

TROPICAL DISCRIMINANTS

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ABSTRACT. Tropical geometry is used to develop a new approach to the theory of discriminants and resultants in the sense of Gel'fand, Kapranov and Zelevinsky. The tropical A -discriminant, which is the tropicalization of the dual variety of the projective toric variety given by an integer matrix A , is shown to coincide with the Minkowski sum of the row space of A and the tropicalization of the kernel of A . This leads to an explicit positive formula for the extreme monomials of any A -discriminant, and to a combinatorial rule for deciding when two regular triangulations of A correspond to the same monomial of the A -discriminant.

1. INTRODUCTION

Let A be an integer $d \times n$ -matrix such that $(1, 1, \dots, 1)$ is in the row span of A . This defines a *projective toric variety* X_A in $\mathbb{C}\mathbb{P}^{n-1}$. Its *dual variety* X_A^* is the closure in the projective space dual to $\mathbb{C}\mathbb{P}^{n-1}$ of the set of hyperplanes that are tangent to X_A at a regular point. The toric variety X_A is called *non-defective* if its dual variety X_A^* has codimension one. In this case, the *A -discriminant* is the irreducible homogeneous polynomial Δ_A which defines the hypersurface X_A^* . The study of these objects is an active area of research in computational algebraic geometry, with the fundamental reference being the monograph by Gel'fand, Kapranov and Zelevinsky [13].

Our main object of interest in this paper is the *tropical A -discriminant* $\tau(X_A^*)$. This is the polyhedral fan in \mathbb{R}^n which is obtained by tropicalizing X_A^* . While it is generally difficult to compute the dual variety X_A^* from A , we show that its tropicalization $\tau(X_A^*)$ can be computed much more easily. In Theorem 1.1, we derive two explicit combinatorial descriptions of the tropical A -discriminant $\tau(X_A^*)$, and in Theorems 1.2 and 1.3 we present two main applications of our tropical approach.

Without loss of generality, we assume that the matrix A has maximal rank d , that the columns of A span the integer lattice \mathbb{Z}^d , and that the point configuration given by the columns of A is not a pyramid. These hypotheses ensure that the toric variety X_A has dimension $d-1$ and that the dual variety X_A^* is not contained in any coordinate hyperplane.

Let $\mathcal{L}(A)$ denote the geometric lattice whose elements are the supports, ordered by inclusion, of the vectors in $\ker(A)$. We write $\mathcal{C}(A)$ for the set of proper maximal chains in $\mathcal{L}(A)$. We represent these chains as $(n-d-1)$ -subsets $\sigma = \{\sigma_1, \dots, \sigma_{n-d-1}\}$ of $\{0, 1\}^n$. A key player in this paper is the tropicalization of the kernel of A . As shown in [1] and [10], this tropical linear space is subdivided both by the *Bergman*

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fan $\mathcal{B}^*(A)$ of the matroid dual to A , and by the *nested set fans* of $\mathcal{L}(A)$. Thus, tropicalizing the kernel of A yields the following subset of \mathbb{R}^n :

$$(1.1) \quad \tau(\text{kernel}(A)) = \text{support}(\mathcal{B}^*(A)) = \bigcup_{\sigma \in \mathcal{C}(A)} \mathbb{R}_{\geq 0} \sigma.$$

The union on the right hand side indicates the finest in a hierarchy of unimodular simplicial fan structures, corresponding to the *nested set complexes* [5, 8, 9, 10, 11].

We obtain the tropical A -discriminant by adding the row space of A to the tropical linear space (1.1). This is the tropical analogue of Kapranov's *Horn uniformization* [16]. Alternatively, we describe the tropical A -discriminant in terms of regular polyhedral subdivisions of the point configuration given by the columns of A .

Theorem 1.1. *For any integer $d \times n$ -matrix A as above, the following sets coincide:*

- (a) *the tropical A -discriminant $\tau(X_A^*)$,*
- (b) *the Minkowski sum of $\tau(\text{kernel}(A))$ and the row space of A ,*
- (c) *the set $\{w \in \mathbb{R}^n : \Pi_w \text{ has a maximal cell } \sigma \text{ without strong co-loops}\}$.*

Here, Π_w denotes the regular polyhedral subdivision of A defined by $w \in \mathbb{R}^n$. Each cell σ of Π_w is a subset of the columns of A , with the induced matroid structure, and $a \in \sigma$ is a *co-loop* of σ if it lies in every basis. A co-loop a in σ is a *strong co-loop* if it is also a co-loop in the superset σ^* which is defined in equation (5.1) of Section 5. Typically, a co-loop is strong; e.g., any maximal simplex in a regular subdivision consists entirely of strong co-loops. In particular, for a generic configuration A , the set in (c) is the union of all codimension one cones in the secondary fan of A . The tropical discriminant $\tau(X_A^*)$ inherits the structure of a fan both from the Gröbner fan of the ideal of X_A^* and from the secondary fan of A . In general, neither of these two fan structures refines the other, as we shall see in Examples 5.5 and 5.6.

The tropicalization of an algebraic variety retains a lot of information about the geometry of the original variety [17, 19, 20, 22, 23]. In Theorem 1.2 below, our tropical approach leads to a formula for the extreme monomials of the A -discriminant Δ_A , and, a fortiori, for the degree of the dual variety X_A^* . An alternating product formula for the extreme monomials of Δ_A was given in [13, §11.3.C] under the restrictive assumption that X_A is smooth. Our formula (1.2) is positive, it is valid for any toric variety X_A regardless of smoothness, and its proof is self-contained.

Theorem 1.2. *If X_A is non-defective and w a generic vector in \mathbb{R}^n then the exponent of x_i in the initial monomial $\text{in}_w(\Delta_A)$ of the A -discriminant Δ_A equals*

$$(1.2) \quad \sum_{\sigma \in \mathcal{C}_{i,w}} |\det(A^t, \sigma_1, \dots, \sigma_{n-d-1}, e_i)|,$$

where $\mathcal{C}_{i,w}$ is the subset of $\mathcal{C}(A)$ consisting of all chains such that the row space of the matrix A has non-empty intersection with the cone $\mathbb{R}_{>0}\{\sigma_1, \dots, \sigma_{n-d-1}, -e_i, -w\}$.

Here, the A -discriminant Δ_A is written as a homogeneous polynomial in the variables x_1, \dots, x_n , and $\text{in}_w(\Delta_A)$ is the w -highest monomial $x_1^{u_1} \cdots x_n^{u_n}$ which appears in the expansion of Δ_A in characteristic zero. Theorem 1.2 generalizes to the defective case, when we take Δ_A as the *Chow form* of the dual variety X_A^* . This generalization is stated in Theorem 4.6. Aiming for maximal efficiency in evaluating (1.2) with a

computer, we can replace $\mathcal{C}_{i,w}$ with the corresponding maximal nested sets of the geometric lattice $\mathcal{L}(A)$, or with the maximal cones in the Bergman fan $\mathcal{B}^*(A)$. Our `maple` implementation of the formula (1.2) will be discussed at the end of Section 4.

Our second application of tropical discriminants concerns the combinatorial problem of characterizing Δ -equivalence for regular triangulations of A . Following Gel'fand, Kapranov and Zelevinsky [13, p. 368], two neighboring regular triangulations of a non-defective point configuration A are called Δ -equivalent if they specify the same initial monomial of the A -discriminant. While Δ -equivalence is fully explained for toric surfaces and three-folds in [13, §11.3.B], the problem of finding a combinatorial characterization in higher dimensions had remained open until now. We present a solution to this problem which generalizes [13, Proposition 11.3.10]:

Theorem 1.3. *Two neighboring regular triangulations of a non-defective point configuration A are Δ -equivalent if and only if the restructuring occurs on a circuit which lies in a facet of A and involves a point of minimal distance from that facet.*

This paper is organized as follows. In Section 2, we review the construction of the tropicalization $\tau(Y)$ of a projective variety Y , and we show how the algebraic cycle underlying any initial monomial ideal of Y can be read off from $\tau(Y)$. In Section 3, we discuss general varieties which are parametrized by a linear map followed by a monomial map. Theorem 3.1 gives a combinatorial description of the tropicalization of the image of such a map. The dual variety X_A^* of any toric variety X_A admits such a parametrization. This is derived in Section 4, and it is used to prove the first half of Theorem 1.1, and Theorem 1.2 in the general form of Theorem 4.6. In Section 5 we turn our attention to regular polyhedral subdivisions and we prove the remaining half of Theorem 1.1. We then resolve the Δ -equivalence problem by proving Theorem 1.3. Finally, Section 6 is devoted to the case when A is an *essential Cayley configuration*. The corresponding dual varieties X_A^* are resultant varieties, and we compute their degrees and initial cycles in terms of mixed subdivisions.

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2. TROPICAL VARIETIES AND THEIR INITIAL CYCLES

Tropical algebraic geometry refers to algebraic geometry over the semi-ring $(\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ with arithmetic operations $x \oplus y := \min\{x, y\}$ and $x \odot y := x + y$. It transfers the objects of classical algebraic geometry into the combinatorial context of polyhedral geometry. Fundamental references include [7, 17, 18, 19, 20, 23].

Tropicalization is an operation that turns complex projective varieties into polyhedral fans. If $Y \subset \mathbb{CP}^{n-1}$ is an irreducible projective variety of dimension $r - 1$ and $I_Y \subset \mathbb{C}[x_1, \dots, x_n]$ its prime ideal, then the *tropicalization* $\tau(Y)$ of Y is the set

$$(2.1) \quad \tau(Y) = \{w \in \mathbb{R}^n : \text{in}_w(I_Y) \text{ does not contain a monomial}\}.$$

The set $\tau(Y)$ carries the structure of a polyhedral fan; namely, it is a subfan of the Gröbner fan of I_Y ; see [22, §9]. By a result of Bieri and Groves [2], the fan $\tau(Y)$ is

pure of dimension r . In [3] it was shown that $\tau(Y)$ is connected in codimension one, and a practical algorithm was given for computing $\tau(Y)$ from polynomial generators of I_Y . We will view the tropicalization $\tau(Y)$ of a projective variety as an $(r-1)$ -dimensional fan in *tropical projective space* $\mathbb{TP}^{n-1} := \mathbb{R}^n / \mathbb{R}(1, 1, \dots, 1)$, which is an $(n-1)$ -dimensional real affine space.

