|\Delta_{\mathcal{T}}(X)|-1) + |\mathfrak{A}(\mathcal{L}) \setminus \lfloor \bigvee \mathcal{T} \rfloor| = |\mathfrak{A}(\mathcal{L})| - |\mathcal{T}| = \operatorname{codim} \mathfrak{V}(\mathcal{T})$ linear independent vectors, thus a basis for the sublattice determined by $V(\mathcal{T})^{\perp}$.

For $\mathcal{T} \in \mathcal{N}(\mathcal{L},\mathcal{G})_{k-1}$, $k \geq 2$, and $v_{A_i} - v_{A_j} \in \mathcal{C}_1$, the relation (2.5.2) reads as

$$\begin{aligned} & r(\mathcal{T}, v_{A_i} - v_{A_j}) \\ &\quad = \sum_{\substack{\mathcal{S} \supset \mathcal{T} \\ \mathcal{S} \in \mathcal{N}(\mathcal{L},\mathcal{G})_k}} < v_{A_i} - v_{A_j}, z_{\mathcal{S},\mathcal{T}} > \; [\mathcal{O}_{\mathcal{S}}] \\ &\quad = \sum_{\substack{Y \in \mathcal{G} \setminus \mathcal{T} \\ \mathcal{T} \cup \{Y\} \in \mathcal{N}(\mathcal{L},\mathcal{G})}} < v_{A_i} - v_{A_j}, v_Y > \; [\mathcal{O}_{\mathcal{T} \cup \{Y\}}] \\ &\quad = [\mathcal{O}_{\mathcal{T}}] \cdot \Big(\sum_{\substack{Y \in \mathcal{G} \setminus \mathcal{T}, Y \geq A_i \\ \mathcal{T} \cup \{Y\} \in \mathcal{N}(\mathcal{L},\mathcal{G})}} [\mathcal{O}_{\{Y\}}] \; - \sum_{\substack{Y \in \mathcal{G} \setminus \mathcal{T}, Y \geq A_j \\ \mathcal{T} \cup \{Y\} \in \mathcal{N}(\mathcal{L},\mathcal{G})}} [\mathcal{O}_{\{Y\}}] \; \Big)\,. \end{aligned}$$

Monomials over non-nested index sets being zero, we may drop the condition $\mathcal{T} \cup \{Y\} \in \mathcal{N}(\mathcal{L},\mathcal{G})$ in both sums. Moreover, if $Y \in \mathcal{T}$, Y either is larger than both A_i and A_j, or not larger than either of them. Thus, both sums in $r(\mathcal{T}, v_{A_i} - v_{A_j})$ are relations of type (2.5.3), hence $r(\mathcal{T}, c)$, $c \in \mathcal{C}_1$, is a consequence of relations of type (2.2.1) and (2.2.2) holding in $\mathrm{Ch}^*(X_\Sigma)$, as claimed.

For $v_A \in \mathcal{C}_2$, the reasoning is similar, but easier. Indeed

$$\begin{aligned} r(\mathcal{T}, v_A) &= \sum_{\substack{\mathcal{S} \supset \mathcal{T} \\ \mathcal{S} \in \mathcal{N}(\mathcal{L},\mathcal{G})_k}} < v_A, z_{\mathcal{S},\mathcal{T}} > \; [\mathcal{O}_{\mathcal{S}}] \\ &= \sum_{\substack{Y \in \mathcal{G} \setminus \mathcal{T} \\ \mathcal{T} \cup \{Y\} \in \mathcal{N}(\mathcal{L},\mathcal{G})}} < v_A, v_Y > \; [\mathcal{O}_{\mathcal{T} \cup \{Y\}}] \\ &= [\mathcal{O}_{\mathcal{T}}] \;\cdot\; \sum_{Y \geq A} [\mathcal{O}_{\{Y\}}]\,, \end{aligned}$$

since no $Y \in \mathcal{T}$ can be larger than A, and again, by monomials over non-nested sets being zero, the condition $\mathcal{T} \cup \{Y\} \in \mathcal{N}(\mathcal{L},\mathcal{G})$ can be dropped. This completes our proof. □

2.6 A GEOMETRIC DESCRIPTION OF $X_{\Sigma(\mathcal{L},\mathcal{G})}$

The goal of this section is to give a geometric description of the variety $X_{\Sigma(\mathcal{L},\mathcal{G})}$. For an arbitrary atomic lattice $\mathcal{L}$, we describe the toric variety $X_{\Sigma(\mathcal{L},\mathcal{G})}$ as the result of a sequence of blowups of closed torus orbits and subsequent removal of a number of open orbits. We start with a more constructive description of the fan $\Sigma(\mathcal{L},\mathcal{G})$ as the result of a sequence of stellar subdivisions and subsequent removal of a number of open cones.

We allow the same setting as for the definition of $\Sigma(\mathcal{L},\mathcal{G})$ in (2.5.1). Let $\mathcal{L}$ be a finite atomic lattice with set of atoms $\mathfrak{A}(\mathcal{L}) = \{\mathfrak{A}_1,\ldots,\mathfrak{A}_n\}$ and $\mathcal{G}$ a building set in $\mathcal{L}$.

Construction of $\Theta(\mathcal{L},\mathcal{G})$**.**
(0) Start with the fan Θ_0 given by the n-dimensional cone spanned by the coordinate vectors in $\mathbb{R}^n$ together with all its faces.
(1) Choose a linear order $\succ$ on $\mathcal{G}$ that is non-increasing with respect to the original partial order on $\mathcal{L}$, i.e., $G \leq G'$ implies $G' \succeq G$. Write $\mathcal{G} = \{G_1 \succ G_2 \succ \cdots \succ G_t\}$. Construct a fan $\widetilde{\Theta}(\mathcal{L},\mathcal{G})$ by successive barycentric stellar subdivisions in faces $V(\lfloor G_i \rfloor)$ of Θ_0 for $i = 1,\ldots,t$, introducing in each step a new ray generated by the characteristic vector v_{G_i}, $i = 1,\ldots,t$.
(2) Remove from $\widetilde{\Theta}(\mathcal{L},\mathcal{G})$ all (open) cones $V(\mathcal{T})$ with index sets of generating vectors $\mathcal{T}$ that are not nested in $\mathcal{G}$ and denote the resulting fan by $\Theta(\mathcal{L},\mathcal{G})$.

Theorem 2.6.1 *The simplicial fan* $\Theta(\mathcal{L},\mathcal{G})$ *constructed above coincides with the fan* $\Sigma(\mathcal{L},\mathcal{G})$ *defined in section 2.5.*

Proof. By construction the fans share the same generating vectors. In fact, due to the removal of cones in step (2) of the construction above, it is enough to show that for any nested set $\mathcal{S} \in \mathcal{N}(\mathcal{L},\mathcal{G})$ there exists a cone in $\widetilde{\Theta}(\mathcal{L},\mathcal{G})$ containing $V(\mathcal{S})$ as a face. Due to the recursive construction of $\widetilde{\Theta}(\mathcal{L},\mathcal{G})$ this statement reduces to the following claim.

Claim. Let $\mathcal{S} = \{X_1,\ldots,X_k\}$ be nested in $\mathcal{L}$ with respect to $\mathcal{G}$, and assume that the indexing is compatible with the linear order $\succ$ on $\mathcal{G}$, i.e., $X_1 \succ \ldots \succ X_k$. For notational convenience, extend the set by $X_{k+1} := \hat{0}$. Then any stellar subdivision in $V(\lfloor G \rfloor)$, $G \in \mathcal{G}$, during the construction of $\widetilde{\Theta}(\mathcal{L},\mathcal{G})$, for $G \succ X_i$, $G \not\succeq X_{i-1}$, $i = 1,\ldots,k+1$, retains a cone W_G with

$$V(\,\{X_1,\ldots,X_{i-1}\} \,\cup\, \lfloor X_i \rfloor \,\cup\, \ldots \,\cup\, \lfloor X_k \rfloor\,)$$

among its faces and for $G = X_i$, $i = 1, \ldots, k$, creates a cone W_{X_i} with

$$V(\{X_1, \ldots, X_i\} \cup \lfloor X_{i+1} \rfloor \cup \ldots \cup \lfloor X_k \rfloor)$$

among its faces.

Proof of the claim. Assume first that $G \succ X_i$, $G \not\succeq X_{i-1}$, for some $i \in \{1, \ldots, k+1\}$ (the second condition being empty for $k = 1$), and assume that the previous subdivision step in $V(\lfloor G' \rfloor)$, $G' \in \mathcal{G}$, has created, resp. retained a cone $W_{G'}$ with $V(\{X_1, \ldots, X_{i-1}\} \cup \lfloor X_i \rfloor \cup \ldots \cup \lfloor X_k \rfloor)$ among its faces.

If $W(G')$ does not contain $V(\lfloor G \rfloor)$, it will not be altered by stellar subdivision in $V(\lfloor G \rfloor)$. Any cone that is to be altered when subdividing $V(\lfloor G \rfloor)$ needs to be contained in star $V(\lfloor G \rfloor)$, hence among its faces needs to contain $V(\lfloor G \rfloor)$.

If $W(G')$ does contain $V(\lfloor G \rfloor)$ among its faces, choose

$$g \in \lfloor G \rfloor \setminus \bigcup_{j=i}^{k} \lfloor X_j \rfloor . \tag{2.6.1}$$

If the set was empty, we would have $\lfloor G \rfloor \subseteq \bigcup_{j \geq i} \lfloor X_j \rfloor$, in particular,

$$G \leq \bigvee_{j \geq i} X_j \leq \bigvee_{\max \mathcal{S}_{\succeq X_i}} X_j .$$

The join on the right hand side is taken over all X_j that are maximal among $X_1, X_2, \ldots, X_i$ with respect to the partial order in $\mathcal{L}$. Since these elements are pairwise incomparable and nested in $\mathcal{L}$ they are the factors of their join. This implies that $G \leq X_{j^*}$ for some $j^* \geq i$ [FK1, Prop. 2.5(i)] contradicting the fact that $G \succ X_{j^*}$.

Hence we can choose g as described in (2.6.1) and, when subdividing $V(\lfloor G \rfloor)$, we replace $W_{G'}$ by W_G by substituting the new ray $\langle v_G \rangle$ for the ray $\langle v_g \rangle$ in $W_{G'}$. Observe that $V(\{X_1, \ldots, X_{i-1}\} \cup \lfloor X_i \rfloor \cup \ldots \cup \lfloor X_k \rfloor)$ remains as a face in the newly created cone W_G.

Assume now that $G = X_i$ and again denote the cone emerging from the previous subdivision step by $W_{G'}$, assuming that it contains $V(\{X_1, \ldots, X_{i-1}\} \cup \lfloor X_i \rfloor \cup \ldots \cup \lfloor X_k \rfloor)$ among its faces. When subdividing $V(\lfloor X_i \rfloor)$ now replace $W_{G'}$ by W_{X_i} by substituting the new ray $\langle v_{X_i} \rangle$ for the generating ray associated with some

$$x_i \in \lfloor X_i \rfloor \setminus \bigcup_{j \geq i+1} \lfloor X_j \rfloor = \lfloor X_i \rfloor \setminus \bigcup_{\substack{j \geq i+1 \\ X_j < X_i}} \lfloor X_j \rfloor = \Delta_{\mathcal{S}}(X_i) ,$$

where the right hand side is non-empty as we observed before (see proof of Thm. 2.5.4).

Note that $V(\{X_1, \ldots, X_i\} \cup \lfloor X_{i+1} \rfloor \cup \ldots \cup \lfloor X_k \rfloor)$ is a face of the newly created cone W_{X_i}. This completes the proof of our claim. □

Corollary 2.6.2 *The toric variety $X_{\Sigma(\mathcal{L},\mathcal{G})}$ can be constructed as follows. Start from the toric variety associated with the n-dimensional cone spanned by the standard lattice basis in $\mathbb{Z}^n$, i.e., from $\mathbb{C}^n$ stratified by torus orbits. Perform a sequence of blowups in orbit closures associated with faces $V(\lfloor G \rfloor)$ of the standard cone for $G \in \mathcal{G}$ in some linear, non-increasing order. Remove from the resulting variety all open torus orbits that correspond to cones in $\widetilde{\Theta}(\mathcal{L},\mathcal{G})$ indexed with non-nested subsets of $\mathcal{L}$.*

It follows immediately from this description that the toric variety $X_{\Sigma(\mathcal{L},\mathcal{G})}$ is smooth.

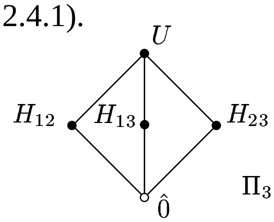
2.7 Examples

We discuss a number of examples to illustrate the central notions of this chapter.

Partition lattices.

Let Π_n denote the lattice of set partitions of $[n]$ ordered by reversed refinement. As we mentioned above, the partition lattice Π_n occurs as the intersection lattice of the braid arrangement $\mathcal{A}_{n-1}$ (compare Example 2.4.1).

For $n = 3$, the only building set is the maximal one, i.e., $\mathcal{G} = \Pi_3 \setminus \{\hat{0}\}$. Denoting elements as in the Hasse diagram depicted on the right, the nested set complex $\mathcal{N}(\Pi_3,\mathcal{G})$ contains the following simplices:



$\Pi_3 = \mathcal{L}(\mathcal{A}_2)$

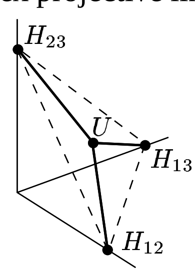
$$\mathcal{N}(\Pi_3,\mathcal{G}) = \{H_{12}, H_{13}, H_{23}, U, H_{12}U, H_{13}U, H_{23}U\} .$$

The algebra $D(\Pi_3,\mathcal{G})$ thus is the following:

$$D(\Pi_3,\mathcal{G}) = \mathbb{Z}[x_{H_{12}}, x_{H_{13}}, x_{H_{23}}, x_U] \Big/ \left\langle \begin{array}{l} x_{H_{12}}x_{H_{13}},\ x_{H_{12}}x_{H_{23}},\ x_{H_{13}}x_{H_{23}} \\ x_{H_{12}} + x_U,\ x_{H_{13}} + x_U,\ x_{H_{23}} + x_U \end{array} \right\rangle .$$

We find that $D(\Pi_3,\mathcal{G}) \cong \mathbb{Z}[x_U] / \langle x_U^2 \rangle$, which illustrates Corollary 2.4.3. The compactification $\overline{Y}_{\Pi_3 \setminus \{\hat{0}\}}$ of the complement of the projectivized braid arrangement $\mathbb{P}\mathcal{A}_2$ (a three times punctured $\mathbb{CP}^1$) is the complex projective line.

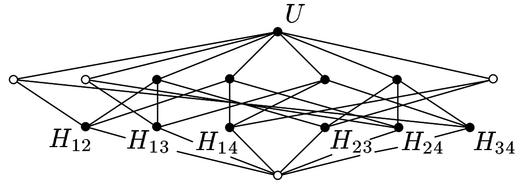
To visualize the fan $\Sigma(\Pi_3,\mathcal{G})$ we choose to depict its intersection with a hyperplane orthogonal to the diagonal ray in the positive octant of $\mathbb{R}^3$. To shorten notation, we denote rays by building set elements.



$\Sigma(\Pi_3,\mathcal{G})$

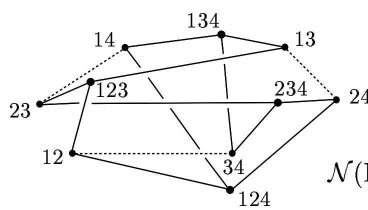
The toric variety $X_{\Sigma(\Pi_3,\mathcal{G})}$ is the blowup of $\mathbb{C}^3$ in 0 with open torus orbits corresponding to cones $V(H_{12},H_{13})$, $V(H_{12},H_{23})$, $V(H_{13},H_{23})$ and $V(H_{12},H_{13},U)$, $V(H_{12},H_{23},U)$, $V(H_{13},H_{23},U)$ removed. What we remove here, in fact, are the proper transforms of the three coordinate axes of $\mathbb{C}^3$ after blowup in 0.

For $n=4$, we have several choices when fixing a building set. The partitions with only one non-trivial block of size ≥ 2 form the minimal building set $\mathcal{F}$. To obtain the others we add any number of 2-block partitions in Π_4.



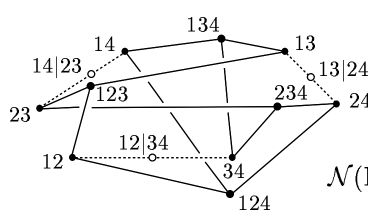
$\Pi_4 = \mathcal{L}(\mathcal{A}_3)$

The nested set complex $\mathcal{N}(\Pi_4,\mathcal{F})$ is a 2-dimensional complex on 11 vertices. It is a cone with apex U, the simplices in its base $\mathcal{N}(\Pi_4,\mathcal{F})_0$ being the ordered subsets in $\mathcal{F}\setminus\{U\}$ together with the pairs $H_{12}H_{34}$, $H_{13}H_{24}$, $H_{14}H_{23}$. We depict below the 1-dimensional base $\mathcal{N}(\Pi_4,\mathcal{F})_0$. To simplify notation we label vertices with the non-trivial block of the corresponding partition. The non-ordered nested pairs are indicated by dotted lines.



$\mathcal{N}(\Pi_4,\mathcal{F})_0$

Choosing instead of $\mathcal{F}$ the maximal building sets $\mathcal{G}$ in Π_4, i.e., including the 2-block partitions into the building set, results in a subdivision of these edges by additional vertices $H_{12|34}$, $H_{13|24}$ and $H_{14|23}$ corresponding to the newly added building set elements.



$\mathcal{N}(\Pi_4,\mathcal{G})_0$

Simplifying the presentation of the algebra $D(\Pi_4, \mathcal{F})$ given in Definition 2.2.3 yields

$$D(\Pi_4, \mathcal{F}) \cong \mathbb{Z}[x_{123}, x_{124}, x_{134}, x_{234}, x_U] \Big/ \left\langle \begin{array}{ll} x_{ijk}\, x_U & \text{for all } 1 \leq i < j < k \leq 4 \\ x_{ijk}\, x_{i'j'k'} & \text{for all } ijk \neq i'j'k' \\ x_{ijk}^2 + x_U^2 & \text{for all } 1 \leq i < j < k \leq 4 \end{array} \right\rangle ,$$

where we index generators corresponding to rank 2 lattice elements by the non-trivial block of the respective partitions. The linear basis described in Corollary 2.3.4 is given by the monomials x_{123}, x_{124}, x_{134}, x_{234}, x_U, and x_U^2.

For completeness, we state the description of $D(\Pi_n, \mathcal{F})$ for general n, where $\mathcal{F}$ again denotes the minimal building set, i.e., the set of 1-block partitions in Π_n. Having in mind that $D(\Pi_n, \mathcal{F})$ is isomorphic to the cohomology of the Deligne-Knudson-Mumford compactification $\overline{M}_{0,n+1}$ of the moduli space of $n{+}1$-punctured complex projective lines (compare Example 2.4.1), the following presentation should be compared with presentations for $H^*(\overline{M}_{0,n+1})$ given earlier by Keel [Ke].

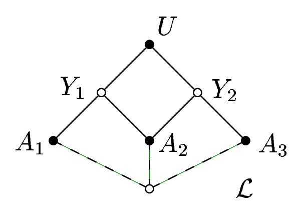
We index generators for $D(\Pi_n, \mathcal{F})$ with subsets of $[n]$ of cardinality larger than two representing the non-trivial blocks in the respective partitions and obtain:

$$D(\Pi_n, \mathcal{F}) \cong \mathbb{Z}[\{x_S\}_{S \subseteq [n], |S| \geq 2}] \Big/ \left\langle \begin{array}{ll} x_S\, x_T & \text{for } S \cap T \neq \emptyset, \\ & \text{and } S \not\subseteq T, T \not\subseteq S\,, \\ \sum_{\{i,j\} \subseteq S} x_S & \text{for } 1 \leq i < j \leq n \end{array} \right\rangle .$$

A non-geometric lattice.

Consider the lattice $\mathcal{L}$ depicted by its Hasse diagram on the right. We obtain the following building sets:

$$\begin{aligned} \mathcal{G}_1 &= \{A_1, A_2, A_3, U\}\,, \\ \mathcal{G}_2 &= \{A_1, A_2, A_3, Y_1, U\}\,, \\ \mathcal{G}_3 &= \{A_1, A_2, A_3, Y_1, Y_2, U\}\,, \end{aligned}$$



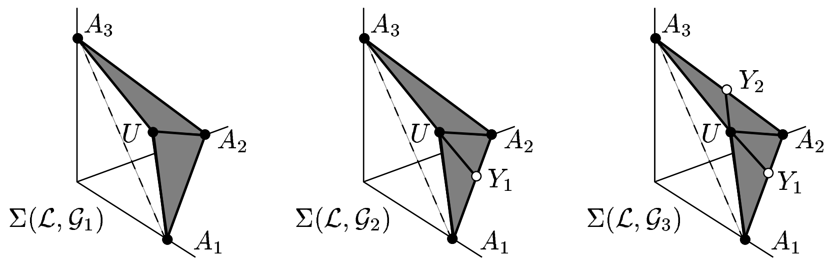
the only other choice being to replace Y_1 by Y_2 in $\mathcal{G}_2$.

For a description of the nested set complexes we refer to the corresponding fans $\Sigma(\mathcal{L}, \mathcal{G}_i)$, $i{=}1,2,3$, shown below. The standard presentations for $D(\mathcal{L}, \mathcal{G}_i)$, $i{=}1,2,3$, according to Definition 2.2.3 simplify so as to reveal the Hilbert functions of the algebras to be

$$H(D(\mathcal{L}, \mathcal{G}_i), t) = 1 + i\,t \qquad \text{for } i = 1, 2, 3\,,$$

with basis in degree 1 being the generators associated to building set elements other than atoms.

We depict the fans $\Sigma(\mathcal{L}, \mathcal{G}_i)$, $i{=}1,2,3$, again by drawing their intersections with a hyperplane orthogonal to the diagonal ray in the positive octant of $\mathbb{R}^3$.



The toric variety $X_{\Sigma(\mathcal{L},\mathcal{G}_1)}$ is the result of blowing up $\mathbb{C}^3$ in the origin, and henceforth removing the open torus orbits corresponding to one original 2-dimensional cone and the unique 3-dimensional cone containing it.

The toric varieties $X_{\Sigma(\mathcal{L},\mathcal{G}_2)}$ and $X_{\Sigma(\mathcal{L},\mathcal{G}_3)}$ differ from $X_{\Sigma(\mathcal{L},\mathcal{G}_1)}$ by blowups in one, resp. two of the original 1-dimensional torus orbits before removing open orbits as above.

CHAPTER 3

ON THE TOPOLOGY OF NESTED SET COMPLEXES

3.1 INTRODUCTION

In the same way as intersection lattices capture the combinatorial essence of hyperplane arrangements, building sets and nested set complexes encode the combinatorics of De Concini-Procesi arrangement models: They prescribe the model construction by sequences of blowups, they describe the incidence combinatorics of the divisor stratification, and they naturally appear in presentations of cohomology algebras for arrangement models in terms of generators and relations (compare [DP3]).

Nested set complexes have been defined in various generalities. The notion of nested sets goes back to the model construction for configurations spaces of algebraic varieties by Fulton & MacPherson [FuM]; the underlying poset in this special case is the lattice of set partitions. De Concini and Procesi [DP3] defined building sets and nested set complexes for intersection lattices of subspace arrangements in real or complex linear space; in this setting they have the broad geometric significance outlined above.

In my joint work with D. Kozlov [FK1], which appears as the first chapter of this thesis, we provided purely order-theoretic definitions of building sets and nested set complexes for arbitrary meet-semilattices. Together with the notion of a combinatorial blowup in a meet-semilattice, a complete combinatorial counterpart to the resolution process of De Concini and Procesi was established. These purely combinatorial notions at hand, we studied abstract algebras that generalize arrangement model cohomology in joint work with S. Yuzvinsky [FY] (appears as chapter 2 of this thesis). In this context, nested set complexes attain yet another geometric meaning as the defining data for certain toric varieties.

In this chapter, we study nested set complexes from the viewpoint of topological combinatorics. Relying on techniques from the homotopy theory of partially ordered sets due to Quillen [Q], we show that, for any building set $\mathcal{G}$ in a meet-semilattice $\mathcal{L}$, the nested set complex $\mathcal{N}(\mathcal{L},\mathcal{G})$ is homotopy equivalent to the order complex of the underlying meet-semilattice without its minimal element, $\mathcal{L}\setminus\{\hat{0}\}$.

For atomic meet-semilattices we can strengthen this result. We consider the realization of nested set complexes $\mathcal{N}(\mathcal{L},\mathcal{G})$ by simplicial fans $\Sigma(\mathcal{L},\mathcal{G})$ proposed in [FY], and we show that, for building sets $\mathcal{H} \subseteq \mathcal{G}$ in $\mathcal{L}$, the simplicial fan $\Sigma(\mathcal{L},\mathcal{G})$ is obtained from $\Sigma(\mathcal{L},\mathcal{H})$ by a sequence of stellar subdivisions. This in particular implies that, for a given atomic meet-semilattice $\mathcal{L}$, the nested set complex for *any* building set is homeomorphic to the order complex of $\mathcal{L}\setminus\{\hat{0}\}$.

After a brief review of the definitions for building sets, nested set complexes, and combinatorial blowups in Section 3.2, we present our result on the homotopy type of nested set complexes in Section 3.3. The strengthening in the case of atomic meet-semilattices is given in Section 3.4.

3.2 Preliminaries on building sets and nested sets

For the sake of completeness we here review the definitions of building sets and nested sets for finite meet-semilattices as proposed in [FK1].

All posets occurring in this chapter are finite. We mostly assume that the posets are meet-semilattices (semilattices, for short), i.e., greatest lower bounds exist for any subset of elements in the poset. Any finite meet-semilattice $\mathcal{L}$ has a minimal element, which we denote by $\hat{0}$. We frequently use the notation $\mathcal{L}_{>\hat{0}}$ to denote $\mathcal{L}$ without its minimal element. For any subset $\mathcal{S}$ in $\mathcal{L}$ we denote the set of maximal elements in $\mathcal{S}$ by $\max\mathcal{S}$. For any $X\in\mathcal{L}$, we set $\mathcal{S}_{\leq X}=\{Y\in\mathcal{S}\,|\,Y\leq X\}$, and we use the standard notation for intervals in $\mathcal{L}$, $[X,Y]:=\{Z\in\mathcal{L}\,|\,X\leq Z\leq Y\}$. The standard simplicial complex built from a poset $\mathcal{L}$ is the *order complex* of $\mathcal{L}$, which we denote by $\Delta(\mathcal{L})$; it is the abstract simplicial complex on the elements of $\mathcal{L}$ with simplices corresponding to linearly ordered subsets in $\mathcal{L}$. As a general reference on posets we refer to [St, Chapter 3].