Every maximal cone σ of the fan $\tau(Y)$ comes naturally with an *intrinsic multiplicity* m_σ , which is a positive integer. The integer m_σ is computed as the sum of the multiplicities of all monomial-free minimal associated primes of the initial ideal $\text{in}_w(I_Y)$ in $\mathbb{C}[x_1, \dots, x_n]$, where w is in the relative interior of the cone σ .

Remark 2.1. A geometric description of the intrinsic multiplicity m_σ arises from the beautiful interplay of degenerations and compactifications discovered by Tevelev [23] and studied by Speyer [19, Chapter 2] and Hacking (unpublished). Let \mathbf{X} denote the toric variety associated with the fan $\tau(Y)$. Consider the intersection Y_0 of Y with the dense torus T in \mathbb{CP}^{n-1} , and let $\overline{Y_0}$ be the closure of Y_0 in \mathbf{X} . By [23, 1.7, 2.5, and 2.7], the variety $\overline{Y_0}$ is complete and the multiplication map $\Psi : T \times \overline{Y_0} \rightarrow \mathbf{X}$ is faithfully flat. It follows that the intersection of $\overline{Y_0}$ with a codimension k orbit has codimension k in $\overline{Y_0}$. In particular, the orbit $\mathcal{O}(\sigma)$ corresponding to a maximal cone σ of $\tau(Y)$ intersects $\overline{Y_0}$ in a zero-dimensional scheme Z_σ . The intrinsic multiplicity m_σ of the maximal cone σ in the tropical variety $\tau(Y)$ is the length of Z_σ .

We list three fundamental examples which will be important for our work.

(1) Let Y be a **hypersurface** in \mathbb{CP}^{n-1} defined by an irreducible polynomial f in $\mathbb{C}[x_1, \dots, x_n]$. Then $\tau(Y)$ is the union of all codimension one cones in the normal fan of the Newton polytope of f . The intrinsic multiplicity m_σ of each such cone σ is the lattice length of the corresponding edge of the Newton polytope of f .

(2) Let $Y = X_A$ be the **toric variety** defined by an integer $d \times n$ -matrix A as above. Its tropicalization $\tau(X_A)$ is the linear space spanned by the rows of A .

(3) Let Y be a **linear subspace** in \mathbb{C}^n or in \mathbb{CP}^{n-1} . The tropicalization $\tau(Y)$ is the Bergman fan of the matroid associated with Y ; see [1, 10, 22] and (3.2) below.

In the last two families of examples, all the intrinsic multiplicities m_σ equal 1.

The tropicalization $\tau(Y)$ can be used to compute numerical invariants of Y . First, the dimension of $\tau(Y)$ coincides with the dimension of Y . In Theorem 2.2 below, we express the multiplicities of the minimal primes in the initial monomial ideals of I_Y in terms of $\tau(Y)$. Equivalently, we compute the algebraic cycle of any initial monomial ideal $\text{in}_w(I_Y)$. This formula tells us the degree of the variety Y , namely, the degree is the sum of the multiplicities of all the minimal primes of $\text{in}_w(I_Y)$.

Let $c := n - r$ denote the codimension of the irreducible projective variety Y in \mathbb{CP}^{n-1} . Assume that Y is not contained in a coordinate hyperplane, and let I_Y be its homogeneous prime ideal in $\mathbb{C}[x_1, \dots, x_n]$. If w is a generic vector in \mathbb{R}^n , the initial ideal $\text{in}_w(I_Y)$ is a monomial ideal of codimension c . Every minimal prime over $\text{in}_w(I_Y)$ is generated by a subset of c of the variables. We write $P_\tau = \langle x_i : i \in \tau \rangle$ for the monomial prime ideal indexed by the subset $\tau = \{\tau_1, \dots, \tau_c\} \subset \{1, 2, \dots, n\}$.

Assume that the cone $w + \mathbb{R}_{>0}\{e_{\tau_1}, \dots, e_{\tau_c}\}$ meets the tropicalization $\tau(Y)$. We may suppose that the generic weight vector $w \in \mathbb{R}^n$ satisfies that the image of w in \mathbb{TP}^{n-1} does not lie in $\tau(Y)$ and that the intersection of the cone $w + \mathbb{R}_{>0}\{e_{\tau_1}, \dots, e_{\tau_c}\}$ with $\tau(Y)$ is finite and contained in the union of the relative interiors of its maximal

cones. Let σ be a maximal cone of the tropical variety and

$$(2.2) \quad \{v\} = (L + w) \cap L',$$

where $L = \mathbb{R}\{e_{\tau_1}, \dots, e_{\tau_c}\}$ and $L' = \mathbb{R}\sigma$ are the corresponding linear spaces, which are defined over \mathbb{Q} . We associate with v the *lattice multiplicity* of the intersection of L and L' , which is defined as the absolute value of the determinant of any $n \times n$ -matrix whose columns consist of a \mathbb{Z} -basis of $\mathbb{Z}^n \cap L$ and a \mathbb{Z} -basis of $\mathbb{Z}^n \cap L'$.

Here is the main result of this section.

Theorem 2.2. *Let $w \in \mathbb{R}^n$ be a generic weight vector. A prime ideal P_τ is associated to the initial monomial ideal $\text{in}_w(I_Y)$ if and only if the cone $w + \mathbb{R}_{>0}\{e_{\tau_1}, \dots, e_{\tau_c}\}$ meets the tropicalization $\tau(Y)$. The number of intersections, each counted with its associated lattice multiplicity times the intrinsic multiplicity, is the multiplicity of the monomial ideal $\text{in}_w(I_Y)$ along the prime P_τ .*

Proof. We work over the Puiseux series field $K = \mathbb{C}\{\{\epsilon\}\}$, and we assume that the coordinates of w are rational numbers. We write $K\mathbb{P}^{n-1}$ for the $(n-1)$ -dimensional projective space over the field K , and we consider Y as a subvariety of $K\mathbb{P}^{n-1}$. We also consider the translated variety $\epsilon^{-w} \cdot Y$ which is defined by the prime ideal

$$\epsilon^w \cdot I_Y = \{f(\epsilon^{w_1}x_1, \dots, \epsilon^{w_n}x_n) : f \in I_Y\} \subset K[x_1, \dots, x_n].$$

Let L be a general linear subspace of dimension c in $K\mathbb{P}^{n-1}$ which is defined over the complex numbers \mathbb{C} . The intersection $\epsilon^{-w} \cdot Y \cap L$ is a finite set of reduced points in $K\mathbb{P}^{n-1}$. The number of these points is the degree of Y . More precisely, each such intersection point can be written in the form

$$\theta \cdot \epsilon^u + \dots = (\theta_1 \epsilon^{u_1} + \dots : \theta_2 \epsilon^{u_2} + \dots : \dots : \theta_n \epsilon^{u_n} + \dots),$$

where $\theta_k \in \mathbb{C}^*$ for all k . There exists a subset $\tau \subset \{1, 2, \dots, n\}$ such that $u_i = 0$ for $i \notin \tau$ and $u_j > 0$ for $j \in \tau$. The genericity in the choice of w and L ensures that τ has cardinality c and that P_τ is a minimal prime of $\text{in}_w(I_Y)$. The intersection number we wish to compute equals the number (possibly with multiplicity) of such points $\theta \cdot \epsilon^u + \dots$ for fixed τ . If we multiply (coordinatewise) the point $\theta \cdot \epsilon^u + \dots$ by the vector ϵ^w then we get a point in Y , and hence $u + w$ lies in the tropical variety $\tau(Y)$. Moreover, $u + w$ lies in the cone $w + \mathbb{R}_{>0}\{e_{\tau_1}, \dots, e_{\tau_c}\}$. Hence the desired points are indexed by the intersection points of that cone with $\tau(Y)$.

By our genericity assumption, each intersection point v lies on some maximal cone σ of $\tau(Y)$ and it is counted with its multiplicity, which is the product of the intrinsic multiplicity m_σ times the lattice multiplicity of the transversal intersection of rational linear spaces in 2.2. This product can be understood by means of the flat family discussed in Remark 2.1. Namely, it follows from the T -invariance of the multiplication map Ψ that the scheme-theoretic fiber of Ψ over any point of $\mathcal{O}(\sigma)$ is isomorphic to $T' \times Z_\sigma$, where T' is the stabilizer of a point in $\mathcal{O}(\sigma)$. Since \mathbf{X} is normal, T' is a torus $(\mathbb{C}^*)^{r-1}$. By [23, 1.7], $(\mathbb{C}^*)^{r-1} \times Z_\sigma$ is the intersection of T with the flat degeneration of Y in $\mathbb{C}\mathbb{P}^{n-1}$ given by the one parameter subgroup of T specified by the rational vector w . Our construction above amounts to computing the intersection of $(\mathbb{C}^*)^{r-1} \times Z_\sigma$ with the corresponding degeneration of L . The lattice index is obtained from the factor $(\mathbb{C}^*)^{r-1}$, and m_σ is obtained from the factor Z_σ . Their product is the desired intersection number. \square

The algebraic cycle of the variety Y is represented by its *Chow form* Ch_Y , which is a polynomial in the bracket variables $[\gamma] = [\gamma_1 \cdots \gamma_c]$; see [13, §3.2.B]. Theorem 2.2 implies that the w -leading term of the Chow form Ch_Y equals $\prod [\gamma]^{u_\gamma}$, where u_γ is the (correctly counted) number of points in $\tau(Y) \cap (w + \mathbb{R}_{>0}\{e_{\gamma_1}, \dots, e_{\gamma_c}\})$. We discuss this statement for the three families of examples considered earlier.

(1) If Y is a hypersurface then $c = 1$ and the bracket variable $[\gamma]$ is simply the ordinary variable x_i for $i = \gamma_1$. The w -leading monomial of the defining irreducible polynomial $f(x_1, \dots, x_n)$ equals $x_1^{u_1} \cdots x_n^{u_n}$ where u_i is the number of times the ray $w + \mathbb{R}_{>0}e_i$ intersects the tropical hypersurface $\tau(Y)$, counted with multiplicity.

(2) If $Y = X_A$ is a toric variety with tropicalization $\tau(Y) = \text{rowspace}(A)$ then Theorem 2.2 implies the familiar result (see [13, Thm. 8.3.3]) that the initial cycles of X_A are the regular triangulations of A . Indeed, the cone $w + \mathbb{R}_{>0}\{e_{\gamma_1}, \dots, e_{\gamma_{n-d}}\}$ intersects the row space of A if and only if the $(d-1)$ -simplex $\tilde{\gamma} = \{a_i : i \notin \gamma\}$ appears in the regular triangulation Π_w of A . The intersection multiplicity is the lattice volume of $\tilde{\gamma}$. Hence $\text{in}_w(Ch_{X_A}) = \prod_{\tilde{\gamma} \in \Pi_w} [\gamma]^{\text{vol}(\tilde{\gamma})}$.

(3) If Y is a linear space then its ideal I_Y is generated by c linearly independent linear forms in $\mathbb{C}[x_1, \dots, x_n]$. The tropical variety $\tau(Y)$ is the Bergman fan, to be discussed in Section 3, and the Gröbner fan of I_Y is the normal fan of the associated matroid polytope [10]. For fixed generic w , there is a unique c -element subset γ of $[n]$ such that $w + \mathbb{R}_{>0}\{e_{\gamma_1}, \dots, e_{\gamma_c}\}$ intersects $\tau(Y)$. The intersection multiplicity is one, and the corresponding initial ideal equals $\text{in}_w(I_Y) = \langle x_{\gamma_1}, \dots, x_{\gamma_c} \rangle$.

3. TROPICALIZING MAPS DEFINED BY MONOMIALS IN LINEAR FORMS

In this section we examine a class of rational varieties Y whose tropicalization $\tau(Y)$ can be computed combinatorially, without knowing the ideal I_Y . We consider a rational map $f : \mathbb{C}^m \dashrightarrow \mathbb{C}^s$ that factors as a linear map $\mathbb{C}^m \rightarrow \mathbb{C}^r$ followed by a Laurent monomial map $\mathbb{C}^r \dashrightarrow \mathbb{C}^s$. The linear map is specified by a complex $r \times m$ -matrix $U = (u_{ij})$, and the Laurent monomial map is specified by an integer $s \times r$ -matrix $V = (v_{ij})$. Thus the i -th coordinate of the rational map $f : \mathbb{C}^m \dashrightarrow \mathbb{C}^s$ equals the following monomial in linear forms:

$$(3.1) \quad f_i(x_1, \dots, x_m) = \prod_{k=1}^r (u_{k1}x_1 + \cdots + u_{km}x_m)^{v_{ik}}, \quad i = 1, \dots, s.$$

Let Y_{UV} denote the Zariski closure of the image of f . Observe that if all row sums of V are equal then f induces a rational map $\mathbb{C}\mathbb{P}^{m-1} \dashrightarrow \mathbb{C}\mathbb{P}^{s-1}$, and the closure of its image is an irreducible projective variety, which we also denote by Y_{UV} .

Our goal is to compute the tropicalization $\tau(Y_{UV})$ of the variety Y_{UV} in terms of the matrices U and V . In particular, we will avoid any reference to the equations of Y_{UV} . The general framework of this section will be crucial for our proofs of the results on A -discriminants and their tropicalization stated in the Introduction.

We list a number of special cases of varieties which have the form Y_{UV} .

(1) If $r = s$, and $V = I_r$ then f is the linear map $x \mapsto Ux$, and Y_{UV} is the image of U . We denote this **linear subspace** of \mathbb{C}^r by $\text{im}(U)$. Its tropicalization $\tau(\text{im}(U))$ is the Bergman fan of the matrix U , to be discussed in detail below.

(2) If $m=r$ and $U=I_m$ then f is the monomial map specified by the matrix V . Hence Y_{UV} coincides with the **toric variety** X_{V^t} which is associated with the transpose V^t of the matrix V . Its tropicalization is the column space of V .

(3) Let $m=2$, suppose the rows of U are linearly independent, and suppose the matrix V has constant row sum. Then $Y_{UV} = \text{image}(\mathbb{CP}^1 \rightarrow \mathbb{CP}^{s-1})$ is a **rational curve**. Every rational projective curve arises from this construction, since every binary form is a monomial in linear forms.

Our next theorem implies that $\tau(Y_{UV})$ consists of the rays in \mathbb{TP}^{s-1} spanned by the rows of V . Theorem 3.1 can also be derived from [23, Proposition 3.1], but we prefer to give a self-contained proof.

Theorem 3.1. *The tropical variety $\tau(Y_{UV})$ equals the image of the Bergman fan $\tau(\text{im}(U))$ under the linear map $\mathbb{R}^r \rightarrow \mathbb{R}^s$ specified by the matrix V .*

Proof. Let $K = \mathbb{C}\{\{\epsilon^{\mathbb{R}}\}\}$ be the field of Puiseux series with complex coefficients and real exponents. The elements of K are also known as *transfinite Puiseux series*, and they form an algebraically closed field of characteristic zero. The *order* of a non-zero element z in $K^* = K \setminus \{0\}$ is the smallest real number ν such that ϵ^ν appears with non-zero coefficient in z . For a vector $z = (z_1, \dots, z_s)$ in $(K^*)^s$ we write $\text{order}(z) := (\text{order}(z_1), \dots, \text{order}(z_s)) \in \mathbb{R}^s$. In what follows we consider all algebraic varieties over the field $K = \mathbb{C}\{\{\epsilon^{\mathbb{R}}\}\}$. Our proof is based on the well-known result that the orders of K^* -valued points of a variety are precisely the points in the tropicalization of that variety (see [7], [19, Theorem 2.1.2], [20, Theorem 2.1]).

Extending scalars, we consider the map $f : K^m \dashrightarrow K^s$. Let $z = f(x)$ be any point in the image of that map. For $k \in \{1, \dots, r\}$ we set $y_k = u_{k1}x_1 + \dots + u_{km}x_m$ and $\alpha_k = \text{order}(y_k)$. The vector $y = (y_1, \dots, y_r)$ lies in the linear space $\text{im}(U)$, and hence the vector $\alpha = (\alpha_1, \dots, \alpha_r)$ lies in $\tau(\text{im}(U))$.

The order of the vector $z = f(x) \in K^s$ is the vector $V \cdot \alpha \in \mathbb{R}^s$, since the order of its i th coordinate z_i equals $\sum_{k=1}^r v_{ik} \cdot \alpha_k$. Hence $z = f(x)$ lies in $V \cdot \tau(\text{im}(U))$, the image of the Bergman fan $\tau(\text{im}(U))$ under the linear map V .

The image of f is Zariski dense in Y_{UV} , i.e., there exists a proper subvariety Y of Y_{UV} such that $Y_{UV} \setminus Y$ contains the image of f . By the Bieri-Groves Theorem [2], $\tau(Y)$ is a polyhedral fan of lower dimension inside the pure-dimensional fan $\tau(Y_{UV})$. From this it follows that $\tau(Y_{UV})$ is the closure of $\tau(Y_{UV} \setminus Y)$ in \mathbb{R}^s . In the previous paragraph we showed that $\tau(Y_{UV} \setminus Y)$ is a subset of $V \cdot \tau(\text{im}(U))$. Since the latter is closed, we conclude that $\tau(Y_{UV}) \subseteq V \cdot \tau(\text{im}(U))$.

It remains to show the converse inclusion $V \cdot \tau(\text{im}(U)) \subseteq \tau(Y_{UV})$. Take any point $\beta \in V \cdot \tau(\text{im}(U))$, and choose $\alpha \in \tau(\text{im}(U))$ such that $V \cdot \alpha = \beta$. There exists a K^* -valued point y in the linear space $\text{im}(U)$ such that $\text{order}(y) = \alpha$. Then the point $y^V = (\prod_{k=1}^r y_k^{v_{1k}}, \dots, \prod_{k=1}^r y_k^{v_{sk}})$ lies in $(K^*)^s \cap Y_{UV}$, and its order clearly equals β . Hence $\beta \in \tau(Y_{UV})$ as desired. \square

Remark 3.2. The intrinsic multiplicity m_σ of any maximal cone σ in the tropical variety $\tau(Y_{UV})$ is a certain lattice index which can be read off from the matrix V . Namely, suppose σ is in the image of a maximal cone σ' of the Bergman fan $\tau(\text{im}(U))$. Then m_σ is the index of the subgroup $V(\mathbb{R}\sigma' \cap \mathbb{Z}^r)$ of the group of $\mathbb{R}\sigma \cap \mathbb{Z}^s$. This follows from Remark 2.1 using standard arguments of toric geometry. The discussion

below implies that $\mathbb{R}\sigma' \cap \mathbb{Z}^r$ is spanned by vectors in $\{0, 1\}^r$ and hence $V(\mathbb{R}\sigma' \cap \mathbb{Z}^r)$ is spanned by sums of columns of V , so the index is easy to compute.

Theorem 3.1 and Remark 3.2 furnish a combinatorial construction for the tropicalization of any variety which is parameterized by monomials in linear forms. Using the results of Section 2, Theorem 3.1 can now be applied to compute geometric invariants of such a variety, such as its dimension, its degree and its initial cycles. To make this computation effective, we need an explicit description of the Bergman fan $\tau(\text{im}(U))$. Luckily, the combinatorics of this object is well understood, thanks to [1, 10], and in the remainder of this section we summarize what is known.

Let M denote the *matroid* associated with the rows of the $r \times m$ -matrix U . Thus M is a matroid of rank at most m on the ground set $[r] = \{1, 2, \dots, r\}$. The *bases* of M are the maximal subsets of $[r]$ which index linearly independent rows of U . Fix a vector $w \in \mathbb{R}^r$. Then the *w-weight* of a basis β of M is $\sum_{i \in \beta} w_i$. Consider the set of all bases of M that have maximal w -weight. This collection is the set of bases of a new matroid which we denote by M_w . Note that each M_w has the same rank and the same ground set as $M = M_0$. An element i of $[r]$ is a *loop* of M_w if it does not lie in any basis of M of maximal w -weight.

We can now describe $\tau(\text{im}(U))$ in terms of the matroid M :

$$(3.2) \quad \tau(\text{im}(U)) = \{w \in \mathbb{R}^r : M_w \text{ has no loop}\}.$$

This representation endows our tropical linear space with the structure of a polyhedral fan. Namely, if $w \in \tau(\text{im}(U))$, then the set of all $w' \in \mathbb{R}^r$ such that $M_{w'} = M_w$ is a relatively open convex polyhedral cone in \mathbb{R}^r . The collection of these cones is denoted $\mathcal{B}(M)$ and is called the *Bergman fan* of the matroid M . Depending on the context, we may also write $\mathcal{B}(U)$ and call it the Bergman fan of the matrix U . For a more geometric construction of Bergman fans as subfans of normal fans of matroid polytopes, see [10, Sect. 2]. Readers of [22, §9] may note that the Bergman fan coincides with the fan structure on $\tau(\text{im}(U))$ gotten by restricting the Gröbner fan of the ideal of $\text{im}(U)$ to the support of $\tau(\text{im}(U))$.