Definition 3.2.1 Let $\mathcal{L}$ be a finite meet-semilattice. A subset $\mathcal{G}$ in $\mathcal{L}_{>\hat{0}}$ is called a *building set* if for any $X\in\mathcal{L}_{>\hat{0}}$ and $\max\mathcal{G}_{\leq X}=\{G_1,\dots,G_k\}$ there is an isomorphism of posets

$$\varphi_x:\ \prod_{j=1}^{k}\,[\hat{0},G_j]\ \stackrel{\cong}{\longrightarrow}\ [\hat{0},X] \tag{3.2.1}$$

with $\varphi_X(\hat{0},\dots,G_j,\dots,\hat{0})=G_j$ for $j=1,\dots,k$. We call $F_{\mathcal{G}}(X):=\max\mathcal{G}_{\leq \mathrm{X}}$ the *set of factors* of X in $\mathcal{G}$.

As a simple example we can take the full semilattice $\mathcal{L}_{>\hat{0}}$ as a building set. Besides this maximal building set, there is a minimal building set consisting of all elements X in $\mathcal{L}_{>\hat{0}}$ which do not allow for a product decomposition of the lower interval $[\hat{0},X]$, the so-called *irreducible elements* in $\mathcal{L}$.

Any choice of a building set $\mathcal{G}$ in $\mathcal{L}$ gives rise to a family of so-called *nested sets*. These are, roughly speaking, subsets of $\mathcal{G}$ whose antichains are sets of factors with respect to the building set $\mathcal{G}$. Nested sets form an abstract simplicial complex on the vertex set $\mathcal{G}$ – the *nested set complex*, which is the main character of this chapter.

Definition 3.2.2 Let $\mathcal{L}$ be a finite meet-semilattice and $\mathcal{G}$ a building set in $\mathcal{L}$. A subset $\mathcal{S}$ in $\mathcal{G}$ is called *nested* (or $\mathcal{G}$*-nested* if specification is needed) if, for any set of incomparable elements $X_1, \ldots, X_t$ in $\mathcal{S}$ of cardinality at least two, the join $X_1 \vee \ldots \vee X_t$ exists and does not belong to $\mathcal{G}$. The $\mathcal{G}$-nested sets form an abstract simplicial complex $\mathcal{N}(\mathcal{L}, \mathcal{G})$, the *nested set complex* with respect to $\mathcal{L}$ and $\mathcal{G}$.

For the maximal building set $\mathcal{L}_{>\hat{0}}$ in $\mathcal{L}$, the nested sets are the chains in $\mathcal{L}_{>\hat{0}}$; in particular, the nested set complex $\mathcal{N}(\mathcal{L}, \mathcal{L}_{>\hat{0}})$ coincides with the order complex $\Delta(\mathcal{L}_{>\hat{0}})$.

We also remind here a construction on semilattices that was proposed in [FK1], the combinatorial blowup of a semilattice $\mathcal{L}$ in an element X in $\mathcal{L}$.

Definition 3.2.3 For a semilattice $\mathcal{L}$ and an element X in $\mathcal{L}_{>\hat{0}}$ we define a poset $(\mathrm{Bl_X}\mathcal{L}, \prec)$ on the set of elements

$$\mathrm{Bl_X}\mathcal{L} \;=\; \{\,\mathrm{Y} \,|\, \mathrm{Y} \in \mathcal{L}, \mathrm{Y} \not\geq \mathrm{X}\} \cup \{\,\hat{\mathrm{Y}} \,|\, \mathrm{Y} \in \mathcal{L}, \mathrm{Y} \not\geq \mathrm{X},\ \text{and } \mathrm{Y} \vee \mathrm{X} \text{ exists in } \mathcal{L}\,\}\,.$$

The order relation $<$ in $\mathcal{L}$ determines the order relation $\prec$ within the two parts of $\mathrm{Bl_X}\mathcal{L}$ described above,

$$\begin{array}{ll} Y \prec Z\,, & \text{for} \quad Y < Z \ \text{ in } \mathcal{L}\,, \\ \hat{Y} \prec \hat{Z}\,, & \text{for} \quad Y < Z \ \text{ in } \mathcal{L}\,, \end{array}$$

and additional order relations between elements of these two parts are defined by

$$Y \prec \hat{Z}\,, \quad \text{for} \quad Y < Z \ \text{ in } \mathcal{L}\,,$$

where in all three cases it is assumed that $Y, Z \not\geq X$ in $\mathcal{L}$. We call $\mathrm{Bl_X}\mathcal{L}$ the *combinatorial blowup* of $\mathcal{L}$ in X.

Let us remark here that $\mathrm{Bl_X}\mathcal{L}$ is again a meet-semilattice. The combinatorial blowup of a semilattice was used in [FK1] to analyze the incidence change of strata in the construction process for De Concini-Procesi arrangement models. In the present chapter we will need combinatorial blowups to describe the incidence change in polyhedral fans under stellar subdivision following an observation in [FK1, Prop. 4.9]:

Proposition 3.2.4 *Let Σ be a polyhedral fan with face poset $\mathcal{F}(\Sigma)$. For a cone σ in Σ, the face poset of the fan obtained by stellar subdivision of Σ in σ, $\mathcal{F}(\mathrm{st}(\Sigma, \sigma))$, can be described as the combinatorial blowup of $\mathcal{F}(\Sigma)$ in σ:*

$$\mathcal{F}(\mathrm{st}(\Sigma, \sigma)) \;=\; \mathrm{Bl}_\sigma(\mathcal{F}(\Sigma))\,.$$

3.3 THE HOMOTOPY TYPE OF NESTED SET COMPLEXES

In this section, we will show that for a given meet-semilattice $\mathcal{L}$ and a building set $\mathcal{G}$ in $\mathcal{L}$ the nested set complex $\mathcal{N}(\mathcal{L},\mathcal{G})$ is homotopy equivalent to the order complex of $\mathcal{L}_{>\hat{0}}$. We will use the following two lemmata on the homotopy type of partially ordered sets going back to Quillen [Q].

Lemma 3.3.1 (Quillen's fiber lemma) *Let $f\colon P \to Q$ be a map of posets such that the order complex of $f^{-1}(Q_{\leq X})$ is contractible for all $X \in Q$, then f induces a homotopy equivalence between the order complexes of P and Q.*

Lemma 3.3.2 *Let P be a poset, and assume that there is an element $X_0\, in\, P$ such that the join $X_0 \vee X$ exists for all $X \in P$. Then the order complex of P is contractible. A poset with the property described above is called* join-contractible *via X.*

Proposition 3.3.3 *Let $\mathcal{L}$ be a finite meet-semilattice, and $\mathcal{G}$ a building set in $\mathcal{L}$. Then the nested set complex $\mathcal{N}(\mathcal{L},\mathcal{G})$ is homotopy equivalent to the order complex of $\mathcal{L}_{>\hat{0}}$,*

$$\mathcal{N}(\mathcal{L},\mathcal{G}) \;\simeq\; \Delta(\mathcal{L}_{>\hat{0}})\,.$$

Proof. We denote by $\mathcal{F}(\mathcal{N})$ the poset of non-empty faces of the nested set complex $\mathcal{N}(\mathcal{L},\mathcal{G})$. Consider the following map of posets:

$$\begin{array}{rccl} \phi\,:\, \mathcal{F}(\mathcal{N}) & \longrightarrow & \mathcal{L}_{>\hat{0}} & \\ \mathcal{S} & \longmapsto & \bigvee \mathcal{S} = \bigvee_{X\in\mathcal{S}} X\,. \end{array}$$

We claim that the order complex of $\mathcal{F}(\mathcal{N})_{\leq X} := \phi^{-1}((\mathcal{L}_{>\hat{0}})_{\leq X})$ is contractible for any $X \in \mathcal{L}_{>\hat{0}}$. An application of the Quillen fiber lemma 3.3.1 will then prove the statement of the proposition.

Case 1: $X \in \mathcal{G}$. We show that $\mathcal{F}(\mathcal{N})_{\leq X}$ is join-contractible via X and, with an application of Lemma 3.3.2, thus prove our claim. Let $\mathcal{S}$ be an element in $\mathcal{F}(\mathcal{N})_{\leq X}$, i.e., a nested set with $\bigvee \mathcal{S} \leq X$. We have to show that $\mathcal{S}\cup\{X\}$ is nested with $\bigvee \mathcal{S}\cup\{X\} \leq X$, hence $\mathcal{S}\cup\{X\} \in \mathcal{F}(\mathcal{N})_{\leq X}$. Either $\bigvee \mathcal{S} = X$, in which case $X \in \mathcal{S}$, and our claim is obvious; or $\bigvee \mathcal{S} < X$, in which case we can add X to $\mathcal{S}$, obtaining a nested set, with $\bigvee \mathcal{S}\cup\{X\} = X$, hence $\mathcal{S}\cup\{X\} \in \mathcal{F}(\mathcal{N})_{\leq X}$.

Case 2: $X \notin \mathcal{G}$. We show that $\mathcal{F}(\mathcal{N})_{\leq X}$ is join-contractible via the set of factors of X, $F_{\mathcal{G}}(X)$. Again, let $\mathcal{S}$ be a nested set with $\bigvee \mathcal{S} \leq X$; we have to show that $\mathcal{S}\cup F_{\mathcal{G}}(X)$ is nested with join less or equal X, hence $\mathcal{S}\cup F_{\mathcal{G}}(X) \in \mathcal{F}(\mathcal{N})_{\leq X}$.

If $\bigvee \mathcal{S} = X$, then $X = \bigvee \max \mathcal{S}$ and $F_{\mathcal{G}}(X) = \max \mathcal{S} \subseteq \mathcal{S}$ by [FK1, Prop. 2.8(2)], which makes our claim obvious.

For $\bigvee \mathcal{S} < X$, assume that $A \subseteq \mathcal{S} \cup F_{\mathcal{G}}(X)$ is an antichain with at least two elements, and $\bigvee A \in \mathcal{G}$. Since the $\mathcal{G}$-factors of X, $F_{\mathcal{G}}(X) = \{G_1, \dots, G_t\}$, give a partition of $\mathcal{G}_{\leq X}$ into subsets $\mathcal{G}_{\leq G_i}$, $i = 1, \dots, t$ [FK1, Prop. 2.5(1)], we find that $\bigvee A \leq G$ for some $G \in F_{\mathcal{G}}(X)$. If A contains any elements of $F_{\mathcal{G}}(X)$, then it must contain G, which contradicts A being an antichain with more than one element. We conclude that A does not contain any factors of X. In particular, it is a subset of the nested set $\mathcal{S}$, thus should have a join outside $\mathcal{G}$, and we again reach a contradiction. We conclude that $\mathcal{S} \cup F_{\mathcal{G}}(X)$ is nested with join X, hence belongs to $\mathcal{F}(\mathcal{N})_{\leq X}$. □

3.4 Simplicial fans realizing nested set complexes

We recall the definition of the simplicial fan $\Sigma(\mathcal{L}, \mathcal{G})$ for a given atomic meet-semilattice $\mathcal{L}$ and a building set $\mathcal{G}$ in $\mathcal{L}$. For details see [FY, Section 5].

Given a finite meet-semilattice $\mathcal{L}$ with set of atoms $\mathfrak{A}(\mathcal{L}) = \{A_1, \dots, A_n\}$, we will frequently use the following notation: For $X \in \mathcal{L}$, define $\lfloor X \rfloor := \{A \in \mathfrak{A}(\mathcal{L}) \mid X \geq A\}$, the set of atoms below a specific element X in $\mathcal{L}$. We define characteristic vectors v_X in $\mathbb{R}^n$ for lattice elements $X \in \mathcal{L}$ by

$$(v_X)_i \;:=\; \begin{cases} 1 & \text{if } A_i \in \lfloor X \rfloor, \\ 0 & \text{otherwise}, \end{cases} \qquad \text{for } i = 1, \dots, n.$$

These characteristic vectors will appear as spanning vectors of simplicial cones in $\mathbb{R}^n$. For a subset $\mathcal{S} \subseteq \mathcal{L}$, we agree to denote by $V(\mathcal{S})$ the cone spanned by the vectors v_X for $X \in \mathcal{S}$.

Definition 3.4.1 Let $\mathcal{L}$ be a finite atomic meet-semilattice and $\mathcal{G}$ a building set in $\mathcal{L}$. We define a rational, polyhedral fan $\Sigma(\mathcal{L}, \mathcal{G})$ in $\mathbb{R}^n$ as the collection of cones $V(\mathcal{S})$ for all nested sets $\mathcal{S}$ in $\mathcal{L}$,

$$\Sigma(\mathcal{L}, \mathcal{G}) \;:=\; \{\, V(\mathcal{S}) \mid \mathcal{S} \in \mathcal{N}(\mathcal{L}, \mathcal{G}) \,\}\,. \tag{3.4.1}$$

By definition, rays in $\Sigma(\mathcal{L}, \mathcal{G})$ are in 1-1 correspondence with elements in $\mathcal{G}$. In fact, the face poset of $\Sigma(\mathcal{L}, \mathcal{G})$ coincides with the face poset of $\mathcal{N}(\mathcal{L}, \mathcal{G})$.

If there is no risk of confusion we will denote the fan in (3.4.1) by $\Sigma(\mathcal{G})$.

Theorem 3.4.2 *Let $\mathcal{L}$ be a finite atomic meet-semilattice, and $\mathcal{G}$, $\mathcal{H}$ building sets in $\mathcal{L}$ with $\mathcal{G} \supseteq \mathcal{H}$. Then, the fan $\Sigma(\mathcal{G})$ is obtained from $\Sigma(\mathcal{H})$ by a sequence of stellar subdivisions. In particular, the supports of the fans $\Sigma(\mathcal{G})$ and $\Sigma(\mathcal{H})$ coincide.*

Proof. For building sets $\mathcal{G} \supseteq \mathcal{H}$ in $\mathcal{L}$ and G minimal in $\mathcal{G} \setminus \mathcal{H}$, set $\overline{\mathcal{G}} := \mathcal{G} \setminus \{G\}$. Obviously, $\max \overline{\mathcal{G}}_{\leq G} = F_{\mathcal{H}}(G)$, and for any $X \in \mathcal{L}$ we find that

$$\max \overline{\mathcal{G}}_{\leq X} \ = \ \begin{cases} F_{\mathcal{G}}(X) & \text{if } G \notin F_{\mathcal{G}}(X)\,, \\ (F_{\mathcal{G}}(X) \setminus \{G\}) \cup F_{\mathcal{H}}(G) & \text{if } G \in F_{\mathcal{G}}(X)\,. \end{cases}$$

Isomorphisms of posets required for the building set property of $\overline{\mathcal{G}}$ expand accordingly in the second case, and we find that $\overline{\mathcal{G}}$ is again a building set for $\mathcal{L}$.

We thus conclude that, for any two building sets $\mathcal{G}$, $\mathcal{H}$ with $\mathcal{G} \supseteq \mathcal{H}$, there is a sequence of building sets

$$\mathcal{G} \ = \ \mathcal{G}_1 \ \supseteq \ \mathcal{G}_2 \ \supseteq \ \ldots \ \supseteq \ \mathcal{G}_t \ = \ \mathcal{H}\,,$$

such that $\mathcal{G}_i$ and $\mathcal{G}_{i+1}$ differ by exactly one element G_i, and G_i is minimal in $\mathcal{G}_i \setminus \mathcal{H}$ for $i = 1, \ldots, t-1$.

We can thus assume that $\mathcal{H} = \mathcal{G} \setminus \{G\}$, and it suffices to show that $\Sigma(\mathcal{G})$ is obtained from $\Sigma(\mathcal{H})$ by a sequence of stellar subdivisions.

In fact, we claim that $\Sigma(\mathcal{G})$ is obtained by a single stellar subdivision of $\Sigma(\mathcal{H})$ in $V(F_{\mathcal{H}}(G))$, introducing a new ray that is generated by the characteristic vector v_G, i.e.,

$$\Sigma(\mathcal{G}) \ = \ \operatorname{st}(\Sigma(\mathcal{H}), V(F_{\mathcal{H}}(G)), v_G)\,. \qquad (3.4.2)$$

Observe that the two fans in (3.4.2) share the same set of generating vectors for rays, so all we have to show is that they have the same combinatorial structure, i.e., their face posets coincide.

The face poset of the subdivided fan can be described as the combinatorial blowup of the face poset $\mathcal{F}(\Sigma(\mathcal{H})) \ = \ \mathcal{F}(\mathcal{N}(\mathcal{H}))$ in the $\mathcal{H}$-nested set $F_{\mathcal{H}}(G)$ (cf. [FK1, Sect. 4.2]), hence we are left to show that

$$\mathcal{F}(\mathcal{N}(\mathcal{G})) \ = \ \mathrm{Bl}_{F_{\mathcal{H}}(G)}(\mathcal{F}(\mathcal{N}(\mathcal{H})))\,. \qquad (3.4.3)$$

Let us abbreviate notation and denote the poset on the right hand side by $\mathrm{Bl}\,\mathcal{F}$.

We first show the left-to-right inclusion in (3.4.3).
Let $\mathcal{S}$ be a $\mathcal{G}$-nested set in $\mathcal{L}$. We need to show that $\mathcal{S}$ is an element in $\mathrm{Bl}\,\mathcal{F}$. For the matter of this proof, we agree to freely switch between sets of atoms and their joins in the respective semilattices.

For $G \notin \mathcal{S}$, we note that $\mathcal{S}$ is $\mathcal{H}$-nested. Moreover, $\mathcal{S}$ does not contain $F_{\mathcal{H}}(G)$, since the latter is certainly not $\mathcal{G}$-nested. We conclude that $\mathcal{S}$ is an element in $\mathrm{Bl}\,\mathcal{F}$.

For $G \in \mathcal{S}$, we need to show that $(S \setminus \{G\}) \cup F_{\mathcal{H}}(G)$ is $\mathcal{H}$-nested. Let A be an antichain with at least two elements that is contained in $(S \setminus \{G\}) \cup F_{\mathcal{H}}(G)$; we need to see that $\bigvee A \notin \mathcal{H}$. If $A \subseteq F_{\mathcal{H}}(G)$ then clearly $\bigvee A$ either equals G or lives between G and its $\mathcal{H}$-factors, hence in any case is not contained in $\mathcal{H}$.

If A does not contain any $\mathcal{H}$-factor of G, then $A \subset \mathcal{S}$ is $\mathcal{G}$-nested, in particular $\bigvee A \notin \mathcal{H}$.

We can thus assume that the antichain A is of the form $A = \{S_1, \ldots, S_t, F_1, \ldots, F_k\}$, where $S_i \in \mathcal{S} \setminus (\{G\} \cup F_{\mathcal{H}}(G))$ for $i = 1, \ldots, t$, and $F_j \in F_{\mathcal{H}}(G)$ for $j = 1, \ldots, k$, and both types of elements occur in A.

Let us assume that $\bigvee A \in \mathcal{H}$. We have

$$\bigvee A \;\leq\; \bigvee_{i=1}^{t} S_i \;\vee\; G \;=\; \bigvee_{\substack{i\in\{1,\ldots t\},\ S_i \text{ in-}\\ \text{comparable with G}}} S_i \;\vee\; G\,, \tag{3.4.4}$$

where the last equality holds since any S_j comparable with G has to be smaller than G, otherwise $S_j \geq G > F_1$ gives a contradiction to A being an antichain.

If there are no S_i, $i \in \{1, \ldots t\}$, that are incomparable with G, the right hand side of (3.4.4) equals G. Assuming that $\bigvee A \in \mathcal{H}$ we find that $\bigvee A \leq F$ for some $F \in F_{\mathcal{H}}(G)$ since the $\mathcal{H}$-factors of G partition the elements of $\mathcal{H}$ below G [FK1, Prop. 2.5.(1)]. We assumed that A contains some of the $\mathcal{H}$-factors of G, and thus conclude that it must contain F. This however contradicts to A being an antichain with at least two elements.

We are left with the case of the join on the right hand side of (3.4.4) being taken over more than one element. Since $\mathcal{S}_0 = \{S_i \in A \mid S_i$ incomparable with $G\} \cup \{G\} \subseteq \mathcal{S}$ is a $\mathcal{G}$-nested antichain, we conclude that $\bigvee \mathcal{S}_0$ is not contained in $\mathcal{G}$ and $\mathcal{S}_0$ is its set of factors. Since these factors partition $\mathcal{G}$-elements below $\bigvee \mathcal{S}_0$ we find that either $\bigvee A \leq S_i$, for some $S_i \in \mathcal{S}_0$, which is a contradiction to A being an antichain, or $\bigvee A \leq G$, which again places $\bigvee A$ below one of the $\mathcal{H}$-factors F of G, and, as argued above, leads to a contradiction. We conclude that $(S \setminus \{G\}) \cup F_{\mathcal{H}}(G)$ is $\mathcal{H}$-nested, thus any $\mathcal{G}$-nested set $\mathcal{S}$ is an element of $\mathrm{Bl}\,\mathcal{F}$ as claimed.

Let us now turn to the right-to-left inclusion in (3.4.3).
Let $\mathcal{S} \in \mathrm{Bl}_{F_{\mathcal{H}}(G)}(\mathcal{F}(\mathcal{N}(\mathcal{H})))$, we have to show that $\mathcal{S}$, respectively the set of atoms below $\mathcal{S}$ in $\mathrm{Bl}\,\mathcal{F}$, is nested with respect to $\mathcal{G}$.

Let us first consider the case when $\mathcal{S}$ is $\mathcal{H}$-nested and does not contain $F_{\mathcal{H}}(G)$, i.e., $\mathcal{S}$ is one of the elements of the face poset $\mathcal{F}(\mathcal{N}(\mathcal{H}))$ that remain after the blowup. Assume that $\mathcal{S}$ is not $\mathcal{G}$-nested, hence there exists an antichain A in $\mathcal{S}$ with $\bigvee A \in \mathcal{G} \setminus \mathcal{H}$, i.e., $\bigvee A = G$. We conclude that A coincides with the set of $\mathcal{H}$-factors of G (cf. [FK1, Prop. 2.8.(2)]), which contradicts our assumption about $\mathcal{S}$ not containing $F_{\mathcal{H}}(G)$.

Let us now consider the remaining case, i.e., $\mathcal{S} = \mathcal{S}' \cup \{G\}$, where $\mathcal{S}'$ is $\mathcal{H}$-nested, $\mathcal{S}' \not\supseteq F_{\mathcal{H}}(G)$, and $\mathcal{S}' \cup F_{\mathcal{H}}(G)$ is $\mathcal{H}$-nested. We have to show that $\mathcal{S}$ is $\mathcal{G}$-nested.

Let A be an antichain contained in $\mathcal{S}$. If $G \notin A$, then $A \subseteq \mathcal{S}'$ and $\bigvee A \in \mathcal{G} \setminus \mathcal{H}$ implies as above that $A = F_{\mathcal{H}}(G)$ contradicting our assumptions.

If $G \in A$, then $A = A' \cup \{G\}$ where A' is an antichain in $\mathcal{S}'$. If $\bigvee A = G$, then A would not be an antichain, hence it suffices to show that $\bigvee A \notin \mathcal{H}$. Consider

$$\bigvee A \;=\; \bigvee A' \vee G \;=\; \bigvee A' \vee \bigvee F_{\mathcal{H}}(G) \;=\; \bigvee A' \vee \bigvee_{\substack{F \in F_{\mathcal{H}}(G),\ F \text{ incom-} \\ \text{parable to elements in } A'}} F\,,$$

where the last equality holds since any $\mathcal{H}$-factor F of G comparable with an element a in the antichain A' must be smaller than a, otherwise $F \geq a$ implies $G > a$ which contradicts to A being an antichain.

We find that $A' \cup \{F \in F_{\mathcal{H}}(G) \mid F$ incomparable to elements in $A'\}$ is an antichain in $\mathcal{S}' \cup F_{\mathcal{H}}(G)$. With the latter being $\mathcal{H}$-nested by assumption, we conclude that $\bigvee A \notin \mathcal{H}$ as required, which completes our proof. □

Corollary 3.4.3 *Let $\mathcal{L}$ be a finite atomic meet-semilattice, and $\mathcal{G}$ a building sets in $\mathcal{L}$. Then the nested set complex $\mathcal{N}(\mathcal{L}, \mathcal{G})$ is homeomorphic to the order complex of $\mathcal{L}_{>\hat{0}}$,*

$$\mathcal{N}(\mathcal{L}, \mathcal{G}) \;\cong\; \Delta(\mathcal{L}_{>\hat{0}})\,.$$

Proof. By Theorem 3.4.2 the simplicial fan $\Sigma(\mathcal{L}_{>\hat{0}})$ is a stellar subdivision of $\Sigma(\mathcal{G})$ for any building set $\mathcal{G}$ in $\mathcal{L}$. This in particular implies that the abstract simplicial complexes encoding the face structure of the respective fans are homeomorphic. The observation that the nested set complex for the maximal building set, $\mathcal{N}(\mathcal{L}, \mathcal{L}_{>\hat{0}})$, coincides with the order complex of $\mathcal{L}_{>\hat{0}}$ finishes our proof. □

CHAPTER 4

ABELIANIZING THE REAL PERMUTATION ACTION VIA BLOWUPS

4.1 INTRODUCTION

Our object of study is an abelianization of the $\mathcal{S}_n$ permutation action on $\mathbb{R}^n$ that is provided by a particular De Concini-Procesi wonderful model for the braid arrangement. Our motivation comes from an analogous construction for finite group actions on complex manifolds, due to Batyrev [B1, B2], and subsequent study of Borisov & Gunnells [BG], where the connection of such abelianizations with De Concini-Procesi wonderful models for arrangement complements was first observed.

Whereas previous studies were restricted to complex manifolds, here we study one of the most natural nontrivial actions of a finite group on a *real* differentiable manifold, namely the permutation action on $\mathbb{R}^n$. The locus of non-trivial stabilizers in this case is provided by the braid arrangement $\mathcal{A}_{n-1}$. We suggest to blow up intersections of subspaces in $\mathcal{A}_{n-1}$, respectively proper transforms of those intersections, in the order of an arbitrary linear extension of the intersection lattice Π_n, so as to exhaust all of the arrangement. That is the same as to take the De Concini-Procesi wonderful model of the arrangement complement with respect to the maximal building set, see [DP3].

Not only do we obtain an abelianization of the real permutation action, we even show that stabilizers of points in the arrangement model are isomorphic to direct products of $\mathbb{Z}_2$. To this end, we develop a combinatorial framework for explicitly describing the stabilizers in terms of automorphism groups of set diagrams over families of cubes.

Moreover, we observe that the natural nested set stratification on the arrangement model is not stabilizer distinguishing with respect to the $\mathcal{S}_n$-action, i.e., stabilizers of points are not in general isomorphic on open strata. Motivated by this structural deficiency, we furnish a new stratification of the De Concini-Procesi arrangement model that distinguishes stabilizers.

Arrangement models have been extensively studied over the last years. They were introduced by De Concini & Procesi in [DP3], one of the motivations being to provide rational models for cohomology algebras of arrangement complements. In [FK1] the De Concini-Procesi model construction was put in a very general combinatorial context, showing that the notions of building sets and nested sets, coined already by Fulton & MacPherson in [FuM], along with the notion of a blowup, have canonical combinatorial counterparts in the theory of semilattices. It was also shown in [FK1] that this combinatorial framework actually traces precisely the step-by-step change in the incidence structure of strata during the De Concini-Procesi resolution process.

On the geometric side, wonderful arrangement models were generalized to wonderful conical compactifications by MacPherson & Procesi [MP], and Gaiffi [Ga2] recently provided a further generalization incorporating mixed real subspace and halfspace arrangements as well as real stratified manifolds as starting points of the construction. Algebraic topological invariants of wonderful models are another focus of interest. Yuzvinsky [Y] provided a monomial basis for the cohomology of wonderful compactifications of hyperplane arrangements that was later generalized by Gaiffi to compactifications of subspace arrangements in [Ga1].

We give a more detailed outline of this chapter: In section 4.2, we begin our investigations with a brief review of De Concini-Procesi wonderful models. Moreover, we describe how an action of a finite group on an arrangement extends to an action on the arrangement model. We then turn to our specific situation, observing that when blowing up the entire locus of non-trivial stabilizers for $\mathcal{S}_n$ acting on $\mathbb{R}^n$, i.e., the entire braid arrangement, the nested set stratification is not sufficient to distinguish stabilizers. That is, we may have two points lying on the same stratum, but having non-isomorphic stabilizers. In fact, this happens already for $n=3$.

In section 4.4, we study the nested set stratification and group actions on De Concini-Procesi models in some detail, so that finally, in section 4.5, we are able to rectify the situation: We define a different stratification on the De Concini-Procesi model such that, on one hand, this stratification is naturally arrived at by tracing a certain, interesting on its own right, subspace arrangement in $\mathbb{R}^n$, on the other hand, this new stratification is *stabilizer distinguishing*.

In section 4.6 we turn to the detailed study of the isomorphism types of stabilizers of points in the De Concini-Procesi resolution of the braid arrangement. Relying on our analysis in the previous sections, we know that the stabilizer of a point in the arrangement model is the intersection of a number of stabilizers of lines and of the stabilizer of one single point in $\mathbb{R}^n$. We develop a combinatorial language to describe stabilizers of points and lines in $\mathbb{R}^n$, namely by representing them as automorphism groups of set diagrams over families of cubes. The crucial property of this representation is that taking intersections of a number of automorphism groups of such diagrams will again yield an automorphism group over a diagram. This new diagram can be combinatorially read of from the original diagrams. Thus, we succeed to represent the stabilizer of a point in the arrangement model as an automorphism group of a set diagram over a family of cubes. By further analysis of this diagram, we are finally able to prove in section 4.7 that, beyond the natural initial expectation that the stabilizers ought to be abelian, they in fact are isomorphic to direct products of $\mathbb{Z}_2$, with the number of factors in each product at most $\lfloor \frac{n}{2} \rfloor$.

4.2 De Concini-Procesi arrangement models

In this section we briefly review the construction and main characteristics of wonderful arrangement models as introduced by De Concini & Procesi in [DP3]. We first remind the notions of building sets and nested sets since they guide the explicit construction and capture the underlying incidence combinatorics of a natural stratification. Moreover, we comment on actions of finite groups on De Concini-Procesi models that are induced from group actions on the arrangement.

4.2.1 Building sets and nested sets

Let $\mathcal{A}$ be an arrangement of linear subspaces in a finite dimensional real or complex vector space, and denote by $\mathcal{L} = \mathcal{L}(\mathcal{A})$ the lattice of intersections of spaces in $\mathcal{A}$ ordered by reverse inclusion, customarily called the *intersection lattice* of $\mathcal{A}$.

Definition 4.2.1 ([DP3, §2]) For $\mathcal{L} = \mathcal{L}(\mathcal{A})$ the intersection lattice of a complex or real subspace arrangement, let $\mathcal{L}^*$ denote the lattice formed by the orthogonal complements of intersections in $\mathcal{A}$ ordered by inclusion.

(1) For $U \in \mathcal{L}^*$, $U = \oplus_{i=1}^k U_i$ with $U_i \in \mathcal{L}^*$, is called a *decomposition* of U if for any $V \subseteq U$, $V \in \mathcal{L}^*$, $V = \oplus_{i=1}^k (U_i \cap V)$ and $U_i \cap V \in \mathcal{L}^*$, for $i = 1, \ldots, k$.

(2) Call $U \in \mathcal{L}^*$ *irreducible* if it does not admit a non-trivial decomposition.

(3) $\mathcal{G} \subseteq \mathcal{L}^* \setminus \{\hat{0}\}$ is called a *building set* for $\mathcal{A}$ if for any $U \in \mathcal{L}^* \setminus \{\hat{0}\}$ and $G_1, \ldots, G_k$ maximal in $\mathcal{G}$ below U, $U = \oplus_{i=1}^k G_i$ is a decomposition (the $\mathcal{G}$-decomposition) of U.

(4) A subset $\mathcal{T} \subseteq \mathcal{G}$ is called *nested* if for any set of non-comparable elements $U_1, \ldots, U_k$ in $\mathcal{T}$, $U = \oplus_{i=1}^k U_i$ is the $\mathcal{G}$-decomposition of U. The nested sets in $\mathcal{G}$ form an abstract simplicial complex, the *nested set complex* $\mathcal{N}(\mathcal{G})$.

We will without further notice consider building sets as subsets of the intersection lattice $\mathcal{L}$, and thus let the consideration of $\mathcal{L}^*$ remain a detour for the sake of providing a transparent definition. Note that for any arrangement $\mathcal{A}$ the set of irreducible elements in $\mathcal{L}(\mathcal{A})\setminus\{\hat{0}\}$ is the minimal building set, whereas $\mathcal{G}=\mathcal{L}(\mathcal{A})\setminus\{\hat{0}\}$ is the maximal building set. For the maximal building set the nested set complex coincides with the order complex of the (non-reduced) intersection lattice.

4.2.2 Arrangement models and the nested set stratification

We are now prepared to give the definition of wonderful arrangement models. Let $\mathcal{A}$ be an arrangement of subspaces in a real or complex vector space V, $\mathcal{L}(\mathcal{A})$ its intersection lattice, and $\mathcal{G}$ a building set for $\mathcal{A}$. On the complement of the arrangement, $\mathcal{M}(\mathcal{A}):=V\setminus\bigcup\mathcal{A}$, consider the map

$$\Phi: \quad \mathcal{M}(\mathcal{A}) \longrightarrow V \times \prod_{G\in\mathcal{G}} \mathbb{P}(V/G)\,, \tag{4.2.1}$$

where in its first coordinate the map is given by inclusion, and in later coordinates by projection to the (real, resp. complex) projectivizations of the respective quotient spaces. Formally,

$$\Phi(x) \;=\; (\,x\,,\,(\Phi_G(x))_{G\in\mathcal{G}}\,)\,,$$

with $\Phi_G(x)=\langle x,G\rangle/G\in\mathbb{P}(V/G)$, for $x\in\mathcal{M}(\mathcal{A})$, where brackets $\langle\cdot\,,\cdot\rangle$ denote the linear span of subspaces or vectors, respectively. This map is an embedding of $\mathcal{M}(\mathcal{A})$, the arrangement model $Y_{\mathcal{G}}$ is defined as the closure of its image in $V\times\prod_{G\in\mathcal{G}}\mathbb{P}(V/G)$:

$$Y_{\mathcal{G}} \;:=\; \mathrm{cl}\,(\mathrm{Im}\,\Phi)\,.$$

Alternatively, $Y_{\mathcal{G}}$ can be described as the result of subsequently blowing up intersections of subspaces in $\mathcal{A}$, and proper transforms of such, corresponding to building set elements $G\in\mathcal{G}$ in some linear extension of the inclusion order.

The arrangement model $Y_{\mathcal{G}}$ is a smooth variety that contains the arrangement complement $\mathcal{M}(\mathcal{A})$ as an open subspace. The complement D of $\mathcal{M}(\mathcal{A})$ in $Y_{\mathcal{G}}$ is a divisor with normal crossings, in fact, it is the union of smooth, irreducible components D_G indexed by building set elements $G\in\mathcal{G}$. The intersections of divisors D_G are smooth and irreducible, naturally, they are indexed with subsets of $\mathcal{G}$. One of the main results of De Concini and Procesi, [DP3], states that an intersection of divisors is non-empty if and only if it is indexed with a nested set in $\mathcal{G}$.

We call the resulting stratification of $Y_{\mathcal{G}}$ by irreducible divisor components D_G and their intersections the *nested set stratification* of $Y_{\mathcal{G}}$, and denote it by $(Y_{\mathcal{G}},\mathfrak{D})$. Note that the poset of strata for $(Y_{\mathcal{G}},\mathfrak{D})$ coincides with the face poset of the nested set complex $\mathcal{N}(\mathcal{G})$.

De Concini & Procesi also provide a projective version of their arrangement models obtained by starting out with the projectivization of the arrangement complement and replacing the first factor on the right hand side of (4.2.1) by $\mathbb{P}(V)$ accordingly. The properties of the resulting projective model $\overline{Y}_{\mathcal{G}}$ are similar to those of $Y_{\mathcal{G}}$, for details we refer to [DP3, §4].

4.2.3 Finite group actions on arrangements and on their wonderful models

Let us now assume that a finite group Γ acts on our vector space V by linear transformations, and that the arrangement $\mathcal{A}$ is invariant under that action. By a standard result from representation theory, any linear action of a finite group is orthogonal [V, 2.3, Thm. 1]. Throughout the chapter, we denote the corresponding Γ-invariant positive definite symmetric bilinear form by the usual scalar product. Since we assume Γ to preserve $\mathcal{A}$, the group acts on the intersection lattice of $\mathcal{A}$,

$$\gamma(A_1 \cap \ldots \cap A_r) \;=\; \gamma(A_1) \cap \ldots \cap \gamma(A_r)\,, \qquad \text{for all } \gamma \in \Gamma,\; A_1, \ldots, A_r \in \mathcal{A}\,,$$

as well as internally on the corresponding intersections of subspaces. Also, Γ acts on the ambient space of the arrangement model corresponding to the maximal building set, that is on $V \times \prod_{G \in \mathcal{G}} \mathbb{P}(V/G)$, where $\mathcal{G} = \mathcal{L}(\mathcal{A}) \setminus \{\hat{0}\}$, by

$$\begin{aligned}\gamma\,(x, (x_G)_{G\in\mathcal{G}}) \;&=\; (\,\gamma(x), (\,\gamma(x_{\gamma^{-1}(G)})\,)_{G\in\mathcal{G}}), \\ &\qquad \text{for all } \gamma \in \Gamma,\;\; (\,x, (x_G)_{G\in\mathcal{G}}) \in V \times \prod_{G\in\mathcal{G}} \mathbb{P}(V/G)\,.\end{aligned}$$

Moreover, the inclusion map $\Phi\;:\;\mathcal{M}(\mathcal{A}) \longrightarrow V \times \prod_{G\in\mathcal{G}} \mathbb{P}(V/G)$ defined in (4.2.1) commutes with the action of Γ:

$$\begin{aligned}\gamma(\Phi(x)) \;&=\; \gamma\,(\,x, (\,\langle x, G\rangle/G\,)_{G\in\mathcal{G}}) \;=\; (\,\gamma(x), (\,\gamma(\,\langle x, \gamma^{-1}(G)\rangle/\gamma^{-1}(G))\,)_{G\in\mathcal{G}}) \\ &=\; (\,\gamma(x), (\,\langle\gamma(x), G\rangle/G)_{G\in\mathcal{G}}) \;=\; \Phi(\gamma(x)), \qquad \text{for } \gamma \in \Gamma,\, x \in \mathcal{M}(\mathcal{A})\,.\end{aligned}$$

We conclude that, since each element of Γ acts continuously on V, the closure of $\operatorname{Im}\Phi$ is Γ-invariant. Hence, Γ acts on the arrangement model $Y_{\mathcal{G}}$ extending the Γ-action on $\mathcal{M}(\mathcal{A}) \subseteq Y_{\mathcal{G}}$.

Note that choosing a Γ-invariant building set $\mathcal{G} \subsetneq \mathcal{L}(\mathcal{A}) \setminus \{\hat{0}\}$ as well yields an action of Γ on the corresponding arrangement model.

4.3 THE ARRANGEMENT MODEL Y_{Π_n}

4.3.1 A candidate for an abelianization of the permutation action

We consider the permutation action of the symmetric group $\mathcal{S}_n$ on $\mathbb{R}^n$,

$$\sigma(x) \;=\; (x_{\sigma(1)}, \ldots, x_{\sigma(n)}), \qquad \text{for all } \sigma \in \mathcal{S}_n,\; x = (x_1, \ldots, x_n) \in \mathbb{R}^n\,.$$

The locus of points in $\mathbb{R}^n$ with non-trivial stabilizer is a union of hyperplanes $H_{i,j}$, $H_{i,j} := \ker(x_i - x_j)$ for $1 \le i < j \le n$. This family of "diagonal hyperplanes" in $\mathbb{R}^n$ is the *braid arrangement* $\mathcal{A}_{n-1}$ of rank $n-1$, its name referring to the fact that the complement of a complexified version in $\mathbb{C}^n$ is the classifying space of the pure braid group on n strands. The braid arrangement is one of the central examples in arrangement theory and has provided a starting point for many investigations and developments in arrangement theory and beyond, see e.g., [OT].

The intersection lattice of $\mathcal{A}_{n-1}$ is the *partition lattice* Π_n, i.e., the poset of set partitions $\pi = (\pi_1 | \ldots | \pi_r)$ of $\{1, \ldots, n\} =: [n]$, $\pi_i \subseteq [n]$ with $\bigcup_{i=1}^r \pi_i = [n]$, ordered by reverse refinement. Clearly, a partition $\pi = (\pi_1 | \ldots | \pi_r)$ in Π_n corresponds to the intersection of hyperplanes $\bigcap_{(i,j) \in J_\pi} H_{i,j}$ with $J_\pi = \{(i,j) \mid 1 \le i < j \le n,\ \{i,j\} \subseteq \pi_k, \text{ for some } 1 \le k \le r\}$. We will freely use this correspondence between partitions and intersections of subspaces in the braid arrangement.

For further considerations, we restrict the permutation action to the $(n-1)$-dimensional real space

$$V \;=\; \{x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = 0\,\}\,.$$

The locus of points in V with non-trivial stabilizers is the intersection of $\mathcal{A}_{n-1}$ with V, an essential arrangement with intersection lattice Π_n, which we still call braid arrangement and denote by $\mathcal{A}_{n-1}$ without further mention.

We propose to study the De Concini-Procesi arrangement model Y_{Π_n} for $\mathcal{A}_{n-1}$ as a candidate for an abelianization of the permutation action. We allow ourselves here to use the shorthand notation Y_{Π_n} instead of $Y_{\Pi_n \setminus \{\hat{0}\}}$. It follows from the general discussion in subsection 4.2.3 that Y_{Π_n} carries a natural $\mathcal{S}_n$-action extending the $\mathcal{S}_n$-action on $\mathcal{M}(\mathcal{A}_{n-1}) \subseteq Y_{\Pi_n}$. It turns out that rather curious phenomena enter the scene already in low dimensions.

4.3.2 *The nested set stratification is not stabilizer distinguishing*

Already for S_3 acting on $\mathbb{R}^3$, the nested set stratification on the De Concini-Procesi model, $(Y_{\Pi_3}, \mathfrak{D})$, is not fine enough to distinguish stabilizers. Let us have a close look at the situation.

As above, we restrict the permutation action of S_3 on $\mathbb{R}^3$ to the subspace $V = \{\, (x_1, x_2, x_3) \mid \sum_{i=1}^3 x_i = 0\,\}$. The arrangement model Y_{Π_3} is the result of blowing up $\{0\}$ in V. Topologically, Y_{Π_3} is an open Möbius band.

As a subspace of $V \times \mathbb{P}(V)$, Y_{Π_3} can be described as follows:

$$Y_{\Pi_3} \;=\; \{\, (x, \langle x \rangle) \mid x \neq 0\,\} \,\cup\, \{\, (0, l) \mid l \in \mathbb{P}(V)\,\} \;\subseteq\; V \times \mathbb{P}(V)\,.$$

In terms of this pointwise description of Y_{Π_3} the divisors D_G, $G \in \Pi_3$, read

$$D_{\{0\}} = D_{(1,2,3)} = \{ (0,l) \mid l \in \mathbb{P}(V) \}$$
$$D_{(1,2)(3)} = \{ (x, \langle x \rangle) \mid x_1 = x_2 \neq 0 \} \cup \{ (0, \langle (1,1,-2) \rangle) \},$$

with $D_{(1,3)(2)}$, $D_{(1)(2,3)}$ having analogous descriptions.

Points on $D_{(1,2)(3)}$ are stabilized by the 2-element subgroup of $\mathcal{S}_3$ generated by the transposition $\tau = (1,2)$: For a generic point on $D_{(1,2)(3)}$, τ fixes the point and thus the generating line. For the single point in $D_{(1,2)(3)} \cap D_{\{0\}}$, τ fixes 0 and the line $\langle (1,1,-2) \rangle$ pointwise. Analogously, we see that points on $D_{(1,3)(2)}$ and on $D_{(1)(2,3)}$ are stabilized by the transpositions $(1,3)$ and $(2,3)$, respectively.

On $D_{\{0\}}$, however, we find points whose stabilizers the nested set stratification does not distinguish: Stabilizers for points on $D_{\{0\}}$ are trivial except for those points on the intersections with one of the other three divisors, *and* for 3 additional points

$$\psi_{12} = (0, \langle (1,-1,0) \rangle) \qquad \psi_{13} = (0, \langle (1,0,-1) \rangle) \qquad \psi_{23} = (0, \langle (0,1,-1) \rangle)$$

The ψ_{ij} are stabilized by transpositions (i,j), $1 \leq i < j \leq 3$, respectively, since the transpositions fix 0 and flip the lines in the second coordinate. In fact, the transposition (i,j), $1 \leq i < j \leq 3$, acts on the open Möbius band Y_{Π_3} like a “central symmetry” with fixed point ψ_{ij}.

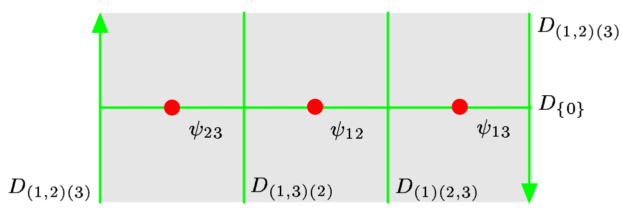


Figure 1. The nested set stratification $(Y_{\Pi_3}, \mathfrak{D})$.

We provide here a glance on the already more complicated situation for $n = 4$. Our picture below shows the stratification of the exceptional divisor $D_{\{0\}}$, a real projective space of dimension 2, as it emerges from the first blowup step in the De Concini-Procesi construction, $\mathrm{Bl}_{\{0\}}V$.

We choose to place the intersection of $D_{\{0\}}$ with the hyperplane $H_{1,2}$ on the equator of the upper hemisphere model, and thus obtain the stratification of $D_{\{0\}}$ by the braid arrangement as depicted above. The double, respectively, triple intersections of hyperplanes in $D_{\{0\}}$, e.g., $H_{1,2} \cap H_{3,4}$, respectively, $H_{1,3} \cap H_{1,4} \cap H_{3,4}$, remain to be blown up in later steps, for triple intersections locally producing the situation that we studied above for $n = 3$.

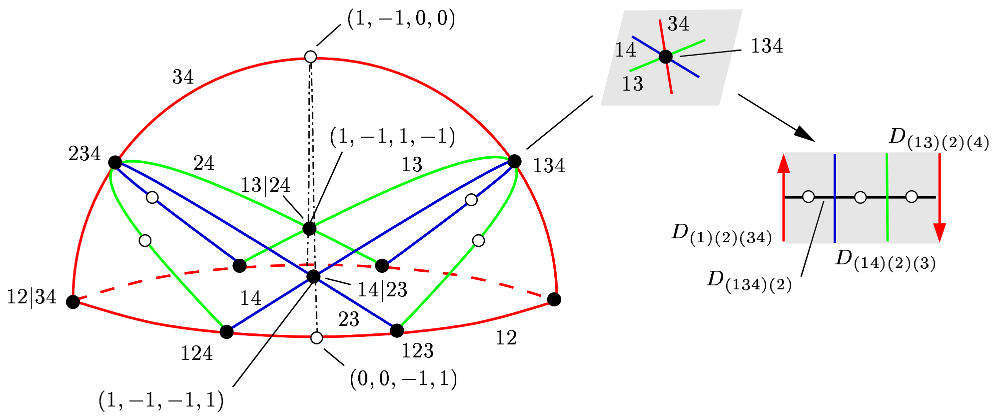


Figure 2. The stratification of $D_{\{0\}}$ after blowup of $\{0\}$ in V.

We mark some points and lines on open strata that ought to be distinguished by a stabilizer distinguishing stratification: For instance, the point on $D_{\{0\}}$ given by the line that is generated by the vector $(0, 0, -1, 1)$ in $H_{1,2}$ should be distinguished from the open stratum corresponding to $H_{1,2}$, since not only the transposition $\tau = (1, 2)$ but also $\sigma = (3, 4)$ stabilizes this line. The same goes for the (dashed) line obtained on $D_{\{0\}}$ as the intersection with the plane spanned by the vectors $(1, -1, 0, 0)$ and $(0, 0, -1, 1)$.

4.4 The nested set stratification of arrangement models

4.4.1 Points in $Y_{\mathcal{G}}$

Let $\mathcal{A}$ be an arrangement of subspaces in a real vector space V, $\mathcal{L}(\mathcal{A})$ its intersection lattice and $\mathcal{G} = \mathcal{L}(\mathcal{A}) \setminus \{\hat{0}\}$ the maximal building set for $\mathcal{A}$. We will encode points in the arrangement model $Y_{\mathcal{G}}$ into tuples of points and lines in V, a description that will prove to be favorable for technical purposes.

A point ω in $Y_{\mathcal{G}}$ will be written as

$$\omega \;=\; (x, H_1, l_1, H_2, l_2, \ldots, H_t, l_t)\,, \qquad (4.4.1)$$

where x is a point in V, the H_i are elements in $\mathcal{G} = \mathcal{L} \setminus \{\hat{0}\}$, and the l_i are lines in V. The point x is the first coordinate of ω when written as an element in the product space on the right hand side of (4.2.1). H_1 is the maximal lattice element that, as a subspace of V, contains x. The line l_1 is orthogonal to H_1 and corresponds to the coordinate entry of ω indexed by H_1 in $\mathbb{P}(V/H_1)$. The lattice element H_2, in turn, is the maximal lattice element that contains both H_1 and l_1. The specification of lines l_i, i.e., lines that correspond to coordinates of ω in $\mathbb{P}(V/H_i)$, and the construction of lattice elements H_{i+1}, continues analogously for $i \geq 2$ until a last line l_t is reached whose span with H_t is not contained in any lattice element other than the full ambient space V. Note, that if H_t is a hyperplane, then the line l_t is uniquely determined. The whole space V can be thought of as H_{t+1}.

Observe that the lattice elements H_i are determined by the point and the sequence of lines; we still choose to include the H_i in order to keep the notation more transparent.

To see that the description (4.4.1) of a point ω in the arrangement model $Y_{\mathcal{G}}$ is sufficient, we need to see that the rest of the coordinates can be read off uniquely from the coordinates $x, l_1, \dots, l_t$. The reconstruction can be explicitly done as follows. Fixing $H_0 := 0$ and $l_0 := \langle x \rangle$, the first coordinate of ω is x, and the coordinate of ω indexed with $H \in \mathcal{G}$, ω_H, can be read from (4.4.1) as

$$\omega_H \;=\; \langle l_j, H \rangle / H \;\in\; \mathbb{P}(V/H)\,, \tag{4.4.2}$$

where j is chosen from the index set $\{1, \dots, t\}$ such that $H \leq H_j$, but $H \not\leq H_{j+1}$.

To prove (4.4.2) we need the following technical lemma.

Lemma 4.4.1 *Let V be a vector space and $\widetilde{H}$, H vector subspaces of V, such that $\widetilde{H} \subseteq H$. Let furthermore $(x_i)_{i=1}^{\infty}$ be a sequence of points in $V \setminus H$ such that the limit $\lim_{i\to\infty} \langle x_i, \widetilde{H} \rangle = \Sigma$ exists in the corresponding Grassmannian.*

Assume that $\Sigma \not\subseteq H$, then $\lim_{i\to\infty} \langle x_i, H \rangle = \langle \Sigma, H \rangle$; again the limit is understood with respect to the topology of the appropriate Grassmannian.

Proof. Let us split V into the direct sum of linear subspaces:

$$V = \widetilde{H} \oplus (\widetilde{H}^{\perp} \cap H) \oplus H^{\perp},$$

where $\widetilde{H}^{\perp}$, resp. $H^{\perp}$, denotes the orthogonal complement of $\widetilde{H}$, resp. of H.

Since $x_i \notin \widetilde{H}$, we have $\dim \langle x_i, \widetilde{H} \rangle = \dim \widetilde{H} + 1$, hence $\dim \Sigma = \dim \widetilde{H} + 1$, and therefore there exists $v \in \widetilde{H}^{\perp}$, $v \neq 0$, such that $\Sigma = \langle \widetilde{H}, v \rangle$.

Writing $x_i = a_i + b_i + c_i$, where $a_i \in \widetilde{H}$, $b_i \in \widetilde{H}^{\perp} \cap H$, and $c_i \in H^{\perp}$, for all i, we have

$$\langle x_i, \widetilde{H} \rangle = \langle b_i + c_i, \widetilde{H} \rangle. \tag{4.4.3}$$

Note that $b_i + c_i \in \widetilde{H}^{\perp}$, and $b_i + c_i \neq 0$. We can scale x_i, such that $|b_i + c_i| = 1$, and, after scaling v and changing x_i to $-x_i$ for some appropriately chosen i, we get that $\lim_{i\to\infty}(b_i + c_i) = v$. Denote $\lim_{i\to\infty} b_i = v_1$ and $\lim_{i\to\infty} c_i = v_2$; these limits exist since b_i and c_i are chosen in mutually orthogonal linear subspaces. We certainly have $\lim_{i\to\infty}(b_i + c_i) = \lim_{i\to\infty} b_i + \lim_{i\to\infty} c_i = v_1 + v_2$, and $v_1 \in \widetilde{H}^{\perp} \cap H$, $v_2 \in H^{\perp}$. Since $v \notin H$, we have $v_2 \neq 0$, hence, for large i, $|c_i| \geq |v_2|/2 > 0$.