We now recall the connection between Bergman fans and *nested set complexes*. The latter encode the structure of wonderful compactifications of hyperplane arrangement complements in the work of De Concini and Procesi [5], and they were later studied from a combinatorial point of view in [8, 9, 11].

A subset $X \subseteq [r]$ is a *flat* of our matroid M if there exists a vector $u \in \text{im}(U)$ such that $X = \{i \in [r] : u_i \neq 0\}$. The set of all flats, ordered by inclusion, is the *geometric lattice* $\mathcal{L} = \mathcal{L}_M$. A flat X in \mathcal{L} is called *irreducible* if the lower interval $\{Y \in \mathcal{L} : Y \leq X\}$ does not decompose as a direct product of posets. Denote by \mathcal{I} the set of irreducible elements in \mathcal{L} . In other contexts, the irreducible elements of a lattice of flats of a matroid were named *connected elements* or *dense edges*. The matroid M is *connected* if the top rank flat $\hat{1} = [r]$ is irreducible, and we assume that this is the case. Otherwise, we artificially add $\hat{1}$ to \mathcal{I} . We call a subset $S \subseteq \mathcal{I}$ *nested* if for any set of pairwise incomparable elements X_1, \dots, X_t in S , with $t \geq 2$, the join $X_1 \vee \dots \vee X_t$ is not contained in \mathcal{I} . The nested subsets in \mathcal{I} form a simplicial complex, the *nested set complex* $\mathcal{N}(\mathcal{L})$. For a definition of nested set complexes in a slightly more general context and for basic properties, see [8, Sect. 2.3].

Feichtner and Yuzvinsky [11, Eqn (13)] introduced the following natural geometric realization of the nested set complex $\mathcal{N}(\mathcal{L})$. Namely, the collection of cones

$$(3.3) \quad \mathbb{R}_{\geq 0}\{e_X : X \in S\} \quad \text{for } S \in \mathcal{N}(\mathcal{L})$$

forms a unimodular fan whose face poset is the face poset of the nested set complex of \mathcal{L} . Here $e_X = \sum_{i \in X} e_i$ denotes the incidence vector of a flat $X \in \mathcal{I}$. We consider this fan in the tropical projective space \mathbb{TP}^{n-1} , and we also denote it by $\mathcal{N}(\mathcal{L})$.

It was shown in [10, Thm 4.1] that the *nested set fan* $\mathcal{N}(\mathcal{L}_M)$ is a simplicial subdivision of the Bergman fan $\mathcal{B}(M)$, and hence of our tropical linear space $\tau(\text{im}(U))$. The Bergman fan need not be simplicial, so the nested set fan can be finer than the Bergman fan. However, in many important cases the two fans coincide. For a characterization when this happens see [10, Thm 5.3].

What we defined above is the coarsest in a hierarchy of nested set complexes associated with the geometric lattice \mathcal{L} . Namely, for certain choices of subsets \mathcal{G} in \mathcal{L} , the same construction gives a nested set complex $\mathcal{N}(\mathcal{L}, \mathcal{G})$ which is also realized as a unimodular simplicial fan. Such \mathcal{G} are called *building sets*; there is one nested set fan for each building set \mathcal{G} in \mathcal{L} , see [8, Sect. 2.2, 2.3]. If two building sets are contained in another, $\mathcal{G}_1 \subseteq \mathcal{G}_2$, then $\mathcal{N}(\mathcal{L}, \mathcal{G}_2)$ is obtained from $\mathcal{N}(\mathcal{L}, \mathcal{G}_1)$ by a sequence of stellar subdivisions [9, Thm 4.2]. The smallest building set is $\mathcal{G} = \mathcal{I}$, the case discussed above, and the largest building set is the set of all flats, $\mathcal{G} = \mathcal{L}$. The corresponding nested set complex $\mathcal{N}(\mathcal{L}, \mathcal{L})$ is the *order complex* $\Delta(\mathcal{L})$ of the lattice, i.e., the simplicial complex on $\mathcal{L} \setminus \{\hat{0}\}$ whose simplices are the totally ordered subsets. Summarizing the situation, we have the following sequence of subdivisions each of which can be used to compute the tropicalization of a linear space.

Theorem 3.3. *Given a matrix U and the matroid M of rows in U , the tropical linear space $\tau(\text{im}(U))$ has three natural fan structures: the Bergman fan $\mathcal{B}(M)$ is refined by the nested set fan $\mathcal{N}(\mathcal{L}_M)$, which is refined by the order complex $\Delta(\mathcal{L}_M)$.*

We present an example which illustrates the concepts developed in this section.

Example 3.4. Let $m=4$, $r=5$, $s=4$, and consider the map $f=(f_1, f_2, f_3, f_4)$ from \mathbb{C}^4 to \mathbb{C}^4 whose coordinates are the following monomials in linear forms:

$$\begin{aligned} f_1 &= (x_1 - x_2)^3(x_1 - x_3)^3 \\ f_2 &= (x_1 - x_2)^2(x_2 - x_3)^2(x_2 - x_4)^2 \\ f_3 &= (x_1 - x_3)^2(x_2 - x_3)^2(x_3 - x_4)^2 \\ f_4 &= (x_2 - x_4)^3(x_3 - x_4)^3. \end{aligned}$$

So f is as in (3.1) with

$$U = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 3 & 3 & 0 & 0 & 0 \\ 2 & 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 & 2 \\ 0 & 0 & 0 & 3 & 3 \end{pmatrix}.$$

The projectivization of the variety $Y_{UV}=\text{cl}(\text{image}(f))$ is a surface in \mathbb{CP}^3 , since U has rank 3. We wish to study the irreducible homogeneous equation $P(z_1, z_2, z_3, z_4) = 0$

which defines this surface. The matroid of U is the graphic matroid of K_4 with one edge removed. From [10, Example 3.4] we know that the Bergman fan $\mathcal{B}(U)$ is the fan over the complete bipartite graph $K_{3,3}$, embedded as a 2-dimensional fan in \mathbb{TP}^4 .

The tropical surface $\tau(Y_{UV})$ is the image of $\mathcal{B}(U)$ under V . This image is a 2-dimensional fan in \mathbb{TP}^3 . It has seven rays: six of them are images of the rays of $\mathcal{B}(U)$, the last one is the intersection of the images of two 2-dimensional faces that occurs due to the non-planarity of $K_{3,3}$. Hence the Newton polytope of the polynomial P is 3-dimensional with 6 vertices, 11 edges, and 7 facets; see Figure 1.

The six extreme monomials of P have been computed (by hand) using Theorem 2.2, namely, by intersecting the rays $w + \mathbb{R}_{>0}e_i$ with $\tau(Y_{UV})$ in \mathbb{TP}^3 . This computation revealed in particular that the degree of the polynomial P is 28. Using linear algebra, it is now easy to determine all 171 monomials in the expansion of P .

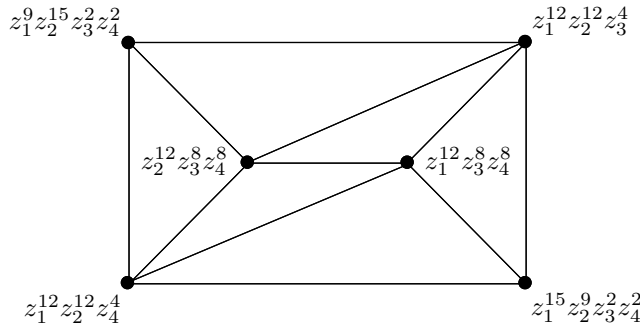


FIGURE 1. The Newton polytope of the polynomial P in Example 3.4.

4. BACK TO A -DISCRIMINANTS

In this section we return to the setting of the Introduction, and we prove Theorem 1.2 and half of Theorem 1.1. Recall that A is an integer $d \times n$ -matrix such that $(1, \dots, 1)$ is in the row span of A , i.e., the column vectors a_1, a_2, \dots, a_n lie in an affine hyperplane in \mathbb{R}^d . We identify the matrix A with the point configuration $\{a_1, a_2, \dots, a_n\}$; the convex hull of the configuration A is a $(d-1)$ -dimensional polytope with $\leq n$ vertices. We assume that the vectors a_1, \dots, a_n span \mathbb{Z}^d .

The projective toric variety X_A is defined as the closure of the image of the monomial map $\psi_A : (\mathbb{C}^*)^d \rightarrow \mathbb{CP}^{n-1}$, $t \mapsto (t^{a_1} : t^{a_2} : \dots : t^{a_n})$. Equivalently, X_A is the set of all points $x \in \mathbb{CP}^{n-1}$ such that $x^u = x^v$ for all $u, v \in \mathbb{N}^n$ with $Au = Av$.

Let $(\mathbb{CP}^{n-1})^*$ denote the projective space dual to \mathbb{CP}^{n-1} . The point $\xi = (\xi_1 : \dots : \xi_n)$ in $(\mathbb{CP}^{n-1})^*$ corresponds to the hyperplane $H_\xi = \{x \in \mathbb{CP}^{n-1} : \sum_{i=1}^n x_i \xi_i = 0\}$. The dual variety X_A^* is defined as the closure in $(\mathbb{CP}^{n-1})^*$ of the set of points ξ such that the hyperplane H_ξ intersects the toric variety X_A at a regular point p and contains the tangent space $T_{X_A}(p)$ of X_A at p .

Kapranov [16] showed that reduced discriminantal varieties are parametrized by monomials in linear forms. This parametrization, called the *Horn uniformization*, will allow us to determine the tropical discriminant $\tau(X_A^*)$ via the results of Section 3. We denote by $\mathbb{CP}(\ker(A))$ the projectivization of the kernel of the linear map given by A , an $(n-d-1)$ -dimensional projective subspace of \mathbb{CP}^{n-1} , and we denote by

$T^{d-1} = (\mathbb{C}^*)^d / \mathbb{C}^*$ the dense torus of X_A . The following result is a variant of [16, Theorem 2.1]; see also [13, §9.3.C].