We finish the proof by writing down two sequences of identities. First,

$$\langle \Sigma, H \rangle = \langle \widetilde{H}, v, H \rangle = \langle v, H \rangle = \langle v_1 + v_2, H \rangle = \langle v_2, H \rangle,$$

where the second equality follows from $\widetilde{H} \subseteq H$, and the fourth equality follows from $v_1 \in H$. Second,

$$\lim_{i\to\infty} \langle x_i, H \rangle = \lim_{i\to\infty} \langle c_i, H \rangle = \langle \lim_{i\to\infty} c_i, H \rangle = \langle v_2, H \rangle,$$

where the first equality follows from (4.4.3) and the fact that $b_i \in H$. The second equality is the most interesting one, it follows from the fact that the points c_i lie in $H^\perp$, and that the projectivization map $\gamma : H^\perp \setminus \{0\} \to \mathbb{P}(H^\perp)$, mapping a point to the line which it spans, is continuous. □

Proof of (4.4.2)**.** Choose a sequence $(x_i)_{i=1}^\infty$, $x_i \in \mathcal{M}(\mathcal{A})$, such that the limit $\lim_{i\to\infty} \Phi(x_i) = w$ in $V \times \prod_{G\in\mathcal{G}} \mathbb{P}(V/G)$. This translates into

$$\begin{cases} x = \lim_{i\to\infty} x_i, \\ \omega_G = \lim_{i\to\infty} \Phi_G(x_i) = \lim_{i\to\infty} \langle x, G \rangle / G. \end{cases}$$

Let us choose $H \in \mathcal{G}$, and $j \in \{1, \dots, t\}$, such that $H \leq H_j$, but $H \not\leq H_{j+1}$. The identity (4.4.2) follows now from the following computation:

$$\lim_{i\to\infty} \langle x_i, H \rangle = \lim_{i\to\infty} \langle \langle x_i, H_j \rangle, H \rangle = \lim_{i\to\infty} \langle \langle l_j, H_j \rangle, H \rangle = \lim_{i\to\infty} \langle l_j, H \rangle,$$

where the first and the third equality are consequences of $H_j \subseteq H$, while the second one follows from Lemma 4.4.1. □

4.4.2 Stabilizers of points in $Y_\mathcal{G}$

We now assume that our subspace arrangement carries the action of a finite group Γ. As we discussed above, the action extends to the arrangement model $Y_\mathcal{G}$. When considering stabilizers of the various actions we will include indices into the notation that indicate the set on which the full group is acting, e.g., we will write $\mathrm{stab}_V(y)$, $\mathrm{stab}_{Y_\mathcal{G}}(y)$ for the stabilizers of y with respect to the Γ-actions on V and on $Y_\mathcal{G}$, respectively.

We take up the encoding of points in $Y_\mathcal{G}$ from subsection 4.4.1, and derive a description for the stabilizer of a point in $Y_\mathcal{G}$:

Proposition 4.4.2 *Let an arrangement model $Y_{\mathcal{G}}$ be equipped with a group action stemming from the action of a finite group Γ on the arrangement. Then for stabilizers of points $\omega = (x, H_1, l_1, H_2, l_2, \ldots, H_t, l_t)$ in $Y_{\mathcal{G}}$ the following description holds:*

$$\mathrm{stab}_{Y_{\mathcal{G}}}(\omega) \;=\; \mathrm{stab}_V(x) \cap \mathrm{stab}_V(l_1) \cap \ldots \cap \mathrm{stab}_V(l_t)\,, \qquad (4.4.4)$$

where $\mathrm{stab}_V(l_i)$, $i=1,\ldots,t$, denotes the subgroup of elements $\gamma \in \Gamma$ with $\gamma(l_i) = l_i$, i.e., elements preserving l_i without necessarily fixing the line pointwise.

Proof. Using the description of points in $Y_{\mathcal{G}}$ given in subsection 4.4.1, and the definition of the group action, we can describe the stabilizer of a point $\omega \in Y_{\mathcal{G}}$ as follows:

$$\mathrm{stab}_{Y_{\mathcal{G}}}(\omega) \;=\; \mathrm{stab}_V(x) \cap \mathrm{stab}_{\mathbb{P}(V/H_1)}(l_1) \cap \ldots \cap \mathrm{stab}_{\mathbb{P}(V/H_t)}(l_t)\,, \quad (4.4.5)$$

where $\mathrm{stab}_{\mathbb{P}(V/H_i)}(l_i)$, $i=1,\ldots,t$, translating from the projective to the original linear setting, means elements $\gamma \in \Gamma$ under which *both* H_i and l_i are invariant:

$$\mathrm{stab}_{\mathbb{P}(V/H_i)}(l_i) \;:=\; \mathrm{stab}_V(H_i) \cap \mathrm{stab}_V(l_i)\,.$$

Again, $\mathrm{stab}_V(H_i)$ denotes group elements that preserve H_i but do not necessarily fix H_i pointwise.

We show that

$$\begin{aligned} \mathrm{stab}_V(x) &\subseteq \mathrm{stab}_V(H_1)\,, \quad \text{and} \\ \mathrm{stab}_V(H_i) \cap \mathrm{stab}_V(l_i) &\subseteq \mathrm{stab}_V(H_{i+1})\,, \quad \text{for } i = 1,\ldots,t-1\,, \end{aligned}$$

which, successively applied for $i = t-1$, $i = t-2$, etc., reduces the right hand side of (4.4.5) to the right hand side of (4.4.4), since $A \cap B = A$, for any two sets A and B, such that $A \subseteq B$.

For $\gamma \in \mathrm{stab}_V(x)$, x in contained in $\gamma(H_1) \cap H_1$. But $H_1 \supseteq \gamma(H_1) \cap H_1$ is assumed to be maximal in $\mathcal{G} = \mathcal{L} \setminus \{\hat{0}\}$ containing x, thus, it follows from the fact that $\mathcal{G}$ is closed under taking intersections, that $\gamma(H_1) = H_1$. Similarly for $\gamma \in \mathrm{stab}_V(H_i) \cap \mathrm{stab}_V(l_i)$: $\gamma(H_{i+1}) \cap H_{i+1}$ contains both H_i and l_i, but H_{i+1} should be maximal in $\mathcal{G} = \mathcal{L} \setminus \{\hat{0}\}$ with this property, hence $\gamma(H_{i+1}) = H_{i+1}$.

Note additionally, that if H_t is a hyperplane, then $\mathrm{stab}_V(H_t) = \mathrm{stab}_V(l_t)$, hence, in this case, $\mathrm{stab}_V(l_t)$ can be removed from the right hand side of (4.4.4) without changing the expression. □

4.4.3 The divisors D_G, $G \in \mathcal{G}$

Recall from Section 4.2 that the nested set stratification $(Y_{\mathcal{G}}, \mathfrak{D})$ on an arrangement model $Y_{\mathcal{G}}$ is given by irreducible components of divisors and their intersections. Our objective is to provide, in our special setting, a description of the divisors D_G, $G \in \mathcal{G}$, that enables us to tell for a given point in the arrangement model on which of these divisors it lies.

De Concini & Procesi give a description of the divisors in terms of affine and projective arrangement models for "smaller" arrangements. To keep track of the respective settings, we provide arrangement models with an additional index that specifies the ambient space of the original arrangement, and we indicate projective models by a bar, e.g., in presence of other arrangement models we will now write $Y_{V,\mathcal{G}}$ for the affine and $\overline{Y}_{V,\mathcal{G}}$ for the projective model of the previously considered arrangement.

In our special setting the description of divisors by De Concini & Procesi reads as follows:

Proposition 4.4.3 *[DP3, Thm. 4.3, Rem. 4.3.(1)] Let $\mathcal{A}$ be an essential arrangement of subspaces, $\mathcal{G}$ the maximal building set, $\mathcal{G} = \mathcal{L}(\mathcal{A}) \setminus \{\hat{0}\}$, and $Y_{V,\mathcal{G}}$ the corresponding arrangement model. For the irreducible divisors D_G, $G \in \mathcal{G}$, there are natural isomorphisms:*

$$D_{\{0\}} \cong \overline{Y}_{V,\mathcal{G}}\,, \tag{4.4.6}$$

$$D_{\{G\}} \cong \overline{Y}_{V/G,\mathcal{G}_{\leq G}} \times Y_{G,\mathcal{G}_{>G}}, \qquad \textit{for } G \neq \{0\}\,. \tag{4.4.7}$$

Here, $\overline{Y}_{V/G,\mathcal{G}_{\leq G}}$ is the projective model for the quotient arrangement $\mathcal{A}/G := \{H/G \mid H \in \mathcal{A}, H \supseteq G\}$ with (maximal) building set $\mathcal{G}_{\leq G} = \{H \in \mathcal{G} \mid H \leq G\}$, and $Y_{G,\mathcal{G}_{>G}}$ is the affine model for the restricted arrangement $\mathcal{A} \cap G := \{H \cap G \mid H \in \mathcal{A}\}$ with (maximal) building set $\mathcal{G}_{>G} = \{H \in \mathcal{G} \mid H > G\}$.

The projective model $\overline{Y}_{V,\mathcal{G}}$, in fact, is isomorphic to the inverse image of $\{0\}$ when projecting $Y_{V,\mathcal{G}}$ to V, the first coordinate of its ambient space [DP3, Thm.4.1]. Hence, $\omega \in D_{\{0\}}$ if and only if $\omega_{\{0\}} = 0$, in other words

$$\omega \in D_{\{0\}} \;\Leftrightarrow\; \omega \in Y_{V,\mathcal{G}} \cap \left(\{0\} \times \prod_{G \in \mathcal{G}} \mathbb{P}(V/G) \right). \tag{4.4.8}$$

It is a description of this type that we want to achieve for the other divisors, D_G, $G \neq \{0\}$, as well.

To this end, note that the right hand side of (4.4.7) can be considered as a subspace of

$$\{0\} \times \prod_{H \in \mathcal{G}_{\leq G}} \mathbb{P}(V/G/H/G) \times G \times \prod_{H \in \mathcal{G}_{>G}} \mathbb{P}(G/H)\,.$$

For $K \in \mathcal{G}_{>G}$, we can "expand" the factor $\mathbb{P}(G/K)$ by a diagonal map

$$\mathbb{P}(G/K) \longrightarrow \prod_{\substack{H \in \mathcal{G} \\ H \vee G = K}} \mathbb{P}(G/(H \vee G)),$$

and thus interpret D_G as a subset of

$$U_G := G \times \prod_{H \in \mathcal{G}_{\not\leq G}} \mathbb{P}(G/(H \vee G)) \times \prod_{H \in \mathcal{G}_{\leq G}} \mathbb{P}(V/H).$$

With $G/(H \vee G) \cong \langle G, H \rangle / H$, U_G can be considered a subspace of the ambient space $V \times \prod_{H \in \mathcal{G}} \mathbb{P}(V/G)$ of the arrangement model.

We thus can state our description of divisors D_G:

Proposition 4.4.4 *Let $\mathcal{A}$ be an essential arrangement of subspaces, $\mathcal{G}$ the maximal building set, $\mathcal{G} = \mathcal{L}(\mathcal{A}) \setminus \{\hat{0}\}$, and $Y_{\mathcal{G}}$ the corresponding arrangement model. The irreducible divisors D_G, $G \in \mathcal{G}$, are intersections of $Y_{\mathcal{G}}$ with the product spaces U_G, where the U_G are obtained by restricting those factors of the original ambient space of Y_G which are indexed with $H \in \mathcal{G}_{\not\leq G}$:*

$$\begin{aligned} D_G &= Y_{\mathcal{G}} \cap U_G \\ &= Y_{\mathcal{G}} \cap \left(G \times \prod_{H \in \mathcal{G}_{\not\leq G}} \mathbb{P}(\langle G, H \rangle / H) \times \prod_{H \in \mathcal{G}_{\leq G}} \mathbb{P}(V/H) \right). \end{aligned}$$

Proof. Observe first that the description for $D_{\{0\}}$ given in (4.4.8) coincides with the one stated in the Proposition: intersecting $Y_{\mathcal{G}}$ with $U_{\{0\}}$ restricts the first coordinate to 0.

For $G \neq \{0\}$, we start with the description of D_G in (4.4.7) and see from the reasoning above that any element in D_G is contained in U_G. For the converse, let $\omega = (x, H_1, l_1, H_2, l_2, \ldots, H_t, l_t)$ be contained in $Y_{\mathcal{G}} \cap U_G$. From $\omega \in U_G$ we conclude that $x \in G$, hence $H_1 \geq G$. Assuming for the moment that $H_1 \gneq G$, we look at the component of ω indexed by H_1. Using the expansion of ω from (4.4.2) and the fact that $\omega \in U_G$, we see that

$$\omega_{H_1} = \langle l_1, H_1 \rangle / H_1 \in \mathbb{P}(G/H_1),$$

hence $l_1 \subseteq G$. This implies that H_2 is larger or equal G, for, if it were not, $H_2 \vee G \gneq H_2$ would contain both H_1 and l_1 in contradiction to H_2 being maximal with this property.

We conclude that there is an index $k \in \{1, \ldots, t\}$ with $H_k = G$, and can thus split the point/lines description of ω into

$$\omega = \Big((x, H_1, l_1, H_2, l_2, \ldots, l_{k-1}, G), (l_k, H_{k+1}, \ldots, H_t, l_t) \Big).$$

The first tuple clearly describes an element in $Y_{G,\mathcal{G}_{>G}}$. We rewrite the second tuple as follows:

$$(0_{V/G}, l_k, H_{k+1}/G, \ldots, H_t/G, l_t)\,.$$

With l_j being orthogonal to G, hence $l_j \in \mathbb{P}(V/G)$, we can then interpret it as an element of $\overline{Y}_{V/G,\mathcal{G}_{\leq G}}$. With (4.4.7) we thus conclude that $\omega \in D_G$. □

4.4.4 Open strata of the nested set stratification

We will provide a characterization of points on open strata of the nested set stratification of Y_G in terms of their point/line encoding described in subsection 4.4.1.

To fix some notation, let us denote by $D^\circ_{G_1,\ldots,G_m}$ the open stratum in $(Y_\mathcal{G}, \mathfrak{D})$ that lies in the intersection of divisors $D_{G_1}, \ldots, D_{G_m}$, but on no other divisors indexed with building set elements. Recall that the index set $\{G_1, \ldots, G_m\}$ is $\mathcal{G}$-nested, which in our context, i.e., for the maximal building set, means that it is a chain in $\mathcal{L}(\mathcal{A})$. We tacitly assume that the G_i are listed in a descending order: $G_1 > \ldots > G_m$.

Proposition 4.4.5 *Let $Y_\mathcal{G}$ be an arrangement model with nested set stratification $\mathfrak{D}$. A point $\omega \in Y_\mathcal{G}$ is contained in the open stratum of $\mathfrak{D}$ indexed with the nested set $\mathcal{T} = \{G_1, \ldots, G_m\}$ if and only if the spaces in $\mathcal{T}$ coincide with the spaces occurring in the point/line description of ω:*

$$\omega \in D^\circ_{G_1,\ldots,G_m} \iff \omega = (x, G_1, l_1, \ldots, G_m, l_m)\,,$$

where on the right hand side the usual restrictions for coordinates of a point/line tuple as in (4.4.1) *apply.*

Proof. First observe that the claim holds for points ω in the big open stratum $Y_\mathcal{G} \setminus D = \mathcal{M}(\mathcal{A})$, that is for $m = 0$: The indexing nested set is empty, and the point/line description for ω reduces to the point entry $x \in \mathcal{M}(\mathcal{A})$.

We can thus assume that $\omega \in D$, in particular, ω is contained in some open stratum in D, say

$$\omega \ \in \ D^\circ_{G_1,\ldots,G_m},$$

where we remind that the G_i are indexed in descending order, and $m \geq 1$.

At the same time, ω has a point/line description, say

$$\omega \ = \ (x, H_1, l_1, \ldots, H_t, l_t)\,,$$

where $H_1, \ldots, H_t \in \mathcal{G}$, $x \in H_1$, and $l_i \in \mathbb{P}(V/H_i)$, for $i = 1, \ldots, t$. We show in the following that the descending chains $G_1 > \ldots > G_m$ and $H_1 > \ldots > H_t$ coincide, in particular implying $m = t$.

Step 1: *The maximal elements of the chains coincide:* $H_1 = G_1$.
With $\omega \in D_{G_1}$, we know by Proposition 4.4.4 that $x \in G_1$; but H_1 is maximal with this property, hence, $H_1 \geq G_1$.

We want to see, that $\omega \in D_{H_1}$. Using again Proposition 4.4.4 and the expansion of ω in (4.4.2), we have to check that $x \in H_1$, and that for any $H \not\leq H_1$ the coordinate $\omega_H = \langle\langle x\rangle, H\rangle/H$ is a point in $\mathbb{P}(\langle H_1, H\rangle/H)$. With $\langle x\rangle \subseteq H_1$ this is obviously the case.

We conclude that $H_1 \in \mathcal{T}$, hence, $H_1 \leq G_1$ by maximality of G_1 in $\mathcal{T}$. This yields our claim. In particular, we see that $t \geq 1$.
Step 2: *Assume* $H_j = G_j$ *for* $j = 1, \ldots, i$, *and* $i \lneq t$. *Then* $m \geq i{+}1$ *and* $H_{i+1} = G_{i+1}$.
Here, we first want to see, that $\omega \in D_{H_{i+1}}$. For this we need to check that $x \in H_{i+1}$, and that for any $H \not\leq H_{i+1}$ the coordinate $\omega_H = \langle l_j, H\rangle/H$ is a point in $\mathbb{P}(\langle H_{i+1}, H\rangle/H)$. The line l_j depends on H (compare (4.4.2)), but for any H in question its index j is strictly less than $i+1$. From the point/line description for ω we see that $x \in H_1 \subseteq H_{i+1}$. With $l_j \subset H_{j+1} \subseteq H_{i+1}$ we conclude that $\langle l_j, H\rangle/H \in \mathbb{P}(\langle H_{i+1}, H\rangle/H)$, hence $\omega \in D_{H_{i+1}}$.

Since H_{i+1} belongs to the nested set $\mathcal{T}$, $H_{i+1} < H_i = G_i$, implies that, in fact, $m \geq i{+}1$ and $H_{i+1} \leq G_{i+1}$.

To obtain equality we write out the condition on the coordinate of ω indexed with H_i that results from $\omega \in D_{G_{i+1}}$: $\omega_{H_i} = \langle l_i, H_i\rangle/H_i \in \mathbb{P}(\langle G_{i+1}, H_i\rangle/H_i) = \mathbb{P}(G_{i+1}/H_i)$.

We conclude that $l_i \subseteq G_{i+1}$. Moreover, $G_i \subseteq G_{i+1}$ by descending order on $\mathcal{T}$. But H_{i+1} is maximal in $\mathcal{G}$ containing both $H_i = G_i$ and l_i, hence $H_{i+1} \geq G_{i+1}$, from which our claim follows.
Step 3: $m = t$, *and hence the chains coincide.*
From Steps (1) and (2) we conclude that $m \geq t$. Let us assume that $m > t$, in particular, $\omega \in D_{G_{t+1}}$. We conclude from the resulting condition on the coordinate indexed by H_t, $\omega_{H_t} = \langle l_t, H_t\rangle/H_t \in \mathbb{P}(\langle G_{t+1}, H_t\rangle/H_t) = \mathbb{P}(G_{i+1}/H_i)$, that both l_t and $H_t = G_t$ are contained in G_{t+1} which contradicts the fact that the point/line description of ω was terminated after the t-th step. Hence $m = t$, and the chains $G_1 > \ldots > G_t$ and $H_1 > \ldots > H_t$ coincide. □

4.5 A STABILIZER DISTINGUISHING STRATIFICATION OF Y_{Π_n}

4.5.1 Adding strata

On our way to construct a stabilizer distinguishing stratification for Y_{Π_n} we first analyze the locus of *lines* in $\mathbb{R}^n$ that are stabilized by a given element in $\mathcal{S}_n$. Let $\pi \in \mathcal{S}_n$, and, restricting the permutation action, consider $\mathbb{R}^n$ as a representation space of the cyclic group $\langle\pi\rangle$. In $\mathbb{R}^n$ we have, on one hand, the linear subspace $T_1(\pi)=\mathrm{Fix}(\pi)$, the locus of lines that are pointwise fixed by π, on the other hand, we have the subspace $T_{-1}(\pi)$, the locus of lines that are flipped by π. We can characterize lines in $\mathbb{R}^n$ that are invariant under $\pi\in\mathcal{S}_n$ as follows:

Proposition 4.5.1 *Let $\pi\in\mathcal{S}_n$ and $S(\pi) := T_1(\pi)\cup T_{-1}(\pi)$. For a given line l in $\mathbb{R}^n$,*

$$\pi \in \mathrm{stab}\,(l) \quad\Longleftrightarrow\quad l\subseteq S(\pi)\,.$$

We would like to emphasize that $S(\pi)$ is defined as a union of $T_1(\pi)$ and $T_{-1}(\pi)$, *not* as their span.

Let us now describe stratifications of the orthogonal complements $G^\perp$ of subspaces G in Π_n. For such G, and for any $\pi\in\mathcal{S}_n$, define $S(\pi,G):=S(\pi)\cap G^\perp$. Then,

$$\mathfrak{S}_{\mathfrak{G}} \;:=\; \left\{\mathfrak{S}(\pi,\mathfrak{G})\right\}_{\pi\in\mathcal{S}_\mathfrak{n}}$$

is a stratification of $G^\perp$. Unlike the restriction of the braid arrangement stratification to $G^\perp$, it distinguishes stabilizers of points as well as stabilizers of lines.

We propose a construction for subsets in real arrangement models $Y_\mathcal{G}$ that takes unions of linear subspaces in $\mathbb{R}^n$ as input data. It is inspired by the description of divisors D_G, $G\in\mathcal{G}$, that we presented in Proposition 4.4.4. Taking spaces $S(\pi,G)\times G$, $G\in\mathcal{G}$, $\pi\in S_n$, with $S(\pi,G)$ as defined above, our construction will provide us with the additional maximal strata in Y_{Π_n} for obtaining a stabilizer distinguishing stratification.

Definition 4.5.2 Let $Y_{V,\mathcal{G}}$ be an arrangement model, and $W=\{W_1,\dots,W_m\}$ a family of real linear subspaces in V. Define a subset $B(W)$ in $Y_\mathcal{G}$ by

$$B(W) \;:=\; Y_\mathcal{G}\;\cap\;\left(\bigcup W \;\times \prod_{\substack{H\in\mathcal{G},H\not\supseteq W_i\\ \text{for any } W_i\in W}} \mathbb{P}(\langle W,H\rangle/H)\;\times \prod_{\substack{H\in\mathcal{G},H\supseteq W_i\\ \text{for some } W_i\in W}} \mathbb{P}(V/H)\right),$$

where $\mathbb{P}(\langle W,H\rangle/H)$ stands for the projectivization of $\bigcup_{i=1}^m\langle W_i,H\rangle/H$.

We now can refine the nested set stratification $\mathfrak{D}$ of Y_{Π_n} so as to obtain a stabilizer distinguishing stratification. As before, we describe the stratification by listing its maximal strata:

$$\mathfrak{B} \;:=\; \left\{\;\big(\mathfrak{D}_\mathfrak{G}\big)_{\mathfrak{G}\in\mathfrak{t}_\mathfrak{n}},\;\big(\mathfrak{B}(\,\mathfrak{S}(\pi,\mathfrak{G})\times\mathfrak{G}\,)\big)_{\mathfrak{G}\in\mathfrak{t}_\mathfrak{n},\pi\in\mathcal{S}_\mathfrak{n}}\right\}, \tag{4.5.1}$$

where in the second family of strata we only consider those with $\{0\}\subsetneq S(\pi,G)\subseteq G^\perp$.

4.5.2 $(Y_{\Pi_n}, \mathfrak{B})$ *is stabilizer distinguishing*

We can now state one of the main results of this chapter:

Theorem 4.5.3 *The stratification $\mathfrak{B}$ for the arrangement model Y_{Π_n} defined in (4.5.1) is stabilizer distinguishing, i.e., the stabilizer of a point $\omega \in Y_{\Pi_n}$ is completely determined by the open stratum of $\mathfrak{B}$ that contains ω.*

Proof. We pick a point $\omega = (x, G_1, l_1, \dots, G_t, l_t)$ in Y_{Π_n}, and assume that we have the complete list of maximal strata in $\mathfrak{B}$ which contain ω. We want to show that the stabilizer of ω is fully determined by this list.

Note first that by Proposition 4.4.5 our list of strata contains the divisors $D_{G_1}, \dots, D_{G_t}$, and no other divisors of this type. This means that we can read of from the list the elements $G_1, \dots, G_t$ for the point/line description of ω.

Assume $\omega \in B(S(\pi, G_i) \times G_i)$, for some G_i, $i \in \{1, \dots, t\}$. With Definition 4.5.2, and $S(\pi, G_i) \times G_i \supseteq G_i$, this puts the following restriction on the coordinate of ω that is indexed by G_i:

$$\omega_{G_i} = \langle l_i, G_i \rangle / G_i \ \in \ \mathbb{P}(\langle S(\pi, G_i) \times G_i, G_i \rangle / G_i) \, .$$

We conclude that $l_i \subseteq S(\pi, G_i)$, in particular, π stabilizes l_i.

From the strata $B(S(\pi, G_i) \times G_i)$, that occur on our list for a fixed space G_i, $i \in \{1, \dots, t\}$, we can read off a subset Γ_i of $\mathrm{stab}(l_i)$. Namely, for each $i \in \{1, \dots, t\}$, Γ_i consists of all π such that $\omega \in B(S(\pi, G_i) \times G_i)$.

Let us assume that, when constructing Γ_i from our list of strata for ω, we actually missed some elements of $\mathrm{stab}(l_i)$: let $\sigma \in \mathrm{stab}(l_i) \setminus \Gamma_i$. Then $l_i \subseteq S(\sigma, G_i)$, but $\omega \notin B(S(\sigma, G_i) \times G_i)$. By definition of the additional maximal strata we conclude that there exists a subspace $H \in \Pi_n$, which does not contain any of the spaces in $S(\sigma, G_i) \times G_i$, such that

$$\omega_H = \langle l_j, H \rangle / H \ \notin \ \mathbb{P}(\langle S(\sigma, G_i) \times G_i, H \rangle / H) \, . \tag{4.5.2}$$

The line index j depends on H, but in any case, $j > i$: for $j < i$, $l_j \subseteq G_i$, and for $j = i$, $l_i \subseteq S(\sigma, G_i)$, and the condition on ω_H for ω being contained in $B(\, S(\sigma, G_i) \times G_i \,)$ would be fulfilled.

It follows from (4.5.2) that $l_j \not\subseteq S(\sigma, G_i)$. Since l_j is orthogonal to G_i, it implies $\sigma \notin \mathrm{stab}(l_j)$, and, in particular, $\sigma \notin \bigcap_{i=1}^{t} \mathrm{stab}(l_i)$. Hence, even if for some i, $\Gamma_i \subsetneq \mathrm{stab}(l_i)$, once the full intersection is taken, this is rectified:

$$\bigcap_{i=1}^{t} \Gamma_i \ = \ \bigcap_{i=1}^{t} \mathrm{stab}(l_i) \, .$$

With the description of $\mathrm{stab}(\omega)$ from Proposition 4.4.2, and $\mathrm{stab}(x)$ being determined by the partition pattern of x, hence by G_1, we can conclude that the list of strata in $\mathfrak{B}$ containing ω actually determines the stabilizers of ω. □

4.5.3 Y_{Π_3} *revisited*

Let us have a look at the stratification $\mathfrak{B}$ on Y_{Π_3} and see how it resolves the problem raised in 4.3.2, namely to distinguish stabilizers of points by means of a stratification.

To start with, we have to identify those spaces $S(\pi, G) \times G$ for $G \in \Pi_3$, $\pi \in \mathcal{S}_3$, that give raise to new strata $B(S(\pi, G) \times G)$. We claim that the only interesting case occurs for π a transposition, $\pi = (i, j)$, $1 \leq i < j \leq 3$, and $G = \{0\}$.

We have $S(\pi) = H_{i,j} \cup H_{i,j}^{-}$, where we denote hyperplanes of $\mathcal{A}_{n-1}$ in V by $H_{i,j}$, just as for the original (non-essential) arrangement in $\mathbb{R}^3$, and their orthogonal complements by $H_{i,j}^{-}$. With $S(\pi, \{0\}) = S(\pi)$, we obtain new strata

$$\begin{aligned} B_{(i,j)} &= B(S((i,j), \{0\}) \times \{0\}) \\ &= Y_{\Pi_3} \cap \big((H_{i,j} \cup H_{i,j}^{-}) \times (\mathbb{P}(H_{i,j}) \cup \mathbb{P}(H_{i,j}-))\big). \end{aligned}$$

In terms of the pointwise description for Y_{Π_3} that we gave in 4.3.2 this reads

$$\begin{aligned} B_{(1,2)} = \{ (x, \langle x \rangle) \mid x_1 = x_2 \neq 0 \text{ or } x_1 = -x_2 \neq 0 \} \\ \cup \{ (0, \langle (1,1,-2) \rangle), (0, \langle (1,-1,0) \rangle) \}, \end{aligned}$$

analogously for $B_{(1,3)}$, $B_{(2,3)}$. Hence, as opposed to the nested set stratification $\mathfrak{D}$, the stratification $\mathfrak{B} = \{(\mathfrak{D}_{\mathfrak{G}})_{\mathfrak{G} \in \mathfrak{t}_3}, \mathfrak{B}_{(1,2)}, \mathfrak{B}_{(1,3)}, \mathfrak{B}_{(2,3)}\}$ distinguishes the points $\psi_{i,j}$, $1 \leq i < j \leq 3$ from the rest of the divisor $D_{\{0\}}$.

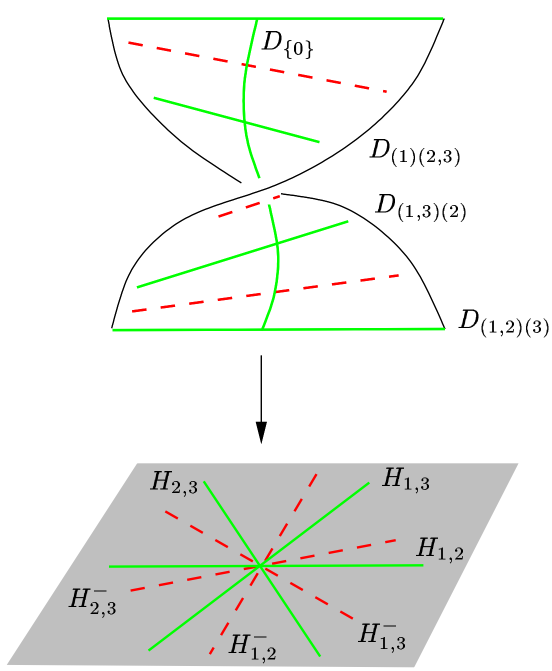


Figure 3. The stratification $(Y_{\Pi_3}, \mathfrak{B})$.

4.6 A COMBINATORIAL FRAMEWORK FOR DESCRIBING STABILIZERS

In this section we develop a combinatorial framework for describing stabilizers of points on the De Concini-Procesi arrangement model Y_{Π_n} with respect to the $\mathcal{S}_n$-action. In section 4.7 we will use this description to prove that the stabilizers of points of Y_{Π_n} are isomorphic to direct products of $\mathbb{Z}_2$.

4.6.1 Diagrams over families of cubes

Definition 4.6.1

1. Let I be a finite, possibly empty set of positive integers. We call the collection of all subsets of I (including the empty subset) an I-**cube**. Reversely, given an I-cube K, we call I the *index set* of K.

2. Let t be a positive integer. A *t-family of cubes* is a collection $\mathcal{C} = \{K_1, \dots, K_p\}$, where, for each $j = 1, \dots, p$, K_j is an $I(j)$-cube, for some $I(j) \subseteq \{1, \dots, t\}$.

One can make use of geometric intuition by thinking of an I-cube as a coordinate $0/1$-cube with I indexing the set of "directions" of the cube. The $\emptyset$-cube is simply the point at the origin. For every $n \geq \max(I)$, the I-cube can be imbedded as a coordinate $0/1$-cube in $\mathbb{R}^n$, and our object is the equivalence class of all these imbeddings.

Let K be an I-cube, to discriminate from other I-cubes, we write elements of K as pairs (K, S), for $S \subseteq I$. We denote $\text{vert}\,(K) = \{(K, S) \mid S \subseteq I\}$, and refer to its elements as vertices of K. When it is clear which cube we are in, we may choose to skip K, and call S itself a vertex of K.

Note also that a t-family of cubes is simply specified by a function $I : [p] \to 2^{[t]}$, and that if $\tilde{t} > t$, then every t-family of cubes is also a $\tilde{t}$-family. For $\mathcal{C} = \{K_1, \dots, K_p\}$ we denote $\text{vert}\,(\mathcal{C}) = \bigcup_{i=1}^{p} \text{vert}\,(K_i)$, and refer to its elements as vertices of $\mathcal{C}$.

Definition 4.6.2

1. Let $\mathcal{C}$ be a t-family of cubes, $\mathcal{C} = \{K_1, \dots, K_p\}$, and let n be a positive integer. An *n-diagram* $\mathcal{D}$ over $\mathcal{C}$ is a partition of the set $[n]$ into $|\text{vert}\,(\mathcal{C})|$ blocks, some blocks may be empty, and an assignment of the blocks of this partition to vertices of $\mathcal{C}$; in other words, it is a function

$$\begin{array}{rcccl} \mathcal{D} & : & [n] & \longrightarrow & \text{vert}\,(\mathcal{C}), \\ & & k & \mapsto & (K_{\alpha(k)}, v_k), \end{array} \tag{4.6.1}$$

where $\alpha(k) \in [p]$ specifies the index of the cube and $v_k \subseteq I(\alpha(k))$ the vertex of $K_{\alpha(k)}$ assigned to k.

2. For a vertex (K, v) of $\mathcal{C}$, we call the set $\mathcal{D}^{-1}(K, v)$ the *fiber* of $\mathcal{D}$ over (K, v). For an I-cube K in $\mathcal{C}$, the fiber of $\mathcal{D}$ over K is defined as the union of the fibers of the vertices of K:

$$\mathcal{D}^{-1}(K) := \bigcup_{v \subseteq I} \mathcal{D}^{-1}(K, v).$$

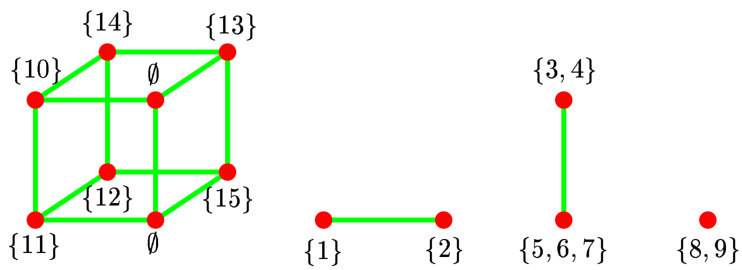


Figure 4. An example of a 15-diagram over a 3-family of cubes.

As yet another piece of notation, let $\rho(\mathcal{D}) \vdash n$ be the set partition with blocks being the fibers of $\mathcal{D}$ over the vertices of $\mathcal{C}$, i.e., $\rho(\mathcal{D}) = \{\mathcal{D}^{-1}(K, v)\}_{(K,v) \in \text{vert}(\mathcal{C})}$, where we disregard all the empty blocks in the set on the right hand side.

4.6.2 Automorphism groups

There is a standard $\mathbb{Z}_2^n$-action on an $[n]$-cube: it is generated by reflections with respect to n hyperplanes, which are parallel to the facets of the cube, and which go through the center of the cube. A technically convenient way to describe this action is to think of the vertices of an $[n]$-cube as vectors in an n-dimensional vector space over the field $\mathbb{F}_2$, again denoted $\mathbb{Z}_2^n$, and the action as parallel translations by vectors in $\mathbb{Z}_2^n$ (i.e., generated by parallel translations with respect to the coordinate vectors).

For a subset $I \subseteq [n]$, let $\mathbb{Z}_2^I$ denote the corresponding coordinate subspace of $\mathbb{Z}_2^n$, and let $\text{proj}_I : \mathbb{Z}_2^n \to \mathbb{Z}_2^I$ denote the projection onto $\mathbb{Z}_2^I$ which simply "forgets" the coordinates with indices outside of I.

The following definition generalizes these actions to the case of diagrams over families of cubes.

Definition 4.6.3 Let an n-diagram $\mathcal{D}$ over a t-family of cubes $\mathcal{C}=\{K_1, \ldots, K_p\}$ be given. We define the *group of automorphisms* of $\mathcal{D}$, which we denote $\text{Aut}(\mathcal{D})$, as follows: $\text{Aut}(\mathcal{D})$ consists of all permutations $\pi \in \mathcal{S}_n$, such that

i) $\pi_{\mathcal{D}^{-1}(K_j)} \in \mathcal{S}_{\mathcal{D}^{-1}(K_j)}$, for all $j = 1, \dots, p$, i.e., π preserves the fibers over cubes;

ii) there exists (not necessarily unique) $\sigma \in \mathbb{Z}_2^t$, such that

$$v_{\pi(k)} = \sigma_{\alpha(k)}(v_k), \quad \text{for all } k \in \{1, \dots, n\}, \tag{4.6.2}$$

where $\sigma_j = \text{proj}_{I(j)}(\sigma)$, for all $j \in \{1, \dots, p\}$, and where v_k and $\alpha(k)$ are as in (4.6.1). In other words, π maps fibers to fibers according to a uniform scheme obtained by restricting σ to the cubes in the family $\mathcal{C}$.

Remark 4.6.4 Maps between fibers of an n-diagram $\mathcal{D}$ over a t-family of cubes $\mathcal{C}$, which are induced by an element $\pi \in \text{Aut}(\mathcal{D})$, must be bijections.

Indeed, let K be an I-cube in $\mathcal{C}$, let $v \subseteq I$, and let $\sigma \in \mathbb{Z}_2^t$ be associated to π by Definition 4.6.3 ii), then, by (4.6.2), we have

$$\pi(\mathcal{D}^{-1}(K, v)) \subseteq \mathcal{D}^{-1}(K, \text{proj}_I(\sigma)(v)),$$

while

$$\pi(\mathcal{D}^{-1}(K, \text{proj}_I(\sigma)(v))) \subseteq \mathcal{D}^{-1}(K, \text{proj}_I(\sigma)^2(v)) = \mathcal{D}^{-1}(K, v).$$

Since π is injective, its restrictions are injective as well, hence we can conclude that π restricts to a bijection between $\mathcal{D}^{-1}(K, v)$ and $\mathcal{D}^{-1}(K, \text{proj}_I(\sigma)(v))$.

Lemma 4.6.5

1. *For $x \in \mathbb{R}^n$, the stabilizer of x under the $\mathcal{S}_n$-action is the Young subgroup of $\mathcal{S}_n$ indexed by the set partition of $[n]$, which is induced by the coordinates of x. One can represent this Young subgroup as an automorphism group of an n-diagram over a 0-family of cubes.*

2. *For a line $l \subseteq \mathbb{R}^n$, the stabilizer of l under the $\mathcal{S}_n$-action can be represented as an automorphism group of an n-diagram over a 1-family of cubes.*

Proof. (1) The first part of the statement is immediate. To construct the necessary n-diagram, group together all the coordinates of x that are equal and assign the corresponding sets of indices to different 0-cubes. This yields an n-diagram $\mathcal{D}$ over a 0-family of cubes, and, obviously, $\text{Aut}(\mathcal{D})$ is exactly the $\mathcal{S}_n$-stabilizer of x in $\mathbb{R}^n$.

(2) Take a nonzero vector $v \in l$. Group together all the equal coordinates of v, and assign corresponding sets of indices to 0-cubes, just like we did for x. Now, whenever there are two groups of coordinates, such that these groups are

of equal cardinality, and the coordinates in the two groups are negatives of each other, we connect the two corresponding 0-cubes with an edge, to form a 1-cube. We orient all these cubes in the same coordinate direction. Clearly, this yields an n-diagram $\mathcal{D}$ over a 1-family of cubes.

Assume first that our diagram consists of a number of 1-cubes and at most one 0-cube, with the fiber over this 0-cube consisting of all the indices of the coordinates of v which are equal to 0. The elements of the group $\mathrm{Aut}(\mathcal{D})$ are of two sorts, depending on which of the two elements of $\mathbb{Z}_2$ they are associated to. We easily verify that those elements of $\mathrm{Aut}(\mathcal{D})$, which are associated to $0 \in \mathbb{Z}_2$, are exactly those $\pi \in \mathcal{S}_n$, which fix v, while those elements of $\mathrm{Aut}(\mathcal{D})$, which are associated to $1 \in \mathbb{Z}_2$, are exactly those $\pi \in \mathcal{S}_n$, which map v to $-v$. Since these are the only two options for mapping v, if l is to be preserved by the element π, we have proven the lemma in this case.

Assume now that $\mathcal{D}$ is a diagram of some other form. Then, there exist no $\pi \in \mathcal{S}_n$ such that $\pi(v) = -v$, i.e., each element of $\mathrm{stab}(l)$ fixes l pointwise. In this case, $\mathrm{stab}(l) = \mathrm{stab}(v)$, thus we are back to case (1) and the diagram can be obtained by splitting all the 1-cubes into 0-cubes. □

4.6.3 Intersections of diagrams

Let $\mathcal{C}_1 = \{K_1, \dots, K_p\}$, resp. $\mathcal{C}_2 = \{L_1, \dots, L_q\}$, be a t_1-, resp. t_2-family of cubes, where K_i is an $I_1(i)$-cube, and L_j is an $I_2(j)$-cube, for all $i \in [p]$, $j \in [q]$.

Let $\mathcal{D}_1$, resp. $\mathcal{D}_2$, be n-diagrams over $\mathcal{C}_1$, resp. $\mathcal{C}_2$:

$$\begin{aligned} \mathcal{D}_1 \ : \ [n] &\longrightarrow \text{vert}\,(\mathcal{C}_1), \\ k &\mapsto (K_{\alpha_1(k)}, v_k^{(1)}), \end{aligned}$$

$$\begin{aligned} \mathcal{D}_2 \ : \ [n] &\longrightarrow \text{vert}\,(\mathcal{C}_2), \\ k &\mapsto (L_{\alpha_2(k)}, v_k^{(2)}). \end{aligned}$$

Definition 4.6.6 The *intersection of diagrams* $\mathcal{D}_1$ and $\mathcal{D}_2$, denoted $\mathcal{D} = \mathcal{D}_1 \cap \mathcal{D}_2$, is an n-diagram over a $(t_1 + t_2)$-family of cubes $\mathcal{C}$ defined as follows:

$$\mathcal{C} = \{M_{i,j}\}_{i\in[p],j\in[q]}, \quad I(i,j) = I_1(i) \cup \{x + t_1 \mid x \in I_2(j)\},$$

here $M_{i,j}$ is an $I(i,j)$-cube, furthermore

$$\begin{aligned} \mathcal{D} \ : \ [n] &\longrightarrow \text{vert}\,(\mathcal{C}), \\ k &\mapsto (M_{\alpha_1(k),\alpha_2(k)}, v_k^{(1)} \cup \{x + t_1 \mid x \in v_k^{(2)}\}). \end{aligned}$$

Note that the fibers over the vertices and cubes of $\mathcal{D}$ are determined by the fibers of $\mathcal{D}_1$ and $\mathcal{D}_2$ as follows:

$$\mathcal{D}^{-1}(M_{i,j}) = \mathcal{D}_1^{-1}(K_i) \cap \mathcal{D}_2^{-1}(L_j),$$

and

$$\mathcal{D}^{-1}(M_{i,j}, v) = \mathcal{D}_1^{-1}(K_i, I_1(i) \cap v) \cap \mathcal{D}_2^{-1}(L_j, \{x - t_1 \mid x \in v, x > t_1\}), \quad (4.6.3)$$

for each $v \subseteq I(i,j)$.

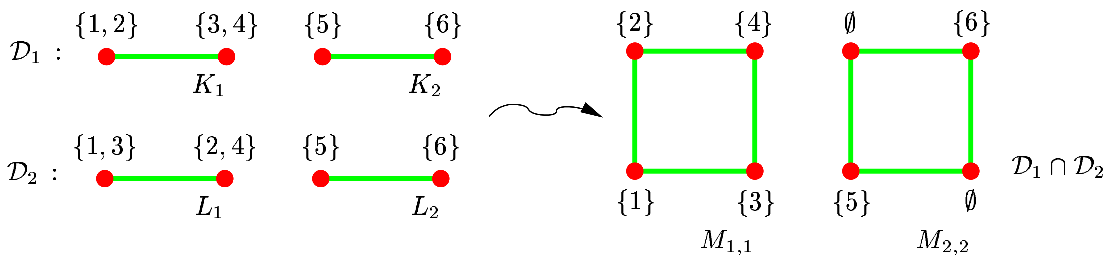


Figure 5. An example of an intersection of two diagrams.

In the above example, observe that $\mathcal{D}_1 \cap \mathcal{D}_2$ actually contains two more cubes, $M_{1,2}$ and $M_{2,1}$, with 2-element index sets $I(1,2)$ and $I(2,1)$, whose fibers, however, are all empty.

Lemma 4.6.7 *For two n-diagrams $\mathcal{D}_1$ and $\mathcal{D}_2$, we have $\rho(\mathcal{D}_1 \cap \mathcal{D}_2) = \rho(\mathcal{D}_1) \wedge \rho(\mathcal{D}_2)$, where $\wedge$ denotes the operation of common refinement of the set partitions.*

Proof. By (4.6.3), the blocks of $\rho(\mathcal{D}_1 \cap \mathcal{D}_2)$ are all nonempty intersections of the blocks of $\rho(\mathcal{D}_1)$ with the blocks of $\rho(\mathcal{D}_2)$, which is precisely the definition of the common refinement operation. □

We shall prove two structural theorems about n-diagrams. The first one asserts that taking intersections of diagrams commutes with passing to the automorphism group.

Theorem 4.6.8 *For two n-diagrams $\mathcal{D}_1$ and $\mathcal{D}_2$ as above, and $\mathcal{D} = \mathcal{D}_1 \cap \mathcal{D}_2$ their intersection, we have*

$$\mathrm{Aut}(\mathcal{D}_1) \cap \mathrm{Aut}(\mathcal{D}_2) = \mathrm{Aut}(\mathcal{D}). \quad (4.6.4)$$

Proof. First we prove that the set on the left hand side of (4.6.4) is a subset of the set on the right hand side.

Let $\pi \in \mathrm{Aut}(\mathcal{D}_1) \cap \mathrm{Aut}(\mathcal{D}_2)$. By Definition 4.6.3 i) we know that π preserves the fibers $\mathcal{D}_1^{-1}(K_i)$, for all $i \in [p]$, and π preserves the fibers $\mathcal{D}_2^{-1}(L_j)$, for all $j \in [q]$. Hence π preserves $\mathcal{D}_1^{-1}(K_i) \cap \mathcal{D}_2^{-1}(L_j) = \mathcal{D}^{-1}(M_{i,j})$, for all $i \in [p], j \in [q]$, and so property i) of Definition 4.6.3 is valid for π.

By Definition 4.6.3 ii), there exist $\sigma^{(1)} \in \mathbb{Z}_2^{t_1}$, and $\sigma^{(2)} \in \mathbb{Z}_2^{t_2}$, such that

$$\sigma^{(1)}_{\alpha_1(k)}(v^{(1)}_k) = v^{(1)}_{\pi(k)}, \quad \text{and} \quad \sigma^{(2)}_{\alpha_2(k)}(v^{(2)}_k) = v^{(2)}_{\pi(k)},$$

for all $k \in [n]$, where $\sigma^{(1)}_{\alpha_1(k)} = \mathrm{proj}_{I_1(\alpha_1(k))}(\sigma)$, and $\sigma^{(2)}_{\alpha_2(k)} = \mathrm{proj}_{I_2(\alpha_2(k))}(\sigma)$.

Define $\sigma \in \mathbb{Z}_2^{t_1+t_2}$ as a concatenation $\sigma = (\sigma^{(1)}, \sigma^{(2)})$, that is the first t_1 coordinates of σ are equal to $\sigma^{(1)}$, and the last t_2 coordinates of σ are equal to $\sigma^{(2)}$. Let $k \in [n]$, and decompose $v_k \subseteq [t_1 + t_2]$ as $v_k = v^{(1)}_k \cup \tilde{v}^{(2)}_k$, where $v^{(1)}_k = v_k \cap \{1, \dots, t_1\}$, and $\tilde{v}^{(2)}_k = v_k \cap \{t_1 + 1, \dots, t_1 + t_2\}$. Then, we have

$$\begin{aligned} \sigma_{\alpha_1(k),\alpha_2(k)}(v_k) &= \sigma_{\alpha_1(k),\alpha_2(k)}(v^{(1)}_k \cup \tilde{v}^{(2)}_k) \\ &= \sigma_{\alpha_1(k)}(v^{(1)}_k) \cup \tilde{\sigma}_{\alpha_2(k)}(\tilde{v}^{(2)}_k) = v^{(1)}_{\pi(k)} \cup \tilde{v}^{(2)}_{\pi(k)} = v_{\pi(k)}\,, \end{aligned}$$

where $\sigma_{\alpha_1(k),\alpha_2(k)} = \mathrm{proj}_{I(\alpha_1(k),\alpha_2(k))}(\sigma)$, $\tilde{\sigma}_{\alpha_2(k)}$ is equal to $\sigma_{\alpha_2(k)}$ in the coordinates $\{t_1 + 1, \dots, t_1 + t_2\}$, and is equal to 0 in the other coordinates, while $\tilde{v}^{(2)}_{\pi(k)} = \{x + t_1 \mid x \in v^{(2)}_{\pi(k)}\}$. In other words, $\tilde{\sigma}_{\alpha_2(k)}$ and $\tilde{v}^{(2)}_{\pi(k)}$ are the t_1-shifted versions of $\sigma_{\alpha_2(k)}$ and $v^{(2)}_{\pi(k)}$. So, we have shown that $\pi \in \mathrm{Aut}(\mathcal{D}_1 \cap \mathcal{D}_2)$.

Now let us prove that the set on the right hand side of (4.6.4) is a subset of the set on the left hand side.

Take $\pi \in \mathrm{Aut}(\mathcal{D})$, then π preserves $\mathcal{D}^{-1}(M_{i,j})$, and therefore π also preserves

$$\begin{aligned} \bigcup_{j=1}^{q} \mathcal{D}^{-1}(M_{i,j}) &= \bigcup_{j=1}^{q} \mathcal{D}_1^{-1}(K_i) \cap \mathcal{D}_2^{-1}(L_j) = \mathcal{D}_1^{-1}(K_i) \cap \bigcup_{j=1}^{q} \mathcal{D}_2^{-1}(L_j) \\ &= \mathcal{D}_1^{-1}(K_i) \cap [n] = \mathcal{D}_1^{-1}(K_i)\,, \qquad \text{for any } i \in [p]\,; \end{aligned}$$

in the same way π preserves $\mathcal{D}_2^{-1}(L_j)$, for any $j \in [q]$. This checks condition i) of Definition 4.6.3.

Finally, by condition ii) of Definition 4.6.3, there exists $\sigma \in \mathbb{Z}_2^{t_1+t_2}$, such that for any $k \in [n]$ we have $\sigma_{\alpha_1(k),\alpha_2(k)}(v_k) = v_{\pi(k)}$. As above, we can decompose $\sigma = (\sigma^{(1)}, \sigma^{(2)})$ and $v_k = v^{(1)}_k \cup \tilde{v}^{(2)}_k$ as a concatenation of the first t_1 and the last t_2 coordinates. Then, in the notations which we used above, we can derive that

$$\sigma^{(1)}_{\alpha_1(k)}(v^{(1)}_k) = v^{(1)}_{\pi(k)}, \quad \text{and} \quad \tilde{\sigma}^{(2)}_{\alpha_2(k)}(\tilde{v}^{(2)}_k) = \tilde{v}^{(2)}_{\pi(k)}.$$

Shifting the second identity down by t_1, we get $\sigma^{(2)}_{\alpha_2(k)}(v^{(2)}_k) = v^{(2)}_{\pi(k)}$. □

4.6.4 A reduction theorem

When $\mathcal{D}$ is an n-diagram over a t-family of cubes, not every element $\sigma \in \mathbb{Z}_2^t$ gives rise to an element $\pi \in \mathrm{Aut}(\mathcal{D})$. The natural obstruction is that, by Remark 4.6.4, fibers with different cardinalities cannot map to each other. It turns out that one can always canonically reduce $\mathcal{D}$ to another n-diagram with the same automorphism group, such that in this new n-diagram all fibers over vertices in the same cube have the same cardinality.