Proposition 4.1. *The dual variety X_A^* of the toric variety X_A is the closure of the image of the map $\varphi_A : \mathbb{C}\mathbb{P}(\ker(A)) \times T^{d-1} \rightarrow (\mathbb{C}\mathbb{P}^{n-1})^*$ which is given by*

$$(4.1) \quad \varphi_A(u, t) = (u_1 t^{a_1} : u_2 t^{a_2} : \cdots : u_n t^{a_n}).$$

Proof. Consider the unit point $\mathbf{1} = (1 : 1 : \cdots : 1)$ on the toric variety X_A . The hyperplane H_ξ contains both the point $\mathbf{1}$ and the tangent space $T_{X_A}(\mathbf{1})$ at this point if and only if ξ lies in the kernel of A . This follows by evaluating the derivative of the parametrization ψ_A of X_A at $(t_1, t_2, \dots, t_d) = (1, 1, \dots, 1)$. If $p = \psi_A(t)$ is any point in the dense torus of X_A , then the tangent space at that point is gotten by translating the tangent space at $\mathbf{1}$ as follows:

$$T_{X_A}(p) = p \cdot T_{X_A}(\mathbf{1}).$$

The hyperplane H_ξ contains p if and only if $p^{-1} \cdot H_\xi = H_{\xi \cdot p}$ contains $\mathbf{1}$, and H_ξ contains $T_{X_A}(p)$ if and only if $H_{\xi \cdot p}$ contains $T_{X_A}(\mathbf{1})$. These two conditions hold, for some p in the dense torus of X_A , if and only if $\xi \in \text{image}(\varphi_A)$. \square

Proposition 4.1 shows that the dual variety X_A^* of the toric variety X_A is parametrized by monomials in linear forms. In the notation of Section 3 we set $m = n$, $r = n + d$, $s = n$, and the two matrices are

$$(4.2) \quad U = \begin{pmatrix} B & 0 \\ 0 & I_d \end{pmatrix} \quad \text{and} \quad V = (I_n \quad A^t),$$

where B is an $n \times (n-d)$ -matrix whose columns span the kernel of A over the integers. Thus the rows of B are *Gale dual* to the configuration A .

Lemma 4.2. *The variety $Y_{UV} \subset \mathbb{C}^n$ defined by (4.2) as in Section 3 is equal to the cone over the dual variety $X_A^* \subset (\mathbb{C}\mathbb{P}^{n-1})^*$ of the toric variety $X_A \subset \mathbb{C}\mathbb{P}^{n-1}$.*

Proof. Let f be the rational map defined by (4.2) as in Section 3, and set $x = (x_1, \dots, x_{n-d})$ and $t = (t_1, \dots, t_d) = (x_{n-d+1}, \dots, x_n)$. Then (3.1) equals

$$f_i(x, t) = (b_{i,1}x_1 + \cdots + b_{i,n-d}x_{n-d}) \cdot t^{a_i},$$

which equals the i -th coordinate of φ_A if we write $\ker(A)$ as the image of B . \square

The following result proves the equivalence of (a) and (b) in Theorem 1.1. The *co-Bergman fan* $\mathcal{B}^*(A)$ of the rank d configuration given by the columns of A equals the Bergman fan of the rank $n-d$ configuration given by the rows of B . Thus $\mathcal{B}^*(A)$ is the Bergman fan of the matroid dual to the matroid given by the columns of A .

Corollary 4.3. *For any configuration A , the tropical A -discriminant $\tau(X_A^*)$ is the Minkowski sum of the co-Bergman fan $\mathcal{B}^*(A)$ and the row space of A .*

Proof. By Theorem 3.3, the support of the co-Bergman fan $\mathcal{B}^*(A)$ is the tropicalization of the linear space $\ker(A) = \text{im}(B)$. Now, if U is taken as in (4.2) then we have the following decomposition in $\mathbb{R}^r = \mathbb{R}^n \oplus \mathbb{R}^d$:

$$\tau(\text{im}(U)) = \tau(\text{im}(B)) \oplus \tau(\text{im}(I_d)) = \mathcal{B}^*(A) \oplus \mathbb{R}^d.$$

The image of this fan under the linear map $V = \begin{pmatrix} I_n & A^t \end{pmatrix}$ is the (Minkowski) sum of $\mathcal{B}^*(A)$ and the image of A^t . Of course, the latter is the row space of A . Hence our assertion follows from Lemma 4.2 and Theorem 3.1. \square

Similarly, a description of the tropicalization of the reduced version of the dual variety can be derived from Theorem 3.1. The *reduced dual variety* Y_A^* is the closure of the image of the rational morphism $\tilde{\varphi}_A : \mathbb{C}\mathbb{P}^{n-d-1} \dashrightarrow \mathbb{C}\mathbb{P}^{n-d-1}$, whose i -th coordinate equals

$$\tilde{\varphi}_A(s_1 : \dots : s_{n-d})_i = \prod_{k=1}^n (b_{k,1}s_1 + \dots + b_{k,n-d}s_{n-d})^{b_{k,i}}, \quad i = 1, \dots, n-d,$$

where B again is an $n \times (n-d)$ -matrix whose columns span the kernel of A over the integers. The variety $Y_{BB^t} \subseteq \mathbb{C}^{n-d}$ equals the cone over Y_A^* in $\mathbb{C}\mathbb{P}^{n-d-1}$; hence the tropicalization $\tau(Y_A^*)$ equals the image of $\mathcal{B}(B) = \mathcal{B}^*(A)$ under the linear map B^t .

We often do not distinguish between the reduced and unreduced version of dual varieties and their tropicalizations, and denote both by X_A^* and $\tau(X_A^*)$, respectively.

We illustrate Corollary 4.3 for the case when X_A is the Veronese surface, regarded as the projectivization of the variety of all symmetric 3×3 matrices of rank ≤ 1 .

Example 4.4. We take $d = 3, n = 6$, and we fix the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{pmatrix}.$$

Note that the more usual matrix

$$A' = \begin{pmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot A,$$

defines the same Veronese embedding of $\mathbb{C}\mathbb{P}^2$ into $\mathbb{C}\mathbb{P}^5$, but in this case the parametrization $\psi_{A'}$ is two-to-one.

Points x in $\mathbb{C}\mathbb{P}^5$ are identified with symmetric 3×3 -matrices

$$X = \begin{pmatrix} 2x_1 & x_2 & x_4 \\ x_2 & 2x_3 & x_5 \\ x_4 & x_5 & 2x_6 \end{pmatrix}.$$

A point u is in $\mathbb{C}\mathbb{P}(\ker(A))$ if and only if the corresponding matrix U has zero row and column sums. If this holds, and t is any point in $(\mathbb{C}^*)^3$, then the symmetric 3×3 -matrix X corresponding to $x = \varphi_A(u, t)$ is singular because it satisfies

$$(1 \quad 1/t_2 \quad 1/t_3) \cdot X = (0 \quad 0 \quad 0).$$

Hence φ_A parametrizes rationally the hypersurface of singular symmetric matrices X , and the A -discriminant equals the classical discriminant $\Delta_A(x) = \frac{1}{2}\det(X)$.

The tropicalization of X_A^* is obtained as follows. We choose a Gale dual B of A ,

$$B^t = \begin{pmatrix} 1 & -2 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -2 & 0 & 1 \end{pmatrix}.$$

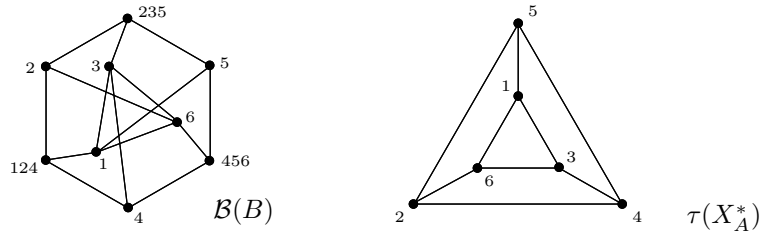


FIGURE 2. Bergman complex and tropical discriminant in Example 4.4

Note that the matroid given by the columns of A is self-dual. The Bergman fan $\mathcal{B}(B) = \mathcal{B}^*(A)$ is a 2-dimensional fan in $\mathbb{R}^6/\mathbb{R}(1, 1, 1, 1, 1, 1)$, or, equivalently, a graph on the 4-sphere. We depict this graph in Figure 2. It has nine vertices, corresponding to the six singletons $1, 2, \dots, 6$ and the three circuits $124, 235, 456$.

Its image under B^t is the tropical discriminant $\tau(X_A^*)$. This is a graph on the 2-sphere, namely, it is the edge graph of a triangular prism as shown in Figure 2. Since A is non-defective, $\tau(X_A^*)$ is the union of codimension 1 cones in the normal fan of the Newton polytope of Δ_A . We conclude that the Newton polytope is a bipyramid, whose five vertices correspond to the five terms in the determinant of X .

Returning to the general case, we note that the dimension of the image of φ_A is at most $\dim(\mathbb{C}\mathbb{P}(\ker(A)) \times T^{d-1}) = n - 2$, so the dual variety X_A^* is a proper subvariety of $\mathbb{C}\mathbb{P}^{n-1}$. If the dimension of X_A^* is less than $n - 2$, that is, if X_A^* is not a hypersurface, we say that the toric variety X_A , respectively the point configuration A , is *defective*. In the non-defective case, there is a unique (up to sign) irreducible polynomial Δ_A with integer coefficients which vanishes on X_A^* . The polynomial Δ_A is the *A-discriminant* as defined in [13, §9.1.A]. In what follows, the dual variety X_A^* itself will be referred to as the *A-discriminant*, even if A is defective.

By the Bieri-Groves Theorem [2, 22], the dimension of the *A-discriminant* X_A^* coincides with the dimension of the tropical *A-discriminant* $\tau(X_A^*)$. Corollary 4.3 furnishes a purely combinatorial formula for that dimension.