Theorem 4.6.9 *Let an n-diagram $\mathcal{D}$ over a t-family of cubes $\mathcal{C} = \{K_1, \dots, K_p\}$ be given. Then, there exists an n-diagram $\widetilde{\mathcal{D}}$ over a $\tilde{t}$-family of cubes $\widetilde{\mathcal{C}} = \{L_1, \dots, L_q\}$, such that*

0) $\tilde{t} \leq t$;

1) $\mathrm{Aut}(\mathcal{D}) = \mathrm{Aut}(\widetilde{\mathcal{D}})$;

2) $|\mathcal{D}^{-1}(L_j, v)| = |\mathcal{D}^{-1}(L_j, v')|$, *for all* $j \in [q]$, *and for all* $v, v' \subseteq I_2(j)$, *where* $I_2(j)$ *is the index set of* L_j.

In the continuation, we shall call an n-diagram satisfying Condition 2) of Theorem 4.6.9 a *reduced diagram*.

Proof of Theorem 4.6.9. Let G be the set of all $\sigma \in \mathbb{Z}_2^t$, such that σ occurs as a $[t]$-cube symmetry for some $\pi \in \mathrm{Aut}(\mathcal{D})$. Clearly, G is a linear subspace of $\mathbb{Z}_2^t$, when both are viewed as vector spaces over the field $\mathbb{F}_2$. Hence, there exists $0 \leq d \leq t$, such that $G \cong \mathbb{Z}_2^d$. Therefore, we can choose an orthogonal linear basis $\{e_1, \dots, e_t\}$ for $\mathbb{Z}_2^t$, such that $\{e_1, \dots, e_d\}$ is an orthogonal linear basis for G.

Let us split each cube $K_i \in \mathcal{C}$ into the orbits of the restriction of the action of G to K_i. We can think of cubes K_i as coordinate subspaces, that is as intersections of coordinate hyperplanes, with respect to the standard basis in the vector space $\mathbb{Z}_2^t$. The orbits themselves however are not coordinate subspaces, rather they are intersections of the coordinate subspaces corresponding to cubes with affine linear subspaces of dimension d obtained from G by parallel translations. Therefore, if we change the linear basis in $\mathbb{Z}_2^t$ from the standard one to $\{e_1, \dots, e_t\}$ at the same time as we split the cubes of $\mathcal{C}$ into the orbits as described above, we end up with a new t-family of cubes $\widetilde{\mathcal{C}} = \{L_1, \dots, L_q\}$, and an n-diagram $\widetilde{\mathcal{D}}$ over this family, which is induced from $\mathcal{D}$.

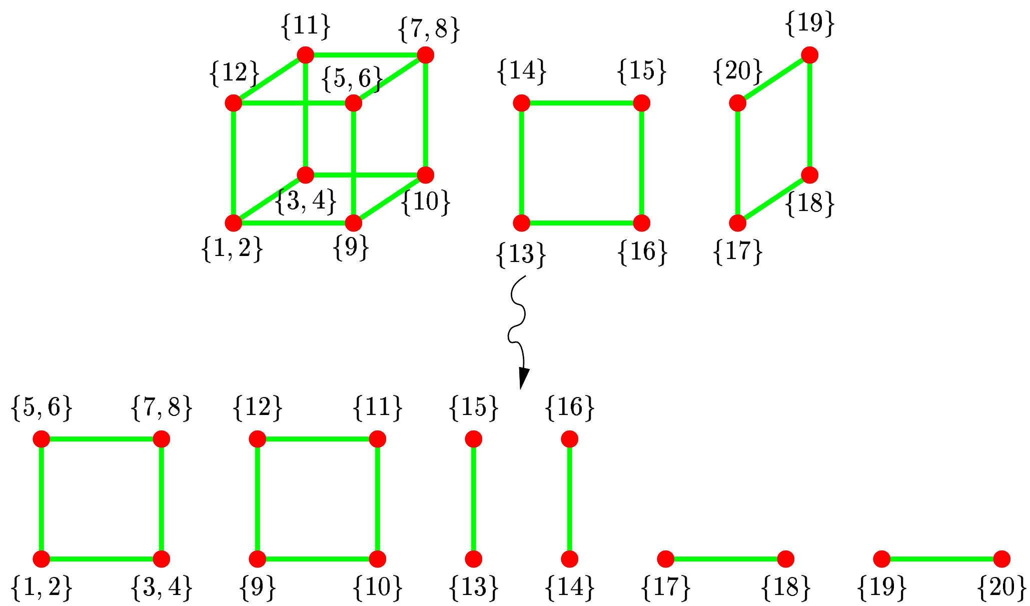


Figure 6. An example of the canonical splitting of a diagram.

By the choice of G and of the basis $\{e_1, \ldots, e_t\}$, we see that all the cubes of $\widetilde{\mathcal{C}}$ actually lie within the coordinate subspace of $\mathbb{Z}_2^t$ corresponding to the first d coordinates. Thus, we might as well think of $\widetilde{\mathcal{C}}$ as a d-family of cubes, with $\mathbb{Z}_2^d$ action induced from the action of $\mathbb{Z}_2^t$, from which condition 0) of the theorem follows.

Also, since the action on the ground set $[n]$ never changed, we still have the equality $\mathrm{Aut}(\mathcal{D}) = \mathrm{Aut}(\widetilde{\mathcal{D}})$, verifying condition 1) of the theorem.

Finally, since G acts transitively on each of its orbits, we can conclude that the cardinalities of the fibers are constant for the vertices of the same cube in $\widetilde{\mathcal{C}}$, thus demonstrating the truth of the last condition, and completing the proof of the theorem. □

4.7 STABILIZERS OF POINTS IN Y_{Π_n}

In this section we show that the stabilizers of points in Y_{Π_n} are not just abelian, but in fact are isomorphic to direct products of $\mathbb{Z}_2$. In view of the already proven results, it merely remains to put the puzzle pieces together.

Theorem 4.7.1 *For Y_{Π_n}, the De Concini-Procesi arrangement model of the braid arrangement, and $\omega \in Y_{\Pi_n}$, the stabilizer of ω with respect to the $\mathcal{S}_n$-action on Y_{Π_n} is a direct product of $\mathbb{Z}_2$'s:*

$$\mathrm{stab}_{Y_{\Pi_n}}(\omega) \;\cong\; \mathbb{Z}_2^h\,, \qquad \textit{for some } 0 \le h \le \lfloor n/2 \rfloor\,.$$

Proof. By (4.4.1) a point in Y_{Π_n} can be written as $\omega = (x, H_1, l_1, H_2, \dots, H_t, l_t)$, where $H_i \in \Pi_n \setminus \{\hat{0}\}$, and there does not exist a subspace $H \in \Pi_n$, $H \neq \mathbb{R}^n$, such that $H \supseteq \langle H_t, l_t \rangle$. By Proposition 4.4.2 we know that

$$\mathrm{stab}_{Y_{\Pi_n}}(\omega) = \mathrm{stab}_{\mathbb{R}^n}(x) \cap \mathrm{stab}_{\mathbb{R}^n}(l_1) \cap \ldots \cap \mathrm{stab}_{\mathbb{R}^n}(l_t). \tag{4.7.1}$$

By Lemma 4.6.5 there exist diagrams $\mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_t$, such that

$$\mathrm{Aut}(\mathcal{D}_0) = \mathrm{stab}_{\mathbb{R}^n}(x), \quad \text{and} \quad \mathrm{Aut}(\mathcal{D}_i) = \mathrm{stab}_{\mathbb{R}^n}(l_i), \quad \text{for each } i \in [t]. \tag{4.7.2}$$

Combining (4.7.1), (4.7.2), and Theorem 4.6.8, we find an n-diagram $\mathcal{D}$, such that $\mathrm{Aut}(\mathcal{D}) = \mathrm{stab}_{Y_{\Pi_n}}(\omega)$. Moreover, by the Reduction Theorem 4.6.9, we can assume that $\mathcal{D}$ is reduced.

If the partition $\rho(\mathcal{D})$ has a block B of cardinality at least 3, then, by Lemma 4.6.7, so do also the partitions $\rho(\mathcal{D}_0), \rho(\mathcal{D}_1), \dots, \rho(\mathcal{D}_t)$. Let H be the linear subspace of $\mathbb{R}^n$ of codimension 2 defined by setting the coordinates with indices in B equal. By construction, $x \in H$, and $l_1 \subseteq H, \dots, l_t \subseteq H$. Since $H \in \Pi_n$, we see that $x \in H$ implies $H_1 \subseteq H$. Further $l_1 \subseteq H$, together with $H_1 \subseteq H$, implies $\langle l_1, H_1 \rangle \subseteq H$. Hence $H_2 \subseteq H$, and so on, until we can conclude that $\langle l_t, H_t \rangle \subseteq H$. This yields a contradiction, since $H \neq \mathbb{R}^n$.

So we proved that all blocks of the partition $\rho(\mathcal{D})$ are of cardinality at most 2. Assume now there exist two different blocks B_1 and B_2 in $\rho(\mathcal{D})$, such that $|B_1| = |B_2| = 2$. Let H be the linear subspace of $\mathbb{R}^n$ of codimension 2 defined by equations $x_{i_1} = x_{i_2}$, $x_{j_1} = x_{j_2}$, where $B_1 = \{i_1, i_2\}$, $B_2 = \{j_1, j_2\}$. Again $H \in \Pi_n$, and by an argument completely analogous to the previous one, we can trace the two blocks B_1 and B_2 through the partitions $\rho(\mathcal{D}_0), \rho(\mathcal{D}_1), \dots, \rho(\mathcal{D}_t)$, and conclude that $\langle l_t, H_t \rangle \subseteq H$. This again yields a contradiction, since $H \neq \mathbb{R}^n$.

Now we know that $\rho(\mathcal{D})$ has at most one block of size 2. In particular, since $\mathcal{D}$ is reduced, all the fibers over I-cubes, for $|I| \geq 1$, are of cardinality 1. Let us say $\mathcal{D}$ is an n-diagram over a t-family of cubes $\mathcal{C} = (K_1, \dots, K_q)$, where t is minimal possible. If $\rho(\mathcal{D})$ has no blocks of size 2, then there exists a group isomorphism between $\mathrm{Aut}(\mathcal{D})$ and $\mathbb{Z}_2^t$, since each element $\pi \in \mathbb{Z}_2^t$ defines the maps between the fibers uniquely. Each I-cube defines at most $|I|$ new directions and has $2^{|I|}$ vertices, hence

$$t \leq |I_1| + \dots + |I_t| \leq 2^{|I_1|-1} + \dots + 2^{|I_t|-1} = n/2.$$

If the partition $\rho(\mathcal{D})$ has one block B of size 2, then, since $\mathcal{D}$ is reduced, B has to be a fiber over a $\emptyset$-cube. With t chosen as above, it is immediate that $\mathrm{Aut}(\mathcal{D}) \cong \mathbb{Z}_2^t \times \mathbb{Z}_2$, where the first factor on the right hand side is the group acting on the $[t]$-cube, and the second factor is acting on the set B. Just as before we get $t \leq (n-2)/2$, hence $t + 1 \leq n/2$. □

CHAPTER 5

A DESINGULARIZATION OF REAL DIFFEOMORPHIC ACTIONS OF FINITE GROUPS

5.1 INTRODUCTION

Abelianizations of finite group actions on complex manifolds appeared prominently in the work of Batyrev [B1], and a connection to the wonderful arrangement models of De Concini and Procesi was observed by Borisov and Gunnells [BG]. In the previous chapter we presented a detailed study of the key example over the reals, the abelianization of the permutation action of the symmetric group $\mathcal{S}_n$ on $\mathbb{R}^n$ given by the maximal De Concini-Procesi model of the braid arrangement. In particular, we showed that stabilizers of points on the arrangement model are elementary abelian 2-groups. We suggest to call an abelianization with this property a *digitalization* of the given action.

In the present chapter, we extend our analysis in two steps. First, for any linear action of a finite group on a real vector space, we define an arrangement of linear subspaces whose maximal De Concini-Procesi model we then show to be a digitalization of the given action. Second, we proceed by analyzing diffeomorphic actions of finite groups on smooth real manifolds. We propose a locally finite stratification of the manifold by smooth submanifolds and, observing that this stratification is actually a local subspace arrangement, we show that the associated maximal De Concini-Procesi model is a digitalization of the given action.

We present examples in the linear and in the non-linear case. First, we consider the permutation action of the symmetric group $\mathcal{S}_n$ on $\mathbb{R}^n$, and we find that our arrangement construction specializes to the rank 2 truncation of the braid arrangement. The resulting digitalization is the one discussed in the previous chapter.

As a non-linear example, we consider the action of $\mathcal{S}_n$ on $\mathbb{RP}^{n-1}$ given by projectivizing the real permutation action on $\mathbb{R}^n$. We show that our manifold stratification, in this case, coincides with the rank 2 truncation of the projectivized braid arrangement. The resulting digitalization thus is the maximal projective De Concini-Procesi model for the braid arrangement.

In considering these examples, a major role is played by the algebro-combinatorial concept of diagrams over families of cubes and their automorphism groups. This convenient algebro-combinatorial framework has been developed in the previous chapter in order to study stabilizers of points on the maximal model for the braid arrangement. It serves again in the present context as a manageable encoding of the occurring groups.

We give a short overview on the material presented in this chapter: In section 5.2 we provide a review on De Concini-Procesi arrangement models in an attempt to keep this exposition fairly self-contained. Our main results are presented in section 5.3. In 5.3.1 we propose a digitalization for any given linear action of a finite group on a real vector space; in 5.3.2 we extend our setting to diffeomorphic actions of finite groups on smooth real manifolds. Section 5.4 is focused on examples. After presenting a brief review on diagrams over families of cubes and their automorphism groups in 5.4.1, we work out details about the proposed digitalizations for the real permutation action in 5.4.2, and for the permutation action on real projective spaces in 5.4.3.

5.2 A review of De Concini-Procesi arrangement models

5.2.1 Arrangement models

We review the construction of De Concini-Procesi arrangement models as presented in [DP3]. Moreover, we recall an encoding of points in maximal arrangement models from [FK2] that is crucial for the technical handling of stabilizers (cf. 5.2.2).

The model construction

Let $\mathcal{A}$ be a finite family of linear subspaces in some real or complex vector space V. The combinatorial data of such subspace arrangement is customarily recorded by its *intersection lattice* $\mathcal{L} = \mathcal{L}(\mathcal{A})$, the partially ordered set of intersections among subspaces in $\mathcal{A}$ ordered by reversed inclusion. We agree on the empty intersection to be the full space V, represented by the minimal element $\hat{0}$ in the lattice. We will frequently use $\mathcal{L}_{>\hat{0}}$ to denote $\mathcal{L} \setminus \{\hat{0}\}$.

There is a family of arrangement models each coming from the choice of a certain subset of the intersection lattice, so-called *building sets*. For the moment we restrict our attention to the maximal model among those, which results from choosing the whole intersection lattice as building set.

We give two alternative descriptions for the maximal De Concini-Procesi model of $\mathcal{A}$. Consider the following map on the complement $\mathcal{M}(\mathcal{A}) := V \setminus \bigcup \mathcal{A}$ of the arrangement,

$$\Psi: \quad \mathcal{M}(\mathcal{A}) \longrightarrow V \times \prod_{X \in \mathcal{L}_{>\hat{0}}} \mathbb{P}(V/X)\,, \tag{5.2.1}$$

where Ψ is the natural inclusion into the first factor and the natural projection to the other factors restricted to $\mathcal{M}(\mathcal{A})$. Formally,

$$\Psi(x) \;=\; (\,x\,,(\langle x,X\rangle/X)_{X\in\mathcal{L}_{>\hat{0}}})\,,$$

where $\langle\cdot,\cdot\rangle$ denotes the linear span of subspaces or vectors, respectively, and $\langle x,X\rangle/X$ is interpreted as a point in $\mathbb{P}(V/X)$ for any $X\in\mathcal{L}_{>\hat{0}}$.

The map Ψ defines an embedding of $\mathcal{M}(\mathcal{A})$ into the product on the right hand side of (5.2.1). The closure of its image, $Y_{\mathcal{A}} := \overline{\mathrm{im}\Psi}$, is the *maximal De Concini-Procesi model* of the arrangement $\mathcal{A}$. If we want to stress the ambient space of the original arrangement, we will use the notation $Y_{V,\mathcal{A}}$ for $Y_{\mathcal{A}}$.

Alternatively, one can describe $Y_{\mathcal{A}}$ as the result of successive blowups of strata in V. Consider the stratification of V given by the linear subspaces in $\mathcal{A}$ and their intersections. Choose some linear extension of the opposite order in $\mathcal{L}$. Then, $Y_{\mathcal{A}}$ is the result of successive blowups of strata, respectively proper transforms of strata, corresponding to the subspaces in $\mathcal{L}$ in the chosen linear extension order.

Let us mention here that there is a projective analogue $\overline{Y}_{\mathcal{A}}$ of the affine arrangement model $Y_{\mathcal{A}}$ (cf. [DP3, §4]). In fact, the affine model $Y_{\mathcal{A}}$ is the total space of a line bundle over $\overline{Y}_{\mathcal{A}}$. We will need to refer to projective arrangement models only in one of our examples in Section 5.4. We therefore stay with the affine setting in the following exposition

Normal crossing divisors and nested set stratification

The term *wonderful* models has been coined for $Y_{\mathcal{A}}$ and its generalizations for other choices of building sets. We summarize the key facts about the maximal model supporting this connotation.

The space $Y_{\mathcal{A}}$ is a smooth algebraic variety with a natural projection onto the original ambient space V, $p:\; Y_{\mathcal{A}} \longrightarrow V$. The map p is the projection onto the first coordinate of the ambient space of $Y_{\mathcal{A}}$ on the right hand side of (5.2.1), respectively the concatenation of blowdown maps of the sequence of blowups resulting in $Y_{\mathcal{A}}$. This projection is an isomorphism on $\mathcal{M}(\mathcal{A})$, while the complement $Y_{\mathcal{A}}\setminus\mathcal{M}(\mathcal{A})$ is a divisor with normal crossings with irreducible components indexed by the elements of $\mathcal{L}_{>\hat{0}}$. An intersection of several irreducible components is non-empty (moreover, transversal and irreducible) if and only if the indexing lattice elements form a totally ordered set, i.e., a chain, in $\mathcal{L}$ [DP3, 3.1,3.2]. The stratification by irreducible components of the divisor and their intersections is called the *nested set stratification* of $Y_{\mathcal{A}}$, denoted $(Y_{\mathcal{A}},\mathfrak{D})$, for reasons that lie in the more general model construction for arbitrary building sets rather than the maximal building set $\mathcal{L}_{>\hat{0}}$.

An encoding of points in maximal arrangement models

Points in $Y_{\mathcal{A}}$ can be described as a sequence of a point and a number of lines in the vector space V according to the form of the ambient space for $Y_{\mathcal{A}}$ given on the right hand side of (5.2.1). However, there is a lot of redundant information in that description. The following compact encoding of points was suggested in [FK2, Sect 4.1].

Proposition 5.2.1 *Let ω be a point in the maximal wonderful model $Y_{\mathcal{A}}$ for a subspace arrangement $\mathcal{A}$ in complex or real space V. Then ω can be uniquely written as*

$$\omega \;=\; (x, H_1, \ell_1, H_2, \ell_2, \ldots, H_t, \ell_t) \;=\; (x, \ell_1, \ell_2, \ldots, \ell_t)\,, \tag{5.2.2}$$

where x is a point in V, the $H_1, \ldots, H_t$ form a descending chain of subspaces in $\mathcal{L}_{>\hat{0}}$, and the ℓ_i are lines in V, all subject to a number of additional conditions.

More specifically, $x = p(\omega)$, and the linear space H_1 is the maximal lattice element that, as a subspace of V, contains x. The line ℓ_1 is orthogonal to H_1 and corresponds to the coordinate entry of ω indexed by H_1 in $\mathbb{P}(V/H_1)$. The lattice element H_2, in turn, is the maximal lattice element that contains both H_1 and ℓ_1. The specification of lines ℓ_i, i.e., lines that correspond to coordinates of ω in $\mathbb{P}(V/H_i)$, and the construction of lattice elements H_{i+1}, continues analogously for $i \geq 2$ until a last line ℓ_t is reached whose span with H_t is not contained in any lattice element other than the full ambient space V. Note that, if H_t is a hyperplane, then the line ℓ_t is uniquely determined. The whole space V can be thought of as H_{t+1}. Observe that the H_i are determined by x and the sequence of lines ℓ_i; we choose to include the H_i at times in order to keep the notation more transparent.

The full coordinate information on ω can be recovered from (5.2.2) by setting $H_0 = \bigcap \mathcal{A}$, $\ell_0 = \langle x \rangle$, and retrieving the coordinate ω_H indexed by $H \in \mathcal{L}_{>\hat{0}}$ as

$$\omega_H \;=\; \langle \ell_j, H \rangle / H \;\in\; \mathbb{P}(V/H)\,, \tag{5.2.3}$$

where j is chosen from $\{1, \ldots, t\}$ such that $H \leq H_j$, but $H \not\leq H_{j+1}$.

For completeness, let us mention here that we can tell the open stratum in the nested set stratification $(Y_{\mathcal{A}}, \mathfrak{D})$ that contains a given point ω from its point/line encoding stated in Proposition 5.2.1.

Proposition 5.2.2 ([FK2, Prop 4.5]) *A point ω in a maximal arrangement model $Y_{\mathcal{A}}$ is contained in the open stratum of $(Y_{\mathcal{A}}, \mathfrak{D})$ indexed by the chain $H_1 > H_2 > \ldots > H_t > \hat{0}$ in $\mathcal{L}$ if and only if its point/line description* (5.2.2) *reads $\omega = (x, H_1, \ell_1, H_2, \ell_2, \ldots, H_t, \ell_t)$.*

5.2.2 Group actions on arrangement models and a description of stabilizers

Provided an arrangement is invariant under the action of a finite group, this action extends to the maximal arrangement model. We review the details, and recall a description for stabilizers of points in the model from [FK2].

Group actions on $Y_{\mathcal{A}}$

Let $\mathcal{A}$ be an arrangement that is invariant under the linear action of a finite group G on the real or complex ambient space V. Without loss of generality, we can assume that this action is orthogonal [V, 2.3, Thm 1]. We denote the corresponding G-invariant positive definite symmetric bilinear form by the usual scalar product.

The group G acts on the ambient space of the arrangement model $Y_{\mathcal{A}}$, i.e., for $(x, (x_X)_{X\in\mathcal{L}_{>\hat{0}}}) \in V \times \prod_{X\in\mathcal{L}_{>\hat{0}}} \mathbb{P}(V/X)$ and $g \in G$, we have

$$g\,(x, (x_X)_{X\in\mathcal{L}_{>\hat{0}}}) \;=\; (\,g(x), (\,g(x_{g^{-1}(X)})\,)_{X\in\mathcal{L}_{>\hat{0}}})\,,$$

where $g(x_{g^{-1}(X)}) \in \mathbb{P}(V/X)$ for $X \in \mathcal{L}_{>\hat{0}}$. Since the inclusion map Ψ of (5.2.1) commutes with the G-action, and G acts continuously on V, we conclude that $Y_{\mathcal{A}} = \overline{\mathrm{Im}\Psi}$ is as well G-invariant. In particular, the G-action on $Y_{\mathcal{A}}$ extends the G-action on the complement of $\mathcal{A}$.

Stabilizers of points on $Y_{\mathcal{A}}$

The point/line description for points in the arrangement model $Y_{\mathcal{A}}$ given in 5.2.1 allows for a concise description of stabilizers with respect to the G-action on $Y_{\mathcal{A}}$.

Proposition 5.2.3 ([FK2, Prop 4.2]) *For a maximal arrangement model $Y_{\mathcal{A}}$ that is equipped with the action of a finite group G stemming from a linear action of G on the arrangement, the stabilizer of a point $\omega = (x, H_1, \ell_1, H_2, \ell_2, \ldots, H_t, \ell_t)$ in $Y_{\mathcal{A}}$ is of the form*

$$\mathrm{stab}_{Y_{\mathcal{A}}}(\omega) \;=\; \mathrm{stab}_V(x) \cap \mathrm{stab}_V(\ell_1) \cap \ldots \cap \mathrm{stab}_V(\ell_t)\,, \tag{5.2.4}$$

where, for $i = 1, \ldots, t$, $\mathrm{stab}_V(\ell_i)$ denotes the elements in G that preserve the line ℓ_i in V as a set.

5.2.3 Models for local subspace arrangements

The arrangement model construction of De Concini & Procesi generalizes to the context of local subspace arrangements.

Definition 5.2.4 Let X be a smooth d-dimensional real or complex manifold and $\mathcal{A}$ a family of smooth real or complex submanifolds in X such that all non-empty intersections of submanifolds in $\mathcal{A}$ are connected, smooth submanifolds. The family $\mathcal{A}$ is called a *local subspace arrangement* if for any $x \in \bigcup \mathcal{A}$ there exists an open neighborhood U of x in X, a subspace arrangement $\tilde{\mathcal{A}}$ in a real or complex d-dimensional vector space V and a diffeomorphism $\phi : U \longrightarrow V$, mapping $\mathcal{A}$ to $\tilde{\mathcal{A}}$.

Local subspace arrangements fall into the class of conically stratified manifolds as appearing in work of MacPherson & Procesi [MP] in the complex and in work of Gaiffi [Ga2] in the real setting.

A generalization of the arrangement model construction of De Concini & Procesi by sequences of blowups of smooth strata for conically stratified complex manifolds is given in [MP]. Details are provided for blowing up so-called irreducible strata, the more general construction for an arbitrary building set in the stratification is outlined in Sect. 4 of [MP].

In this chapter, we will be concerned with *maximal wonderful models* for conically stratified real manifolds X, in the special case of local subspace arrangements $\mathcal{A}$. The maximal model $Y_{\mathcal{A}} = Y_{X,\mathcal{A}}$ results from successive blowups of *all* initial strata, respectively their proper transforms, according to some linear order on strata which is non-decreasing in dimension.

In fact, local subspace arrangements consisting of a *finite* number of submanifolds implicitly appear already in the arrangement model constructions of De Concini & Procesi [DP3]. A single blowup in a subspace arrangement leads to the class of local arrangements, and it is due to the choice of blowup order on building set strata that this class is closed under blowups that occur in the inductive construction of the arrangement models (compare the discussion in [FK1, 4.1.2], in particular, Example 4.6).

We will encounter the case of local subspace arrangements $\mathcal{A}$ in a smooth real manifold X that are invariant under the diffeomorphic action of a finite group G on X. The G-action can be extended to the maximal model $Y_{\mathcal{A}}$, observing that we can simultaneously blow up orbits of strata, thereby lifting the G-action step by step through the construction process. In particular, the concatenation of blowdown maps $p : Y_{\mathcal{A}} \to X$ is G-equivariant.

5.3 Digitalizing finite group actions

5.3.1 *Finite linear actions on* $\mathbb{R}^n$

In this subsection we assume G to be a finite subgroup of the orthogonal group $O(n)$ acting effectively on $\mathbb{R}^n$. As pointed out before, assuming the action to be orthogonal is not a restriction (compare 5.2.2).

We construct an abelianization of the given action. For any subgroup H in G (we use the notation $H \leq G$ in the sequel), define

$$L(H) \;:=\; \langle\, \ell \mid \ell \text{ line in } \mathbb{R}^n \text{ with } h \cdot \ell = \ell \text{ for all } h \in H \,\rangle\,,$$

the linear span of lines in $\mathbb{R}^n$ that are invariant under H, i.e., the span of lines that are either fixed or flipped by any element h in H. Denote by $\mathcal{A}$ the arrangement given by the *proper* subspaces $L(H) \subsetneq \mathbb{R}^n$, H subgroup in G. Set $Y := Y_{\mathcal{A}}$, the maximal De Concini-Procesi wonderful model for $\mathcal{A}$ as discussed in 5.2.1. If we want to stress the particular group action that gives rise to the arrangement $\mathcal{A}$ we write $\mathcal{A}(G)$ and $Y_{\mathcal{A}(G)}$, or $\mathcal{A}(G \circlearrowright \mathbb{R}^n)$ and $Y_{\mathcal{A}(G \circlearrowright \mathbb{R}^n)}$, respectively.

We will now propose $Y_{\mathcal{A}}$ as an abelianization of the given linear action. Recall that we use the term *digitalization* for an abelianization with stabilizers that are not merely abelian but elementary abelian 2-groups, i.e., are isomorphic to $\mathbb{Z}_2^k$ for some $k \in \mathbb{N}$.

Theorem 5.3.1 *Let an effective action of a finite subgroup G of $O(n)$ on $\mathbb{R}^n$ be given. Then the wonderful arrangement model $Y_{\mathcal{A}}$, as defined above, is a digitalization of the given action.*

Proof. As a first step we prove that

$$L(\operatorname{stab}\omega) \;=\; \mathbb{R}^n \qquad \text{for any } \omega \in Y\,.$$

Let $\omega \in Y$. Using the encoding of points in arrangement models as sequences of point and lines from 5.2.1, we have $\omega = (x, \ell_1, \ldots, \ell_t)$, the associated sequence of building set spaces being $V_1, \ldots, V_t$. The description of $\operatorname{stab}\omega$ from Proposition 5.2.3,

$$\operatorname{stab}\omega \;=\; \operatorname{stab}x \,\cap\, \operatorname{stab}\ell_1 \,\cap\, \ldots \,\cap\, \operatorname{stab}\ell_t\,,$$

implies that $x \in L(\operatorname{stab}\omega)$, and $\ell_i \subseteq L(\operatorname{stab}\omega)$ for $i = 1, \ldots, t$.

The building set element V_1 is the smallest subspace among intersections of spaces $L(H)$ in $\mathcal{A}$ such that $x \in V_1$, in particular, $V_1 \subseteq L(\operatorname{stab}\omega)$. Similarly, the building set element V_2 is the smallest subspace among intersections of spaces $L(H)$ in $\mathcal{A}$ such that $\langle V_1, \ell_1\rangle \subseteq V_2$; since $\langle V_1, \ell_1\rangle \subseteq L(\operatorname{stab}\omega)$, so is V_2: $V_2 \subseteq L(\operatorname{stab}\omega)$.

By analogous arguments we conclude that $V_3, \ldots, V_{t+1} \subseteq L(\operatorname{stab}\omega)$. However, by the description of ω as a sequence of point and lines we know that $V_{t+1} = \mathbb{R}^n$, which proves our claim.

With $L(\operatorname{stab}\omega) = \mathbb{R}^n$, we can now choose a basis $v_1, \ldots, v_n$ in $\mathbb{R}^n$ such that any $\langle v_i\rangle$, for $i = 1, \ldots, n$, is invariant under the action of $\operatorname{stab}\omega$.

Consider the homomorphism

$$\begin{array}{rccl} \alpha : & \operatorname{stab}\omega & \longrightarrow & \mathbb{Z}_2^n \\ & h & \longmapsto & (\epsilon_1, \dots, \epsilon_n)\,, \end{array}$$

with $\epsilon_i \in \mathbb{Z}_2$ defined by $h(v_i) = \epsilon_i\, v_i$ for $i = 1, \dots, n$. Since we assume the action to be effective, α is injective. Hence $\operatorname{stab}\omega \cong \mathbb{Z}_2^k$ for some $k \leq n$. □

5.3.2 Finite diffeomorphic actions on manifolds

We now generalize the results of the previous subsection to diffeomorphic actions of finite groups on smooth manifolds. To this end, we first propose a stratification of the manifold and show that the stratification locally coincides with the arrangement stratifications on tangent spaces that arise from the induced linear actions as described in the previous section. We can assume, without loss of generality, that the manifold is connected, since we can work with connected components one at a time.

The $\mathcal{L}$-stratification

Let X be a smooth manifold, G a finite group that acts diffeomorphically on X. For any point $x \in X$, and any subgroup $H \leq \operatorname{stab} x$, H acts linearly on the tangent space T_xX of X in x. Consider as above

$$L(x, H) \;:=\; \langle\, \ell \,|\, \ell \text{ line in } T_xX \text{ with } h \cdot \ell = \ell \text{ for all } h \in H \,\rangle\,,$$

the linear subspace in T_xX spanned by lines that are invariant under the action of H. Denote the arrangement of proper subspaces $L(x, H)$ in T_xX, $\mathcal{A}(\operatorname{stab} x \circlearrowright T_xX)$, by $\mathcal{A}_x$.

For any subgroup H in $\operatorname{stab} x$, we take up the homomorphism that occurred in the proof of Theorem 5.3.1, and define

$$\alpha_{x,H} :\; H \;\longrightarrow\; \mathbb{Z}_2^{\dim \mathrm{L(x,H)}}$$

by choosing a basis $v_1, \dots, v_t$, $t := \dim \mathrm{L(x,H)}$, for $L(x, H)$, and setting

$$\alpha_{x,H}(h) \;=\; (\epsilon_1, \dots, \epsilon_t)\,,$$

for $h \in H$, with $\epsilon_i \in \mathbb{Z}_2$ determined by $h(v_i) = \epsilon_i v_i$ for $i = 1, \dots, t$.

Moreover, we define

$$F(x, H) \;:=\; \ker \alpha_{\mathrm{x,H}}\,.$$

Note that $F(x,H)$ is the normal subgroup of elements in H that fix all of $L(x,H)$ pointwise. We denote by $\mathcal{L}(x,H)$ the connected component of $\mathrm{Fix(F(x,H)} \circlearrowleft \mathrm{X)}$ in X that contains x.

Consider the stratification of X by the collection of submanifolds $\mathcal{L}(x,H)$ for $x \in X$, $H \leq \mathrm{stab\,x}$,

$$\mathcal{L} \;:=\; \{\mathcal{L}(x,H)\}_{x\in X,\, H\leq \mathrm{stab}\, x}\,.$$

We will refer to this stratification as the *$\mathcal{L}$-stratification* of X. Observe that $\mathcal{L}$ is a locally finite stratification.

We recall the following fact from the theory of group actions on smooth manifolds:

Proposition 5.3.2 *Let G be a compact Lie group acting diffeomorphically on a smooth manifold X, and let $x_0 \in X$. Then there exists a* $\mathrm{stab}\,x_0$*-equivariant diffeomorphism Φ_{x_0} from an open neighborhood U of x_0 in X to the tangent space $T_{x_0}X$ of X in x_0.*

This is a special case of the so-called *slice theorem* [A, tD] that originally appeared in work of Bochner [Bo].

We return to our setting of G being a finite group.

Proposition 5.3.3 *The diffeomorphism Φ_{x_0} maps the $\mathcal{L}$-stratification of X to the arrangement stratification on $T_{x_0}X$ given by $\mathcal{A}_{x_0}$, i.e.,*

$$\Phi_{x_0}(\mathcal{L}(x_0,H)) \;=\; L(x_0,H) \qquad \textit{for any}\;\; H \leq \mathrm{stab}\,x_0\,.$$

Proof. By definition, $\mathcal{L}(x_0,H) = \mathrm{Fix}\,(F(x_0,H) \circlearrowleft X)$, which, using the $\mathrm{stab}\,x_0$-equivariance of Φ_{x_0}, implies that $\Phi_{x_0}(\mathcal{L}(x_0,H)) = \mathrm{Fix}\,(F(x_0,H) \circlearrowleft T_{x_0}X)$. We are left to show that

$$\mathrm{Fix}\,(F(x_0,H) \circlearrowleft T_{x_0}X) \;=\; L(x_0,H)\,.$$

Obviously, $L(x_0,H) \subseteq \mathrm{Fix}\,(F(x_0,H) \circlearrowleft T_{x_0}X)$, and we need to see that $\mathrm{Fix}\,(F(x_0,H) \circlearrowleft T_{x_0}X)$ does not exceed $L(x_0,H)$.

Note that H acts on $L(x_0,H)$. By definition, $F(x_0,H)$ is a normal subgroup of H with quotient $H/F(x_0,H) \cong \mathbb{Z}_2^d$ for some $d \leq t = \mathrm{dim\,L(x_0,H)}$, and we find that H acts on $\mathrm{Fix}\,(F(x_0,H) \circlearrowleft T_{x_0}X)$: For $x \in \mathrm{Fix}\,(F(x_0,H) \circlearrowleft T_{x_0}X)$, $h \in H$, and $h_1 \in F(x_0,H)$, we have $h_1hx = h\tilde{h}_1x$ for some $\tilde{h}_1 \in F(x_0,H)$, thus $h_1hx = hx$, i.e., $hx \in \mathrm{Fix}\,(F(x_0,H) \circlearrowleft T_{x_0}X)$.

Instead of considering the action of H on $\mathrm{Fix}\,(F(x_0,H) \circlearrowleft T_{x_0}X)$, we consider the induced action of $H/F(x_0,H)$ on $\mathrm{Fix}\,(F(x_0,H) \circlearrowleft T_{x_0}X)$. Since $H/F(x_0,H) \cong \mathbb{Z}_2^d$ for some $d \leq t$, $\mathrm{Fix}\,(F(x_0,H) \circlearrowleft T_{x_0}X)$ decomposes into 1-dimensional representation spaces, which, as lines that are invariant under the action of H, must be contained in $L(x_0, H)$ by definition. This shows that Fix $(F(x_0, H) \circlearrowright T_{x_0}X)$ does not exceed $L(x_0, H)$, and thus completes our proof. □

In particular, Proposition 5.3.3 shows that the submanifolds $\mathcal{L}(x, H)$ in the $\mathcal{L}$-stratification form a local subspace arrangement in X. Moreover, the $\mathcal{L}$-stratification is invariant under the action of G since $g(\mathcal{L}(x, H)) = \mathcal{L}(g(x), gHg^{-1})$ for any $x \in X$, $H \leq \operatorname{stab} x$, and any $g \in G$. Hence, we have at hand the maximal G-equivariant wonderful model $Y_{\mathcal{L}} = Y_{X,\mathcal{L}}$ of the local subspace arrangement $\mathcal{L}$ in X as outlined in 5.2.3.

Digitalizing manifolds

We propose the maximal wonderful model of X with respect to the $\mathcal{L}$-stratification as a digitalization of the manifold X.

Theorem 5.3.4 *Let G be a finite group acting diffeomorphically and effectively on a smooth manifold X. Then the maximal wonderful blowup of X with respect to the $\mathcal{L}$-stratification $Y_{X,\mathcal{L}}$ is a digitalization of the given action.*

Proof. Let x be a point in $Y_{X,\mathcal{L}}$, $x_0 = p(x)$ its image under the blowdown map $p:\ Y_{X,\mathcal{L}} \longrightarrow X$. Since p is G-equivariant, stab $x \subseteq \operatorname{stab} x_0$, hence we can restrict our attention to stab x_0 when determining the stabilizer of x in G.

Consider the stab x_0-equivariant diffeomorphism Φ_{x_0} as discussed above (Proposition 5.3.2),

$$\Phi_{x_0} : \quad U \longrightarrow T_{x_0}X \, ,$$

where U is an open neighborhood of x_0 in X, such that Φ_{x_0} maps the $\mathcal{L}$-stratification on U to the arrangement stratification on the tangent space at x_0. Since the De Concini-Procesi model is defined locally, the diffeomorphism Φ_{x_0} induces a stab x_0-equivariant diffeomorphism between the inverse image of U under the blowdown map, $p^{-1}U = Y_{U,\mathcal{L}}$, and the De Concini-Procesi model for the arrangement $\mathcal{A}_{x_0}$ in the tangent space $T_{x_0}X$,

$$\widetilde{\Phi}_{x_0} : \quad Y_{U,\mathcal{L}} \longrightarrow Y_{T_{x_0}X, \mathcal{A}_{x_0}} \, .$$

In particular,

$$\operatorname{stab} x \;\cong\; \operatorname{stab} \widetilde{\Phi}_{x_0}(x) \, ,$$

which, by our analysis of the linear setting, is an elementary abelian 2-group, provided we can see that stab x_0 acts effectively on $T_{x_0}X$. To settle this remaining point, assume that there exists a group element $g \neq e$ in stab x_0 that fixes all of $T_{x_0}X$. By Proposition 5.3.2, g then fixes an open neighborhood of x_0 in X, which implies that g fixes all of X, contrary to our assumption of the action being effective. □

5.4 Permutation actions on linear and on projective spaces

One of the most natural linear actions of a finite group is the action of the symmetric group $\mathcal{S}_n$ permuting the coordinates of a real n-dimensional vector space. This action induces a diffeomorphic action of $\mathcal{S}_n$ on $(n-1)-$dimensional real projective space $\mathbb{RP}^{n-1}$. Our goal in this section is to give explicit descriptions of the $\mathcal{L}$-stratifications and the resulting digitalizations in both cases.

To this end, we will first review the algebro-combinatorial setup of diagrams over families of cubes and their automorphism groups from [FK2]. We will then show that, in the case of the real permutation action, the arrangement $\mathcal{A}(\mathcal{S}_n)$ coincides with the rank 2 truncation of the braid arrangement, $\mathcal{A}_{n-1}^{\mathrm{rk}\geq 2}$, i.e., the braid arrangement $\mathcal{A}_{n-1}$ without its hyperplanes. We can thus conclude that the abelianization construction proposed in the present chapter specializes to the maximal model of the braid arrangement discussed in [FK2].

For the permutation action on $\mathbb{RP}^{n-1}$, we show that the $\mathcal{L}$-stratification coincides with the rank 2 truncation of the projectivized braid arrangement, $\mathbb{P}\mathcal{A}_{n-1}^{\mathrm{rk}\geq 2}$, thus the digitalization proposed in 5.3.2 coincides with the maximal projective arrangement model for $\mathcal{A}_{n-1}$ (cf. [DP3, §4]).

5.4.1 Automorphism groups of diagrams over families of cubes

For the sake of completeness, we here review the setup of diagrams over families of cubes and their automorphism groups as developed in [FK2, Sect. 6].

Definition 5.4.1 A *t-family of cubes* is a collection $\mathcal{C}=\{C_1,\dots,C_k\}$ of sets where each C_j is the set of all subsets of a (possibly empty) index set $I_j\subseteq\{1,\dots,t\}$ for $j=1,\dots,k$. We think of the C_j as copies of $0/1$ cubes which are $|I_j|$-dimensional faces of the t-dimensional $0/1$ cube placed in the coordinate directions prescribed by $I_j\subseteq\{1,\dots,t\}$. Following this interpretation, we talk about subsets of I_j as *vertices* $\mathrm{vert}(C_j)$ of cubes C_j in $\mathcal{C}$, and about *vertices* of the family of cubes, $\mathrm{vert}(\mathcal{C})=\biguplus_{j=1}^{k}\mathrm{vert}(C_j)$. To specify particular vertices, we use the notation (C_j,S), $C_j\in\mathcal{C}$, $S\subseteq I_j$, where the first coordinate names a cube in $\mathcal{C}$ and the second coordinate specifies the vertex of the cube.

An *n-diagram* $\mathcal{D}$ over a t-family of cubes $\mathcal{C}$ is a partition of the set $[n]:=\{1,\dots,n\}$ into $|\mathrm{vert}(\mathcal{C})|$ (possibly empty) blocks, and a bijection between the blocks of this partition and the vertices of $\mathcal{C}$; in other words, it is a function

$$\begin{array}{rcl} \mathcal{D} : [n] & \longrightarrow & \mathrm{vert}(\mathcal{C}) \\ i & \longmapsto & (C_{\alpha(i)}, S_i)\,, \end{array}$$

where $\alpha(i) \in \{1, \ldots, k\}$ specifies the index of the cube, and $S_i \subseteq I_{\alpha(i)}$ the vertex of $C_{\alpha(i)}$ assigned to i.

For a vertex (C_j, S) of $\mathcal{C}$, we call the set $\mathcal{D}^{-1}(C_j, S)$ the *(vertex) fiber* of $\mathcal{D}$ over (C_j, S). For a cube C_j in $\mathcal{C}$, the *(cube) fiber* of $\mathcal{D}$ over C_j is defined as $\mathcal{D}^{-1}(C_j) := \bigcup_{S \subseteq I_j} \mathcal{D}^{-1}(C_j, S)$. We denote the partition of $[n]$ into vertex fibers over $\mathcal{C}$ by $\rho(\mathcal{D})$.

For a given n-diagram $\mathcal{D}$ over a t-family of cubes $\mathcal{C}$ the *group of automorphisms* of $\mathcal{D}$, $\mathrm{Aut}\,\mathcal{D}$, consists of all permutations $\pi \in \mathcal{S}_n$, such that

(i) $\pi_{\restriction \mathcal{D}^{-1}(C_j)} \in \mathcal{S}_{\mathcal{D}^{-1}(C_j)}$ for $j = 1, \ldots, k$;

(ii) there exists a group element $\sigma(\pi) \in \mathbb{Z}_2^t$ such that

$$S_{\pi(i)} = \sigma(\pi)_{\alpha(i)}(S_i), \qquad \text{for all} \quad i \in [n],$$

where $\sigma(\pi) \in \mathbb{Z}_2^t$ is interpreted as a bijection on the vertices of the t-dimensional $0/1$ cube and $\sigma(\pi)_j = \mathrm{proj}_{I_j}(\sigma(\pi))$ is its projection to the vertices of the face C_j for $j = 1, \ldots, k$.

Informally speaking, automorphisms of an n-diagram $\mathcal{D}$ over $\mathcal{C}$ are permutations π of $[n]$ that preserve cube fibers in $\mathcal{D}$ and, restricted to any cube fiber, map vertex fibers to vertex fibers according to an overall scheme that is obtained by restricting a bijection $\sigma(\pi)$ on the vertices of the t-dimensional $0/1$ cube to the respective cube in $\mathcal{C}$.

In [FK2] we proved two structure theorems about n-diagrams and their automorphism groups which we cite here for future use:

Theorem 5.4.2 (1) ([FK2, Theorem 6.8, Lemma 6.7]) *The set of automorphism groups of n-diagrams is closed under intersection, i.e., for any two n-diagrams $\mathcal{D}_1$, $\mathcal{D}_2$ there exists an n-diagram $\mathcal{D}$, canonically depending on $\mathcal{D}_1$ and $\mathcal{D}_2$, such that*

$$\mathrm{Aut}\,\mathcal{D}_1 \,\cap\, \mathrm{Aut}\,\mathcal{D}_2 \;=\; \mathrm{Aut}\,\mathcal{D}\,.$$

For the partitions of $[n]$ by vertex fibers associated with the respective diagrams,

$$\rho(\mathcal{D}) \;=\; \rho(\mathcal{D}_1) \,\wedge\, \rho(\mathcal{D}_2)\,,$$

where $\wedge$ denotes the meet-operation on the partition lattice Π_n.
(2) ([FK2, Theorem 6.9]) *For any n-diagram $\mathcal{D}$ over a t-family of cubes $\mathcal{C}$ there exists an n-diagram $\widetilde{\mathcal{D}}$ over a $\tilde{t}$-family of cubes $\widetilde{\mathcal{C}}$, canonically depending on $\mathcal{D}$, such that*

(i) $\mathrm{Aut}\,\mathcal{D} = \mathrm{Aut}\,\widetilde{\mathcal{D}}$

(ii) $|\widetilde{\mathcal{D}}^{-1}(C,S)| = |\widetilde{\mathcal{D}}^{-1}(C,T)|$ *for any* $C \in \widetilde{\mathcal{C}}$ *and* $S,T \in \mathrm{vert}(C)$.

Moreover, $\rho(\widetilde{\mathcal{D}}) = \rho(\mathcal{D})$ *in the partition lattice* Π_n.
We call a diagram reduced *if it has equicardinal vertex fibers over the vertices of any fixed cube as described in* (ii).

We remark here that the equality $\rho(\widetilde{\mathcal{D}}) = \rho(\mathcal{D})$ in Theorem 5.4.2(2) was not explicitly stated in [FK2], however it follows directly from the proof of [FK2, Theorem 6.9].

To illustrate the context in which the setup of diagrams over families of cubes and their automorphism groups proved useful, we provide the following examples.

Example 5.4.3 Consider the action of the symmetric group $\mathcal{S}_n$ on $\mathbb{R}^n$ by permuting coordinates.

(1) ([FK2, Lemma 6.5 (1)]) For $x \in \mathbb{R}^n$, let $\pi(x) = (B_1|\ldots|B_k)$ be the partition of $[n]$ given by the index sets with equal coordinate entries in x. Let $\mathcal{D}(x)$ be the n-diagram over the 0-family of cubes consisting of k cubes $C_1,\ldots,C_k$ of dimension 0 with (cube) fibers $\mathcal{D}(x)^{-1}(C_j) = B_j$ for $j = 1,\ldots,k$. Then, the stabilizer of x in $\mathbb{R}^n$ under the permutation action, i.e., the Young subgroup of $\mathcal{S}_n$ corresponding to the partition $\pi(x)$, is isomorphic to the automorphism group of $\mathcal{D}(x)$.

(2) ([FK2, Lemma 6.5 (2)]) Let ℓ be a line in $\mathbb{R}^n$ generated by a non-zero vector $x \in \mathbb{R}^n$. If the blocks in $\pi(x)$ can be arranged into pairs $B^{(1)}_{i_1}, B^{(2)}_{i_1}, \ldots, B^{(1)}_{i_s}, B^{(2)}_{i_s}$ with $|B^{(1)}_{i_j}| = |B^{(2)}_{i_j}|$ and coordinate entries in x corresponding to $B^{(1)}_{i_j}$, $B^{(2)}_{i_j}$ have the same absolute value for $j = 1,\ldots,s$, and one remaining block B_{i_0} with corresponding coordinate entries in x being 0, denote by $\mathcal{D}(\ell)$ the 1-diagram with s cubes $C_1,\ldots,C_s$ of dimension 1 and one cube C_0 of dimension 0, where $\mathcal{D}(\ell)^{-1}(C_j,\emptyset) = B^{(1)}_{i_j}$ and $\mathcal{D}(\ell)^{-1}(C_j,\{1\}) = B^{(2)}_{i_j}$ for $j = 1,\ldots,s$ and $\mathcal{D}(\ell)^{-1}(C_0) = B_{i_0}$. If such a construction is not possible, due to multiplicities of coordinate entries in x, set $\mathcal{D}(\ell) = \mathcal{D}(x)$ as described in (1). Then, the stabilizer of ℓ in $\mathbb{R}^n$ under the permutation action, i.e., the subgroup of permutations that either fix or flip the line ℓ, is isomorphic to the automorphism group of $\mathcal{D}(\ell)$.

(3) ([FK2, Theorem 7.1]) Stabilizers of points in the maximal De Concini-Procesi arrangement model $Y_{\mathcal{A}_{n-1}}$ for the braid arrangement with respect to the natural $\mathcal{S}_n$-action are automorphism groups of (reduced) n-diagrams with at most one (vertex) fiber of cardinality greater or equal 2. In particular, the stabilizers are elementary abelian 2-groups.

5.4.2 Digitalizing the real permutation action

As outlined above, we will now recover the rank 2 truncation of the braid arrangement as the arrangement $\mathcal{A}(\mathcal{S}_n)$ arising from the real permutation action.

Let us fix some notation. Any set partition $\pi = (B_1 | \ldots | B_k) \vdash [n]$ gives rise to an intersection of hyperplanes U_π in the braid arrangement $\mathcal{A}_{n-1} = \{H_{ij}\colon x_j - x_i = 0 \mid 1 \leq i, j \leq n\} \subseteq \mathbb{R}^n$, namely

$$U_\pi \;:=\; \bigcap_{r=1}^{k} \bigcap_{i,j\in B_r} H_{ij}\,.$$

We call U_π the *braid space* associated to π.

We find that braid spaces occur in the arrangement $\mathcal{A}(\mathcal{S}_n)$. They arise from particular subgroups of $\mathcal{S}_n$, namely automorphism groups of diagrams over families of cubes as presented in 5.4.1.

Lemma 5.4.4 *Let $\mathcal{D}$ be a reduced diagram over a family of cubes, and assume that fibers over 0-cubes are either singleton sets or have cardinality at least 3. Then the space determined by* $\operatorname{Aut}\mathcal{D}$ *in the arrangement $\mathcal{A}(\mathcal{S}_n)$ is the braid space associated to the set partition $\rho(\mathcal{D})$,*

$$L(\operatorname{Aut}\mathcal{D}) \;=\; U_{\rho(\mathcal{D})}\,.$$

Proof. Let us first assume that the underlying family of cubes for the diagram $\mathcal{D}$ consists of a single t-dimensional cube C. In particular, the partition $\rho(\mathcal{D}) \vdash [n]$ has 2^t equicardinal blocks $B_1, \ldots, B_{2^t}$. For the following discussion, we identify the set $[n]$ with the index set for the coordinates of vectors in $\mathbb{R}^n$.

A line ℓ in V that is invariant under the action of $\operatorname{Aut}\mathcal{D}$ must have equal coordinate entries within every vertex fiber of $\mathcal{D}$, that is within every block of the partition $\rho(\mathcal{D})$, since for any such fiber B_j the full symmetric group $\mathcal{S}_{B_j}$ is a subgroup of $\operatorname{Aut}\mathcal{D}$. A sign change within a fiber would only be possible if it were a 2-element fiber over a 0-dimensional cube which we excluded by our assumptions.

We can thus consider coordinates of generating lines for $L(\operatorname{Aut}\mathcal{D})$ blockwise, and can conclude at this point that

$$\dim\left(L(\operatorname{Aut}\mathcal{D})\right) \;\leq\; 2^t\,.$$

Moreover, coordinates of a generating line ℓ in $L(\operatorname{Aut}\mathcal{D})$ must all have the same absolute value since $\operatorname{Aut}\mathcal{D}$ acts transitively on the full set of coordinates $[n]$.