Corollary 4.5. *The dimension of the A-discriminant X_A^* in $\mathbb{C}\mathbb{P}^{n-1}$ is one less than the largest rank of any matrix $(A^t, \sigma_1, \dots, \sigma_{n-d-1})$ where σ runs over the set $\mathcal{C}(A)$.*

Here $\mathcal{C}(A)$ is the subset of $\{0, 1\}^n$ defined in the Introduction. That definition is now best understood using the matroid-theoretic concepts which we reviewed in the second half of Section 3, where we take U to be the $n \times (n - d)$ -matrix B as in (4.2) and, hence, M to be the rank $n - d$ matroid associated with the rows of B . In fact, M is the matroid dual to the matroid given by the columns of the $d \times n$ -matrix A , and $\mathcal{L}(A)$ coincides with the lattice of flats \mathcal{L}_M . The set $\mathcal{C}(A)$ corresponds to the facets of the order complex $\Delta(\mathcal{L}_M)$. In light of Theorem 3.3, one could reformulate Corollary 4.5 with σ ranging over the facets of the nested set complex $\mathcal{N}(\mathcal{L}_M)$ or the Bergman fan $\mathcal{B}(M) = \mathcal{B}^*(A)$. The `maple` program discussed below takes advantage of that formulation for efficiency reasons.

We are now prepared to state and prove the general version of Theorem 1.2. Let \mathcal{C}^c denote the subset of $\{0, 1\}^n$ consisting of all chains of length $n - d - c$ in

$\mathcal{L}(A) = \mathcal{L}_M$. Equivalently, \mathcal{C}^c is the set of $(n-d-c)$ -element subsets of the elements of $\mathcal{C} = \mathcal{C}(A)$. We write $\text{in}_w(X_A^*) = \text{in}_w(I_{X_A^*})$ for the initial ideal, with respect to some $w \in \mathbb{R}^n$, of the homogeneous prime ideal $I_{X_A^*}$ of the A -discriminant X_A^* .

Theorem 4.6. *Suppose that the A -discriminant X_A^* has codimension c and let $\tau = \{\tau_1, \dots, \tau_c\} \subset \{1, \dots, n\}$. If w is a generic vector in \mathbb{R}^n then the multiplicity of the initial monomial ideal $\text{in}_w(X_A^*)$ along the prime $P_\tau = \langle x_i : i \in \tau \rangle$ equals*

$$(4.3) \quad \sum_{\sigma \in \mathcal{C}_{i,w}^c} |\det(A^\dagger, \sigma_1, \dots, \sigma_{n-d-c}, e_{\tau_1}, \dots, e_{\tau_c})|,$$

where $\mathcal{C}_{i,w}^c$ is the subset of \mathcal{C}^c consisting of all chains σ such that

$$(4.4) \quad \text{rowspan}(A) \cap \mathbb{R}_{>0}\{\sigma_1, \dots, \sigma_{n-d-c}, -e_{\tau_1}, \dots, -e_{\tau_c}, -w\} \neq \emptyset.$$

Theorem 1.2 is the special case of Theorem 4.6 when A is non-defective, i.e., $c = 1$ and $I_{X_A^*}$ is the principal ideal generated by Δ_A . In that case, the initial monomial ideal $\text{in}_w(X_A^*)$ is generated by the initial monomial $\text{in}_w(\Delta_A)$.

Proof of Theorem 4.6. According to Theorem 2.2, the prime P_τ is associated to $\text{in}_w(X_A^*)$ if and only if the polyhedral cone $w + \mathbb{R}_{>0}\{e_{\tau_1}, \dots, e_{\tau_c}\}$ meets the tropicalization $\tau(X_A^*)$, which was described in Corollary 4.3 as $\mathcal{B}^*(A) + \text{rowspan}(A)$.

The collection of cones $\mathbb{R}_{\geq 0}\sigma$ for $\sigma \in \mathcal{C}$ forms a unimodular triangulation of the co-Bergman fan $\mathcal{B}^*(A)$. This was proved by Ardila and Klivans [1], and we discussed it in Theorem 3.3. Therefore, (4.4) characterizes when $w + \mathbb{R}_{>0}\{e_{\tau_1}, \dots, e_{\tau_c}\}$ meets $\mathbb{R}_{\geq 0}\sigma + \text{rowspan}(A)$ for some $\sigma \in \mathcal{C}^c$. The multiplicity of this intersection is precisely the stated $n \times n$ -determinant. This can be derived from Remarks 2.1 and 3.2. \square

Our degree formula for the A -discriminant (Theorem 1.2) can now be rephrased in the following manner which is more conceptual and geometric.

Corollary 4.7. *A monomial prime P_τ is associated to $\text{in}_w(X_A^*)$ if and only if the cone $w + \mathbb{R}_{>0}\{e_{\tau_1}, \dots, e_{\tau_c}\}$ meets the fan $\mathcal{B}^*(A) + \text{rowspan}(A)$. The number of intersections, counted with multiplicity, is the multiplicity of $\text{in}_w(X_A^*)$ along P_τ .*

We close this section with a brief discussion of some computational issues. The co-Bergman fan of the matrix A can be computed efficiently by gluing local Bergman fans, as explained in [10, Algorithm 5.5]. See Examples 5.7, 5.8 and 5.9 in [10] for some non-trivial computations. Extending the software used for those computations, we prepared a `maple` program for evaluating the formula (1.2) in Theorem 1.2. The input for our program consists of three positive integers d, n, R , and a $d \times n$ -matrix A which is assumed to be non-defective. The output is a list of initial monomials $\text{in}_w(\Delta_A)$ of the A -discriminant Δ_A , for R randomly chosen vectors w in \mathbb{N}^n . Our `maple` implementation is available upon request from any of the three authors.

Example 4.8. Let $d = 4$ and $n = 8$, and consider the matrix below. What follows is the output of our `maple` program on this input. On a fast workstation, our code takes about half a second to compute the co-Bergman fan, and afterwards it takes about one second per initial monomial. So the total running time for this matrix is about R seconds, where R is the number of iterations specified by the user:


```

      [ 1   1   1   1   0   0   0   0]
      [                                     ]
A := [ 0   0   0   0   1   1   1   1]
      [                                     ]
      [ 2   3   5   7  11  13  17  19]
      [                                     ]
      [53  47  43  41  37  31  29  23]

```

Computing the Co-Bergman fan of A...

DONE. Time elapsed = 0.520

The number of maximal cones in the Co-Bergman fan is 60
 48 of these cones map to codimension one in the tropical discriminant.

What follows are 10 pairs of weight vectors and initial monomials:

```

                                20  36  36  2  36  58  4
[382,171,298,564,677,339,186,669], x1  x2  x3  x4  x6  x7  x8

```

```

                                36  2  56  66  32
[122, 710, 356, 64, 358, 261, 675, 686], x1  x2  x4  x6  x7

```

From this output we see that the degree of the A -discriminant equals $192 = 36 + 2 + 56 + 66 + 32$. We discuss the geometric meaning of this example in Section 6.

5. SUBDIVISIONS

Gel'fand, Kapranov and Zelevinsky [13] established the relationship between the A -discriminant and the *secondary polytope*. This polytope parametrizes the regular polyhedral subdivisions of A . We now explain how this relationship is derived using our tropical approach. After introducing the notion of a strong co-loop, we prove the remaining part of Theorem 1.1, namely the identification of $\tau(X_A^*)$ with a certain fan of regular subdivisions of A . Thereafter, we turn our attention to the problem of characterizing Δ -equivalence of regular triangulations, and we prove Theorem 1.3.

Recall that any vector $w \in \mathbb{R}^n$ defines a regular polyhedral subdivision Π_w of the configuration $A = \{a_1, \dots, a_n\}$. The cells of Π_w are the subsets of A corresponding to the lower facets of the convex hull of the points $(a_1, w_1), \dots, (a_n, w_n)$ in \mathbb{R}^{d+1} . By a cell σ of Π_w we always mean a subset σ of $A = \{a_1, \dots, a_n\}$ whose associated matroid $M(\sigma)$ has maximal rank $d = \text{rank}(M(A))$. An element of a cell σ is a *co-loop of σ* if it is in every basis of the matroid $M(\sigma)$. If a cell σ has k co-loops then it is a k -fold pyramid over a configuration of rank $d - k$. The assumption stated in the Introduction that A is not a pyramid means that $M(A)$ has no co-loops.

Let σ be a cell of Π_w . There exists a unique affine-linear function ψ_σ on \mathbb{R}^d such that $\psi_\sigma(a_i) = 1$ if $a_i \in \sigma$ is a co-loop and $\psi_\sigma(a_j) = 0$ if $a_j \in \sigma$ is not a co-loop. With the cell σ we associate the *enlarged cell* σ^* which is defined as follows:

$$(5.1) \quad \sigma^* = \sigma \cup \{a \in A : \psi_\sigma(a) < 1\}.$$

Note that $\sigma^* = A$ if σ has no co-loops, and $\sigma^* = \sigma$ if σ is a simplex (i.e., every point of σ is a co-loop). Otherwise, σ^* is constructed geometrically as follows: Consider the hyperplane which passes through all co-loops and is parallel to the affine span of the non-co-loops. Then σ^* consists of the co-loops of σ together with all points of A which lie in the open halfspace containing the non-co-loops. A co-loop of σ is called a *strong co-loop* if it remains a co-loop in the enlarged cell σ^* .

Readers of [13] will find it instructive to look at Figure 49 on page 367. In diagrams (a) and (c) of that figure, the point ω is a strong co-loop of the pyramidal cell, and in diagram (b) both ω_1 and ω_2 are strong co-loops of the tetrahedral cell.

The proof of Theorem 1.1 will be completed by establishing the following lemma.

Lemma 5.1. *The tropical A -discriminant $\tau(X_A^*)$ equals the set*

$$(5.2) \quad \{w \in \mathbb{R}^n : \Pi_w \text{ has a maximal cell } \sigma \text{ with no strong co-loops}\}.$$

The proof is given after Example 5.3. An important corollary to this lemma is that membership in the tropical A -discriminant depends only on the regular subdivision specified by the vector w , i.e., if $w \in \tau(X_A^*)$ and $\Pi_w = \Pi_{w'}$ then $w' \in \tau(X_A^*)$.

Corollary 5.2. *$\tau(X_A^*)$ is a subfan of the secondary fan $\Sigma(A)$.*

The statement of this corollary is new and rather non-trivial when $\text{codim}(X_A^*) \geq 2$. For $\text{codim}(X_A^*) = 1$, which is the case studied in [13], it follows from the known fact that the secondary polytope is the Newton polytope of the principal A -determinant.