Describing the sign pattern for a generating line in $L(\operatorname{Aut}\mathcal{D})$ on the fibers in the t coordinate directions of the underlying cube determines the sign for the remaining fibers. We want to show that, by fixing the sign pattern in the coordinate directions of the cube, we obtain 2^t linearly independent generating lines for $L(\mathrm{Aut}\,\mathcal{D})$ and, to this end, we formalize our description slightly.

We write generating vectors for the lines in n-dimensional space as vectors $v = (v_S)_{S\subseteq[t]}$ with coordinates indexed by subsets S of $[t]$ and with entries ± 1, where v_S stands for the coordinate entries on the fiber $\mathcal{D}^{-1}(C, S)$ over the vertex $S \subseteq [t]$ of C. A function $\sigma : [t] \to \{\pm 1\}$, the choice of signs in the coordinate directions of the cube mentioned above, determines such a vector $v^{(\sigma)}$ by

$$v_S^{(\sigma)} := \prod_{i\in S} \sigma(i) \quad \text{for } S \subseteq [t]\,.$$

We claim that the 2^t generating lines $\langle v^{(\sigma)}\rangle$, $\sigma : [t] \to \{\pm 1\}$, for $L(\mathrm{Aut}\,\mathcal{D})$ are linearly independent, and we verify this fact by showing that the vectors $v^{(\sigma)}$ are pairwise orthogonal.

For functions $\sigma, \tau : [t] \to \{\pm 1\}$ denote by $D(\sigma, \tau)$ the subset of $[t]$ on which the functions differ. Writing out the scalar product, we obtain

$$v^{(\sigma)} v^{(\tau)} = \sum_{S\subseteq[t]} \prod_{i\in S} \sigma(i)\tau(i) = \sum_{S\subseteq[t]} (-1)^{|S\cap D(\sigma,\tau)|}\,.$$

Since $D(\sigma, \tau)$ is non-empty for distinct functions σ, τ there is a bijection between subsets of $[t]$ containing a fixed element x of $D(\sigma, \tau)$ and those not containing x. Pairs of subsets linked by this bijection give contributions of opposite sign to the sum above, and we conclude that $v^{(\sigma)} v^{(\tau)} = 0$ for distinct functions σ, τ.

Thus the 2^t generating lines $\langle v^{(\sigma)}\rangle$ in $L(\mathrm{Aut}\,\mathcal{D})$ are linearly independent and, by the dimension bound given above, they actually form a basis for $L(\mathrm{Aut}\,\mathcal{D})$. Obviously, $L(\mathrm{Aut}\,\mathcal{D}) = U_{B_1|\ldots|B_{2^t}}$, which concludes our proof in the special case of a diagram over a family consisting of only one cube.

Let us now assume that the underlying family of cubes for $\mathcal{D}$ consists of more than one cube, $\mathcal{C} = \{C_1, \ldots, C_s\}$ for $s \geq 1$, and the partition $\rho(\mathcal{D})$ is of the form $(B_1^{(1)}|\ldots|B_{2^{\dim C_1}}^{(1)}|\ \ldots\ |B_1^{(s)}|\ldots|B_{2^{\dim C_s}}^{(s)})$, where the $B_j^{(i)}$ are the (equicardinal) vertex fibers over the cube C_i, for $j = 1, \ldots, \dim C_i$, and $i = 1, \ldots, s$. Again, a line that is invariant under the action of $\mathrm{Aut}\,\mathcal{D}$ must have equal coordinate entries on every (vertex) fiber of $\mathcal{D}$. Hence, the number of blocks in $\rho(\mathcal{D})$, $\sum_{i=1}^{s} 2^{\dim C_i}$, is an upper bound for $\dim L(\mathrm{Aut}\,\mathcal{D})$.

For subsets $T \subseteq [n]$, we denote characteristic vectors in $\mathbb{R}^n$ by e_T. In analogy to our considerations for diagrams over a single cube, we see that

$$\langle e_{B_j^{(i)}}\rangle \qquad \text{for } i = 1, \ldots, s,\ \ j = 1, \ldots, 2^{\dim C_i},$$

are generating lines for $L(\operatorname{Aut}\mathcal{D})$. These lines are linearly independent and, by the upper bound for the dimension of $L(\operatorname{Aut}\mathcal{D})$ given above, they form a basis for $L(\operatorname{Aut}\mathcal{D})$, which obviously coincides with the braid space $U_{\rho(\mathcal{D})}$. □

Theorem 5.4.5 *The arrangement $\mathcal{A}(\mathcal{S}_n)$ associated with the real permutation action as described in* 5.3.1 *coincides with the rank* 2 *truncation of the braid arrangement. In particular, the digitalization $Y_{\mathcal{A}(\mathcal{S}_n)}$ of* Theorem 5.3.1 *coincides with the maximal wonderful model of the braid arrangement as discussed in* [FK2].

Proof. Let $\mathcal{D}_{ijk}$ be a diagram over a 0-family of cubes with all fibers consisting of singletons other than one 3-element fiber containing i, j, k for some $1 \le i < j < k \le n$. For $1 \le i, j, k, l \le n$, the i, j, k, l pairwise distinct, let $\mathcal{D}_{ij|kl}$ be a diagram over a 1-family of cubes with a single 1-dimensional cube with fibers $\{i, j\}$ and $\{k, l\}$ over its vertices and 0-dimensional cubes with singleton fibers otherwise. With Lemma 5.4.4 we see that

$$L(\mathcal{D}_{ijk}) \; = \; U_{ijk} \qquad \text{and} \qquad L(\mathcal{D}_{ij|kl}) \; = \; U_{ij|kl}\,.$$

Thus, the rank 2 truncation of the braid arrangement is contained in $\mathcal{A}(\mathcal{S}_n)$, and it remains to show that all other proper subspaces $L(H)$ arising from subgroups H of $\mathcal{S}_n$ are braid spaces of codimension at least 2.

Let us remark here that hyperplanes never occur in arrangements $\mathcal{A}(G)$ induced by some linear effective action of a finite group G on a real vector space V, since, if $L(H)$ were a hyperplane for some subgroup H in G, then both $L(H)$ and its orthogonal line ℓ in V would be invariant under H. In particular, $\ell \subseteq L(H) = V$.

Claim: All subspaces in $\mathcal{A}(\mathcal{S}_n)$ are of the form $L(\operatorname{Aut}\mathcal{D})$ for some n-diagram $\mathcal{D}$.

Proof of the Claim: For any subgroup H of $\mathcal{S}_n$ define

$$d(H) \; := \bigcap_{\substack{\mathcal{D}\ n-\text{diagram} \\ \operatorname{Aut}\mathcal{D} \supseteq H}} \operatorname{Aut}\mathcal{D}\,.$$

Since automorphism groups of n-diagrams over families of cubes are closed under intersection (cf. Theorem 5.4.2(1)), $d(H)$ itself is an automorphism group of an n-diagram. We claim that

$$L(H) \; = \; L(d(H)) \qquad \text{for any } H \le \mathcal{S}_n\,. \tag{5.4.1}$$

Recall that for a line ℓ in $\mathbb{R}^n$ and $g \in \mathcal{S}_n$, ℓ is invariant under the action of g if and only if $g \in \operatorname{Aut}(\mathcal{D}(\ell))$, where $\mathcal{D}(\ell)$ denotes the n-diagram described in

Example 5.4.3(2). The subgroup H preserves a line ℓ if and only if H is contained in Aut $\mathcal{D}(\ell)$. The latter being equivalent to $d(H) \subseteq \mathrm{Aut}\,\mathcal{D}(\ell)$, we conclude that H preserves ℓ if and only if $d(H)$ preserves ℓ. Hence, (5.4.1) follows, which proofs our claim.

Given a diagram $\mathcal{D}$, we can assume without change of $\mathrm{Aut}\,\mathcal{D}$ that it is reduced (cf. Theorem 5.4.2(2)). Moreover, we can assume that $\mathcal{D}$ contains no 0-dimensional cubes with 2-element fibers. For if it did, we could place the two elements into singleton fibers over the vertices of a 1-cube, which uses a coordinate direction that did not occur previously in the family of cubes underlying $\mathcal{D}$. This operation does not alter the automorphism group of the diagram.

Referring to Lemma 5.4.4, we now find that *all* subspaces in $\mathcal{A}(\mathcal{S}_n)$ are actually braid spaces, which completes our proof. □

5.4.3 Digitalizing the permutation action on real projective space

We will consider the $\mathcal{L}$-stratification on $\mathbb{RP}^{n-1}$ induced by the permutation action of $\mathcal{S}_n$ and give a description of the digitalization proposed in 5.3.2.

Theorem 5.4.6 *The $\mathcal{L}$-stratification on $\mathbb{RP}^{n-1}$ induced by the permutation action of $\mathcal{S}_n$ coincides with the rank 2 truncation of the projectivized braid arrangement $\mathbb{P}\mathcal{A}_{n-1}$. In particular, the digitalization $Y_{\mathbb{RP}^{n-1},\mathcal{L}}$ coincides with the maximal projective arrangement model for $\mathbb{P}\mathcal{A}_{n-1}$.*

Proof. For any $\ell \in \mathbb{RP}^{n-1}$, $\ell = \langle v \rangle$ a line in $\mathbb{R}^n$ with generating vector v of unit length, we will describe the induced linear action of the stabilizer $\mathrm{stab}_{\mathbb{RP}^{n-1}}\ell$ on the tangent space $T_\ell\mathbb{RP}^{n-1}$.

First observe that the stabilizer of a line ℓ is an automorphism group of an n-diagram $\mathcal{D}(\ell)$ as described in Example 5.4.3(2),

$$\mathrm{stab}_{\mathbb{RP}^{n-1}}\ell \;=\; \mathrm{Aut}\,\mathcal{D}(\ell)\,.$$

We interpret the tangent space $T_\ell\mathbb{RP}^{n-1}$ as the orthogonal hyperplane to ℓ in $\mathbb{R}^n$ placed at $v \in \mathbb{R}^n$,

$$T_\ell\mathbb{RP}^{n-1} \;=\; \ell^\perp + v \;:=\; T\,.$$

With this identification, we can give an explicit description of the Bochner map Φ_ℓ (cf. Proposition 5.3.2) that maps a neighborhood U of ℓ in $\mathbb{RP}^{n-1}$ diffeomorphically and $\mathrm{stab}\,\ell$-equivariantly to the tangent space $T_\ell\mathbb{RP}^{n-1}$,

$$\begin{array}{rccl} \Phi_\ell : & U & \longrightarrow & T_\ell\mathbb{RP}^{n-1} \\ & u & \longmapsto & u \cap (\ell^\perp + v)\,. \end{array}$$

Claim: For any $\ell \in \mathbb{RP}^{n-1}$ the arrangement induced by the action of $\mathrm{Aut}\,\mathcal{D}(\ell)$ on $T_\ell\mathbb{RP}^{n-1}$ coincides with the rank 2 truncation of the braid arrangement intersected with T,

$$\mathcal{A}_v(\mathrm{Aut}\,\mathcal{D}(\ell) \circlearrowright T) \;=\; \mathcal{A}_{n-1}^{\mathrm{rk}\geq 2} \cap T\,. \tag{5.4.2}$$

By Proposition 5.3.3 we retrieve the $\mathcal{L}$-stratification by taking the inverse image of $\mathcal{A}_v(\mathrm{Aut}\,\mathcal{D}(\ell) \circlearrowright T)$ under Φ_ℓ for any $\ell \in \mathbb{RP}^{n-1}$. Due to our description of Φ_ℓ, we easily conclude that the $\mathcal{L}$-stratification of $\mathbb{RP}^{n-1}$ coincides with the rank 2 truncation of the projectivized braid arrangement, $\mathbb{P}\mathcal{A}_{n-1}^{\mathrm{rk}\geq 2}$.

The rest of our argument is a proof of the Claim (5.4.2), which we break into a number of steps.

(1) *The* $\mathrm{Aut}\,\mathcal{D}(\ell)$*-action on* T. Let $w = w_0 + v \in \ell^\perp + v = T$, and $\pi \in \mathrm{Aut}\,\mathcal{D}(\ell)$. Recall from the definition of automorphisms of diagrams in 5.4.1 that, for any automorphism π of a diagram over a t-family of cubes, there is a group element $\sigma(\pi) \in \mathbb{Z}_2^t$ describing the automorphism on the t-cube underlying the cubes of the t-family. Note that $\sigma(\pi) \in \mathbb{Z}_2 = \{+1, -1\}$ by construction of $\mathcal{D}(\ell)$.

Writing out the action $\circ_T$ of $\mathrm{Aut}\,\mathcal{D}(\ell)$ on T in detail, we obtain:

$$\begin{aligned} \pi \circ_T w \;=\; \langle \pi \cdot w_0 + \pi \cdot v \rangle \cap T \;&=\; \begin{cases} \pi \cdot w_0 + v & \text{if } \pi \cdot v = v \\ -\pi \cdot w_0 + v & \text{if } \pi \cdot v = -v \end{cases} \\ &=\; \sigma(\pi)\; \pi \cdot w_0 \,+\, v\,. \end{aligned}$$

Observe that $\pi \cdot w_0 \in \ell^\perp$ and $\pi \cdot v = \sigma(\pi)\, v$. For easier distinction, we have chosen $\cdot$ to denote the permutation action on points in $\mathbb{R}^n$.

(2) *Mapping* T *to* $\mathbb{R}^n/\langle x_1 = \ldots = x_n \rangle$. We shall map T $\mathrm{Aut}\,\mathcal{D}(\ell)$-equivariantly to the $(n-1)$-dimensional quotient space $V := \mathbb{R}^n/\Delta$, where $\Delta := \langle x_1 = \ldots = x_n \rangle$ denotes the small diagonal in $\mathbb{R}^n$.

To this end we first define an action $\circ_V$ of $\mathrm{Aut}\,\mathcal{D}(\ell)$ on V by:

$$\begin{aligned} \pi \circ_V \big((x_1, \ldots, x_n) + \Delta\big) \;:=\; & \sigma(\pi)\;\; \pi \cdot ((x_1, \ldots, x_n) + \Delta) \\ & \text{for } \pi \in \mathrm{Aut}\,\mathcal{D}(\ell),\; (x_1, \ldots, x_n) + \Delta \in V\,. \end{aligned}$$

Moreover, we define a map

$$\begin{aligned} q:\quad T &\longrightarrow V \\ w &\longmapsto w + \Delta \end{aligned}$$

by restricting the projection $\mathbb{R}^n \longrightarrow \mathbb{R}^n/\Delta$ to T.

We check that q is $\operatorname{Aut}\mathcal{D}(\ell)$-equivariant with respect to the actions $\circ_T$ and $\circ_V$. Indeed, for $\pi \in \operatorname{Aut}\mathcal{D}(\ell)$, and $w = w_0 + v \in T$, we have

$$\begin{aligned} \pi \circ_V (q(w)) &= \pi \circ_V (w_0 + v + \Delta) = \sigma(\pi) \left(\pi \cdot w_0 + \pi \cdot v + \Delta\right) \\ &= \sigma(\pi)\, \pi \cdot w_0 + v + \Delta = q\,(\pi \circ_T w)\,. \end{aligned}$$

We conclude that, unless $\ell \subseteq \Delta^\perp$ (a case that we will settle separately in step (4)) we have an $\operatorname{Aut}\mathcal{D}(\ell)$-equivariant isomorphism from T to V by restriction from the standard projection on V. This implies that we can retrieve the arrangement $\mathcal{A}_v(\operatorname{Aut}\mathcal{D}(\ell) \circlearrowright T)$ as the inverse image of the arrangement $\mathcal{A}_{\bar{v}}(\operatorname{Aut}\mathcal{D}(\ell) \circlearrowright V)$, where $\bar{v} = q(v)$.

(3) *A description of the arrangement* $\mathcal{A}_{\bar{v}}(\operatorname{Aut}\mathcal{D}(\ell) \circlearrowright V)$. We will show that the arrangement induced by the $\operatorname{Aut}\mathcal{D}(\ell)$-action $\circ_V$ on V, $\mathcal{A}_{\bar{v}}(\operatorname{Aut}\mathcal{D}(\ell) \circlearrowright V)$, coincides with the rank 2 truncation of the braid arrangement $\mathcal{A}_{n-1}^{\mathrm{rk}\geq 2} \cap V$ in a neighborhood of $\bar{v}$.

First, observe that the action $\circ_V$ differs from the permutation action of $\operatorname{Aut}\mathcal{D}(\ell)$ on V by at most a sign, which in particular implies that the construction of $L(H)$ for $H \leq \operatorname{Aut}\mathcal{D}(\ell)$ yields the same subspaces with respect to both actions. Hence, we can freely switch to consider the permutation action of $\operatorname{Aut}\mathcal{D}(\ell)$ on V.

For any subgroup H in $\operatorname{Aut}\mathcal{D}(\ell)$, we argue as in the proof of Theorem 5.4.5 and first observe that we can replace H by $d(H) = \operatorname{Aut}\widetilde{\mathcal{D}}$, the intersection of all automorphism groups of n-diagrams containing H. We can assume that the diagram $\widetilde{\mathcal{D}}$ satisfies the assumptions of Lemma 5.4.4, as we did before in the proof of Theorem 5.4.5. We conclude that

$$L(\operatorname{Aut}\widetilde{\mathcal{D}}) = U_{\rho(\widetilde{\mathcal{D}})}\,,$$

with $\rho(\widetilde{\mathcal{D}}) \leq \rho(\mathcal{D}(\ell))$ in the permutation lattice Π_n, since $\widetilde{\mathcal{D}}$ is an intersection of diagrams with one of the factors being $\mathcal{D}(\ell)$ (cf. Theorem 5.4.2 (1)).

Let π be any partition of rank ≥ 2 in Π_n with $\pi \leq \rho(\mathcal{D}(\ell))$. Consider a diagram $\mathcal{D}_\pi$ over a family of 0-cubes with the blocks of π as fibers. Obviously, $\operatorname{Aut}\mathcal{D}_\pi \leq \operatorname{Aut}\mathcal{D}(\ell)$, the assumptions of Lemma 5.4.4 are satisfied as before, and we conclude that $L(\operatorname{Aut}\mathcal{D}_\pi) = U_\pi$. Thus, any braidspace U_π with $\operatorname{rk}\pi \geq 2$ and $\pi \leq \rho(\mathcal{D}(\ell))$ occurs in the arrangement $\mathcal{A}_{\bar{v}}(\operatorname{Aut}\mathcal{D}(\ell) \circlearrowright V)$.

With $\rho(\mathcal{D}(\ell))$ being the partition type $\pi(v)$ of $\bar{v}$, we can conclude that $\mathcal{A}_{\bar{v}}(\operatorname{Aut}\mathcal{D}(\ell) \circlearrowright V)$ coincides with the rank 2 truncation of the braid arrangement $\mathcal{A}_{n-1}^{\mathrm{rk}\geq 2} \cap V$ in a neighborhood of $\bar{v}$.

Taking the inverse image of $\mathcal{A}_{n-1}^{\mathrm{rk}\geq 2} \cap V$ under q, we conclude that the arrangement $\mathcal{A}_v(\operatorname{Aut}\mathcal{D}(\ell) \circlearrowright T)$ coincides with the rank 2 truncation of the braid arrangement $\mathcal{A}_{n-1}^{\mathrm{rk}\geq 2}$ intersected with T, and have thus proved our claim (5.4.2) for any $\ell \in \mathbb{RP}^{n-1}$, which, as a line in $\mathbb{R}^n$ is not contained in $\Delta^\perp$.

(4) *Settling the remaining case.* Let us now assume that the line $\ell = \langle v \rangle$ is contained in $\Delta^\perp$. Then the tangent space $T = \ell^\perp + v$ at ℓ decomposes as a direct sum into

$$T = (\Delta + v) \oplus T \cap V .$$

The stabilizer of ℓ, $\mathrm{Aut}\,\mathcal{D}(\ell)$, acts on T by

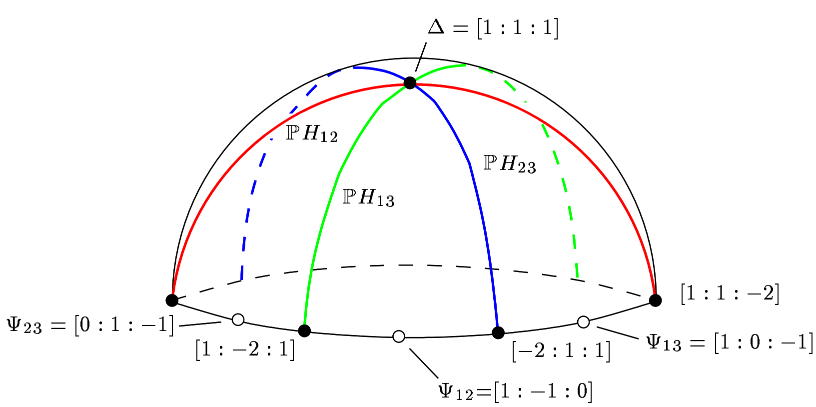
$$\pi \circ_T (d + v, w) = (\sigma(\pi)\, d + v, \pi \circ_V w)$$
$$\text{for } \pi \in \mathrm{Aut}\,\mathcal{D}(\ell)\,,\ d \in \Delta,\ \text{and } w \in T \cap V .$$

We can modify $\circ_T$ so as to act trivially on the first coordinate, since such modification does not change the spaces $L(H)$ in T that arise from subgroups H in $\mathrm{Aut}\,\mathcal{D}(\ell)$.

As in (3), we see that the arrangement $\mathcal{A}_v(\mathrm{Aut}\,\mathcal{D}(\ell) \circlearrowright T \cap V)$ is a restriction of $\mathcal{A}_{n-1}^{\mathrm{rk}\geq 2}$ to $T \cap V$. With the $\mathrm{Aut}\,\mathcal{D}(\ell)$-action on the first coordinate of T being trivial, we can take the direct product of $\mathcal{A}_v(\mathrm{Aut}\,\mathcal{D}(\ell) \circlearrowright T \cap V)$ with Δ and conclude that $\mathcal{A}_v(\mathrm{Aut}\,\mathcal{D}(\ell) \circlearrowright T)$ is the restriction of $\mathcal{A}_{n-1}^{\mathrm{rk}\geq 2}$ to T. This proves our claim in the remaining case. □

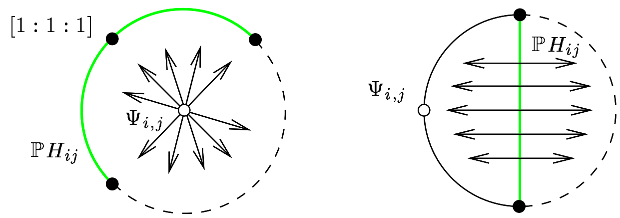
Example 5.4.7 To illustrate our theorem on the $\mathcal{L}$-stratification induced by the permutation action on real projective space and the resulting digitalization we look at $\mathcal{S}_3$ acting on $\mathbb{RP}^2$ in some detail.

We depict $\mathbb{RP}^2$ using the upper hemisphere model, where we place $\mathbb{P}\Delta^\perp = \mathbb{P}V$ on the equator.



The locus of points in $\mathbb{RP}^2$ with non-trivial stabilizer groups consists of the three lines $\mathbb{P}H_{ij}$, $1 \leq i, j \leq 3$, which are projectivizations of the hyperplanes in $\mathcal{A}_2$, intersecting in $\mathbb{P}\Delta = [1{:}1{:}1]$, and points Ψ_{ij} on $\mathbb{P}\Delta^\perp$, where Ψ_{ij} is the line orthogonal to H_{ij} in $\mathbb{R}^3$ for $1 \leq i, j \leq 3$.

Observe that the transposition $(i, j) \in \mathcal{S}_3$ acts on $\mathbb{RP}^2$ as a central symmetry in $\Psi_{i,j}$, respectively, as a reflection in $\mathbb{P}H_{i,j}$.

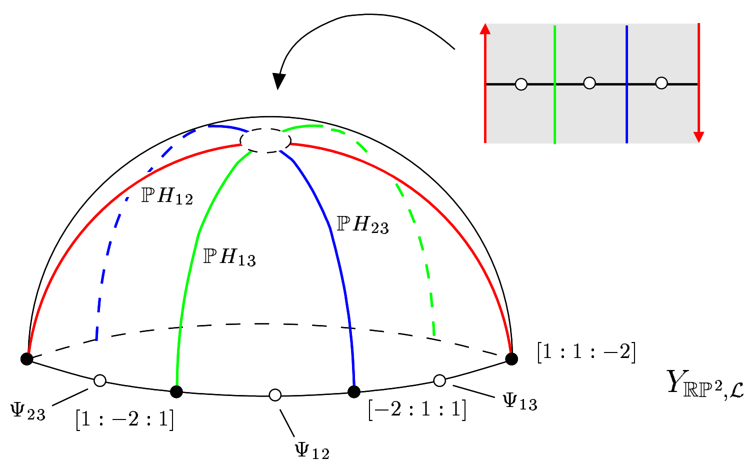


We find that the arrangements $\mathcal{A}_\ell(\operatorname{stab}\ell \circlearrowright T_\ell\mathbb{RP}^2)$ associated with the induced linear actions of the stabilizers on tangent spaces for $\ell \in \mathbb{RP}^2$ are empty unless $\ell = [1{:}1{:}1]$. In this case, we see that $\mathcal{S}_3 \circlearrowright T_{[1:1:1]}\mathbb{RP}^2$ coincides with the standard action of $\mathcal{S}_3$ on $\mathbb{R}^3/\Delta$, since transpositions, as we observed above, act as reflections in the hyperplanes of the projectivized braid arrangement. Thus, $\mathcal{A}_{[1:1:1]}(\mathcal{S}_3 \circlearrowright T_{[1:1:1]}\mathbb{RP}^2)$ coincides with the rank 2 truncation of the braid arrangement consisting of the origin of the tangent space.

We conclude that the $\mathcal{L}$-stratification is given by the single point $[1{:}1{:}1]$ in $\mathbb{RP}^2$, hence the digitalization we propose is the blowup of $\mathbb{RP}^2$ in this point,

$$Y_{\mathbb{RP}^2,\mathcal{L}} \;=\; \mathrm{Bl}_{[1:1:1]}(\mathbb{RP}^2)\,.$$

Topologically, this means to glue a Möbius band into a pointed $\mathbb{RP}^2$, in other words, to glue two Möbius bands along their boundaries. The resulting space hence is a Klein bottle.



Remark 5.4.8 As already the low-dimensional Example 5.4.7 shows, the $\mathcal{L}$-stratification associated with the permutation action of $\mathcal{S}_n$ on $\mathbb{RP}^{n-1}$ is different from the codimension 2 truncation of the stabilizer stratification.

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