The secondary fan $\Sigma(A)$ of a configuration A is usually described in terms of its maximal cones, which are indexed by the regular triangulations of A . The tropical discriminant $\tau(X_A^*)$ lies in the complement of these open maximal cones. It consists of non-maximal cones, so the subdivisions Π_w for $w \in \tau(X_A^*)$ are certainly not triangulations. We illustrate Corollary 5.2 for the configuration in Example 4.4.

Example 5.3. Let A be the 3×6 -matrix in Example 4.4 whose toric variety is the Veronese surface in \mathbb{CP}^5 and whose tropical discriminant $\tau(X_A^*)$ was depicted in Figure 2. We now identify $\tau(X_A^*)$ as a subfan of the secondary fan $\Sigma(A)$. Both of these two-dimensional fans are drawn as planar graphs in Figure 3. The graph $\Sigma(A)$ is dual to the three-dimensional associahedron, and its 14 regions are labeled with the 14 regular triangulations of the configuration A . The tropical discriminant $\tau(X_A^*)$ is the subgraph which is indicated by solid lines. Edges of $\Sigma(A)$ that do not belong to $\tau(X_A^*)$ are dashed. The 14 regular triangulations of A occur in five Δ -equivalence classes corresponding to the open cells in the complement of $\tau(X_A^*)$.

The magnification on the left in Figure 3 shows a portion of the (dual) secondary polytope of A with corresponding polyhedral subdivisions of the 6-point configuration indicated next to the faces. The enlarged points are strong co-loops with respect to the shaded (non-simplicial) faces, showing that neither of the edges of the square face in the secondary polytope nor its interior corresponds to a face of the tropical discriminant. The solid edge, however, corresponds to a cell in $\tau(X_A^*)$ since the only maximal cell of its associated polyhedral subdivision has no strong co-loop.

Proof of Lemma 5.1 (and Theorem 1.1). We first prove the inclusion of $\tau(X_A^*)$ in (5.2). Since the set (5.2) is invariant under translation by an element in the row span of A , it suffices to consider any vector w in the Bergman fan $\mathcal{B}^*(A)$. After adding

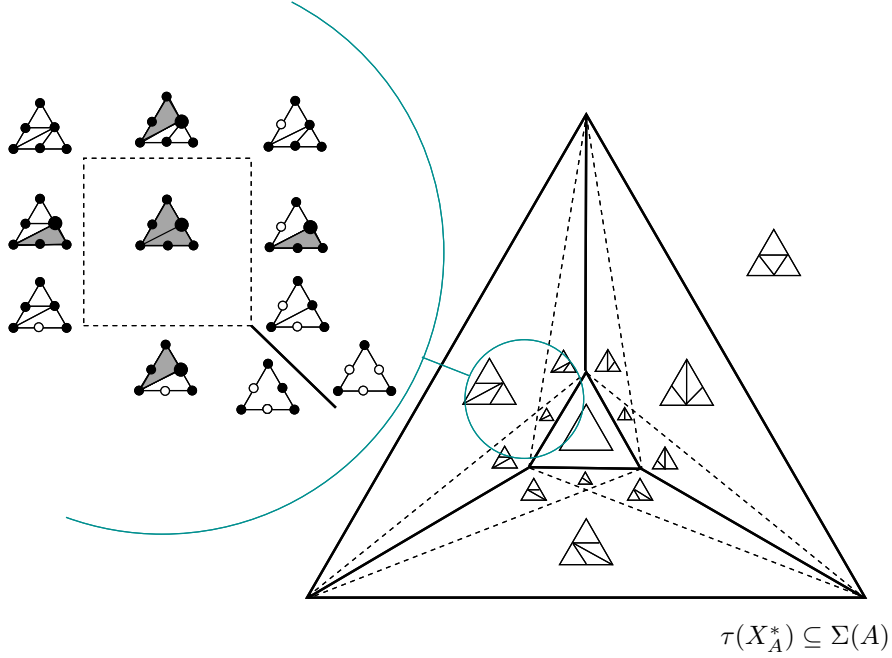


FIGURE 3. The tropical discriminant is a subfan of the secondary fan

a constant vector, we may assume that w is non-negative and $\tau = \{i : w_i = 0\}$ is non-empty. Clearly, τ is a face (of some rank) in the regular subdivision Π_w of A . Now, since $w \in \mathcal{B}^*(A)$, the initial matroid $M(A)_{-w}$ has no co-loops. If τ has rank d then $M(A)_{-w} = M(\tau)$, so τ is a cell of Π_w and has no co-loops, and we are done. Otherwise, if τ has rank less than d , consider any cell σ of Π_w which contains τ . Every co-loop a_i of σ is contained in $\sigma \setminus \tau$. We must show that such an a_i cannot be a strong co-loop. Indeed, since a_i is not a co-loop of $M(A)_{-w}$ there exists an element a_k of $A \setminus \sigma$ which can replace a_i in some basis of $M(A)_{-w}$. This means that $w_i = w_k$, and we can conclude that a_k lies in the enlarged cell σ^* . Hence a_i is not a co-loop of $M(\sigma^*)$, and we are done.

For the converse direction, let w be any vector in \mathbb{R}^n such that some cell σ of Π_w has no strong co-loop. After changing w by an element in the row space of A , we may assume that $w_i = 0$ for $i \in \sigma$ and $w_j > 0$ for $j \notin \sigma$. This ensures $M(A)_{-w} = M(\sigma)$. If σ has no co-loop, we are done. Otherwise, we consider the affine-linear function ψ_σ and we identify it with a vector in the row space of A . Among all elements a_k in σ^* pick one such that $\lambda_k := w_k / (1 - \psi_\sigma(a_k))$ is positive but as small as possible. Now replace w by $w + \lambda_k \psi_\sigma$, and replace σ by its proper superset $\sigma \cup \{a_j : \lambda_j = \lambda_k\}$, which is a subset of σ^* . At least one of the old co-loops is no longer a co-loop of the new matroid $M(\sigma)$. We now iterate this process until all co-loops have disappeared. At the end, we have replaced the original vector w by an element of $w + \text{rowspace}(A)$ which lies in the Bergman fan $\mathcal{B}^*(A)$. This proves that w lies in the tropical discriminant $\tau(X_A^*)$. \square

Corollary 4.3 gives a piecewise-linear parametrization of the tropical discriminant:

$$\begin{aligned} \mathcal{B}^*(A) \times \text{rowspace}(A) &\rightarrow \tau(X_A^*) \\ (u, v) &\mapsto u + v \end{aligned}$$

If X_A is non-defective then this map is generically one-to-one, but some special fibers may have more than one element. The arguments in the proof of Lemma 5.1 show how to invert the map on a dense subset of $\tau(X_A^*)$. Namely, if $w \in \tau(X_A^*)$ is such that the corresponding subdivision Π_w has a cell σ without any co-loops, the inversion amounts to the following steps. Pick the unique element v in the row space of A such that $w_j = v_j$ for all $a_j \in \sigma$, and then set $u = w - v$. Observe that $u \in \mathcal{B}^*(A)$ by definition of the co-Bergman fan and our assumption that σ has no co-loop. If all cells in Π_w do have co-loops, elements $v \in \text{rowspace}(A)$ and $u \in \mathcal{B}^*(A)$ are picked according to the procedure described in the second half of the proof of Lemma 5.1.

We now turn to the proof of Theorem 1.3, which gives our combinatorial characterization of the notion of Δ -equivalence [13, §11.3, page 368]. This equivalence relation on regular triangulations of a non-defective configuration A is defined as follows. Let Π_w and $\Pi_{w'}$ be two regular triangulations which are neighbors in the secondary fan $\Sigma(A)$. This means that their cones in $\Sigma(A)$ share a common face of codimension one. We call Π_w and $\Pi_{w'}$ Δ -equivalent if they specify the same leading monomial of the A -discriminant, i.e., $\text{in}_w(\Delta_A) = \text{in}_{w'}(\Delta_A)$. Lemma 5.1 implies:

Corollary 5.4. *The neighboring regular triangulations Π_w and $\Pi_{w'}$ of the configuration A are Δ -equivalent if and only if every cell in the common coarsening of Π_w and $\Pi_{w'}$ has a strong co-loop.*

Theorem 1.3 expresses the same condition in geometric terms. Here is the proof:

Proof of Theorem 1.3. Let Π be a subdivision which refines to two neighboring regular triangulations. Then Π contains a unique circuit C as a face, and the link of C in Π is a ball or sphere of complementary dimension. Consider any cell σ of Π which contains C . The co-loops of σ are the elements in $\sigma \setminus C$. If $a \in \sigma \setminus C$ is a strong co-loop then every point in $\sigma^* \setminus \{a\}$ must lie in the hyperplane spanned by $\sigma \setminus \{a\}$. Hence this hyperplane is a facet of A , and, among all points of A that are not on that facet, the point a has minimum distance to that facet. Conversely, suppose that a is a point with that property. Then $\sigma^* \setminus \{a\}$ is contained in that facet and hence a is a co-loop of σ^* . We conclude that a is a strong co-loop of σ . \square

We now present two examples that illustrate Theorem 1.1. In particular, they highlight the fact that there are different fan structures of the tropical discriminant.

Example 5.5. Let $d = 3, n = 6$, and consider the non-defective configuration

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 2 & 3 & 0 & 1 \end{pmatrix}.$$

Here, $\Delta_A = x_1x_6^3 - x_2x_5x_6^2 + x_3x_5^2x_6 - x_4x_5^3$. Modulo the row space of A , the secondary fan has eight rays and eighteen 2-cones, corresponding to a planar graph with eight vertices and eighteen edges. The tropical discriminant $\tau(X_A^*)$ corresponds to the induced subgraph on six of the vertices, namely, $e_1, e_2, e_3, e_4, 2e_1 + e_2$ and $e_3 + 2e_4$.

Here, the secondary fan strictly refines the Gröbner fan on $\tau(X_A^*)$. The latter is the complete graph K_4 with vertices e_1, e_2, e_3 and e_4 , while the former has the edges e_1, e_2 and e_3, e_4 subdivided by the vertices $2e_1 + e_2$ and $e_3 + 2e_4$, respectively.

Example 5.6. We take $d = 4, n = 9$, and consider the defective configuration

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \end{pmatrix}.$$

The tropical discriminant $\tau(X_A^*)$ is a 7-dimensional fan in \mathbb{R}^9 , regarded as a 6-dimensional polyhedral complex. Combinatorially, it is an immersion of the complete tripartite hypergraph $K_{3,3,3}$. The Gröbner fan subdivision has 51 maximal cones and it strictly refines the secondary fan subdivision which has only 49 cones. Indeed, the vector $w = (0, 1, 0, 0, 1, 0, 0, 1, 0)$ lies in the relative interior of a maximal cone of the secondary fan subdivision which breaks into three maximal cones in the Gröbner fan subdivision. This example was verified by applying the **Gfan** software presented in [3] to the equations defining X_A^* . In particular, we found that $\text{in}_w(X_A^*)$ is the codimension two primary ideal generated by the determinant of the 3×3 -matrix

$$\begin{pmatrix} x_1 & x_4 & x_7 \\ x_2 & x_5 & x_8 \\ x_3 & x_6 & x_9 \end{pmatrix}$$

plus the square of the ideal of 2×2 -minors of the 2×3 -matrix

$$\begin{pmatrix} x_1 & x_4 & x_7 \\ x_3 & x_6 & x_9 \end{pmatrix}.$$

Remark 5.7. A worthwhile next step would be to relate our results to the celebrated work of Mikhalkin [17] on Gromov-Witten invariants. The point is that the regular polyhedral subdivision Π_w of A is geometrically dual to the tropical hypersurface defined by the tropical polynomial $\bigoplus_{i=1}^n w_i \odot x^{\odot a_i}$. This tropical hypersurface is considered to be *singular* whenever the vector w lies in the tropical discriminant $\tau(X_A^*)$. Thus our object of study in this paper is the *space of singular tropical hypersurfaces*. What has been accomplished in Theorem 1.2 is to give a refined formula for the degree of that space. Our formula is consistent with the lattice paths count in [17] in the case $d = 3$, and it would be interesting to explore possible applications of our combinatorial approach to Gromov-Witten theory. The work of Gathmann and Markwig [12] offers an algebraic setting for such a study.

6. CAYLEY CONFIGURATIONS AND RESULTANT VARIETIES

One of the main applications of A -discriminants is the study of resultants in *elimination theory*. The configurations A which arise in elimination theory have a special combinatorial structure arising from the *Cayley trick*. See [13, §3.2.D] for a geometric introduction. Based on the results of the earlier sections, we here study the combinatorics and geometry of *tropical resultants*, and we generalize the positive degree formula for resultants in [21] to resultant varieties of arbitrary codimension.

Let A_1, \dots, A_m be finite subsets of \mathbb{Z}^r . Their *Cayley configuration* is defined as

$$(6.1) \quad A = \{e_1\} \times A_1 \cup \{e_2\} \times A_2 \cup \dots \cup \{e_m\} \times A_m \subset \mathbb{Z}^m \times \mathbb{Z}^r,$$

where e_1, \dots, e_m is the standard basis of \mathbb{Z}^m . To be consistent with our notation in Sections 1–5, we can regard A as a $d \times n$ -matrix with $d = m + r$ and $n = |A_1| + |A_2| + \dots + |A_m|$. As in [13, §8.1] and [21], the Cayley configuration A represents the following system of m Laurent polynomial equations in r unknowns:

$$(6.2) \quad \sum_{u \in A_1} x_{1,u} z^u = \sum_{u \in A_2} x_{2,u} z^u = \dots = \sum_{u \in A_m} x_{m,u} z^u = 0.$$

Here $z = (z_1, \dots, z_r)$ are coordinates on $(\mathbb{C}^*)^r$ and we use multi-index notation $z^u = z_1^{u_1} \dots z_r^{u_r}$. Our earlier examples include the following Cayley configurations:

- In Example 4.8 we have $m = 2, r = 2$, and the system (6.2) takes the form

$$\begin{aligned} x_1 z_1^2 z_2^{53} + x_2 z_1^3 z_2^{47} + x_3 z_1^5 z_2^{43} + x_4 z_1^7 z_2^{41} &= 0, \\ x_5 z_1^{11} z_2^{37} + x_6 z_1^{13} z_2^{31} + x_7 z_1^{17} z_2^{29} + x_8 z_1^{19} z_2^{23} &= 0. \end{aligned}$$

- In Example 5.5 we have $m = 2, r = 1, A_1 = \{0, 1, 2, 3\}$ and $A_2 = \{0, 1\}$. The

$$A\text{-discriminant is the Sylvester resultant } \Delta_A = \det \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_5 & x_6 & 0 & 0 \\ 0 & x_5 & x_6 & 0 \\ 0 & 0 & x_5 & x_6 \end{pmatrix}.$$

- In Example 5.6 we have $m = 3, r = 1$ and $A_1 = A_2 = A_3 = \{0, 1, 2\}$, and the system (6.2) consists of three quadratic equations in one unknown z :

$$x_1 + x_2 z + x_3 z^2 = x_4 + x_5 z + x_6 z^2 = x_7 + x_8 z + x_9 z^2 = 0.$$

The solvable systems of this form constitute the codimension two variety X_A^* .

Returning to the general case, we say that the Cayley configuration A is *essential* if the Minkowski sum $\sum_{i \in I} A_i$ has affine dimension at least $|I|$ for every subset I of $\{1, \dots, m\}$ with $|I| \leq r$. The *resultant variety* of the Cayley configuration A is the Zariski closure in $\mathbb{C}\mathbb{P}^{n-1}$ of the set of all points $(x_1 : x_2 : \dots : x_n)$ whose corresponding system (6.2) has a solution z in $(\mathbb{C}^*)^r$. The following result is a generalization of Proposition 1.7 in [13, §9.1.A] and of Proposition 5.1 in [4].

Proposition 6.1. *The resultant variety of any Cayley configuration A contains the A -discriminant X_A^* . If $m \geq r + 1$ and the Cayley configuration A is essential then the resultant variety and the A -discriminant coincide.*

Proof. Consider the hypersurface in $(\mathbb{C}^*)^m \times (\mathbb{C}^*)^r$ defined by the Laurent polynomial

$$\sum_{i=1}^m \sum_{u \in A_i} x_{i,u} \cdot t_i z^u = 0.$$

If $(t, z) \in (\mathbb{C}^*)^{m+r}$ is a singular point on this hypersurface then $z \in (\mathbb{C}^*)^r$ is a solution to (6.2). This proves the inclusion. If A is essential then a linear algebra argument as in [13, §9.1.A] shows that every solution z of (6.2) arises in this way. \square

The hypothesis that A be essential is necessary for the equality of the resultant variety and the A -discriminant, even when $m = r + 1$, the situation of classical elimination theory. The following simple example gives a hint of the general behavior.

Example 6.2. Let $r = 2, m = 3, A_1 = A_2 = \{(0, 0), (1, 0)\}$, and $A_3 = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$. The Cayley matrix A represents a toric 4-fold X_A in $\mathbb{C}\mathbb{P}^7$. It is not essential since $A_1 + A_2$ is one-dimensional. The system (6.2) equals

$$x_1 + x_2 z_1 = x_3 + x_4 z_1 = x_5 + x_6 z_1 + x_7 z_2 + x_8 z_1 z_2 = 0.$$

The resultant variety has codimension one, with equation $x_1 x_4 = x_2 x_3$, but the A -discriminant X_A^* has codimension three. In fact, we have $X_A^* = X_A$ in this case.

For the rest of this section we consider an essential Cayley configuration as in (6.1) with $m \geq r + 1$ blocks and we set $c = m - r$. We then have the following result.

Lemma 6.3. *The resultant variety X_A^* has codimension c .*

Proof. Let W denote the incidence variety consisting of all pairs (x, z) in $\mathbb{C}\mathbb{P}^{n-1} \times (\mathbb{C}^*)^r$ such that (6.2) holds. Let $\pi_1 : W \rightarrow \mathbb{C}\mathbb{P}^{n-1}$ be the projection to the first factor. By Proposition 6.1, the resultant variety X_A^* coincides with the closure of $\pi_1(W)$. Looking at the second projection $\pi_2 : W \rightarrow (\mathbb{C}^*)^r$, which is surjective and whose fibers are linear spaces of dimension $n - 1 - m$, we deduce that W is irreducible and has dimension $(n - 1 - m) + r = n - 1 - c$. Then, $\dim(X_A^*) \leq n - 1 - c$.

Given a generic choice of coefficients x_i for the first r polynomials, it follows from the hypothesis of essentiality and Bernstein's Theorem, that the first r equations in (6.2) have a common solution $z \in (\mathbb{C}^*)^r$. We can freely choose all but one of the coefficients of the last c polynomials so that z solves (6.2). This implies that $\dim(X_A^*) \geq n - 1 - c$, and the lemma follows. \square

Corollary 4.5 asserts that there exists a chain $\sigma_1, \dots, \sigma_{n-2m}$ of $(0, 1)$ -vectors representing the supports of vectors in the kernel of A such that the rank of the matrix $(A^t, \sigma_1, \dots, \sigma_{n-2m})$ is precisely $n - c$. We present an explicit way of choosing such a chain. By performing row operations, we can assume that each set A_i contains the origin. Set $B_i = A_i \setminus \{0\}$. Let $b_i \in B_i$ for $i = 1, \dots, r$ such that b_1, \dots, b_r are linearly independent. Such elements exist because the family of supports is essential. Now, for any other element a in $B = (B_1 \cup \dots \cup B_{r+c}) \setminus \{b_1, \dots, b_r\}$, we can find an element v_a in $\ker(A)$ with support corresponding to the origin in each A_i for i from 1 to r , union the variables corresponding to b_1, \dots, b_r and $a \in B_j$, plus the origin of A_j in case $j > r$. Choose any such $a \in B_{r+1}$ and let σ_1 be the support of v_a ; it will have $2r + 2$ non-zero coordinates. Add a new point a' in B . We can assume that the support of $v_a + v_{a'}$ equals the union of their supports. Let σ_2 be the associated support vector. We continue in this manner, adding a new point in B at a time, and considering a new element in the chain of support vectors, but avoiding to pick all of $B_1 \cup \dots \cup B_{r+1}$ and all of each of B_{r+2}, \dots, B_{r+c} . This produces precisely

$$1 + (n_1 - 2 + \dots + n_{r+1} - 2) - 1 + (n_{r+2} - 2) + \dots + (n_{r+c} - 2) = n - 2r - 2c = n - 2m$$

vectors $\sigma_1, \dots, \sigma_{n-2m}$ in $\mathcal{C}(A)$. It is straightforward to check that the rank of the submatrix of $(A^t, \sigma_1, \dots, \sigma_{n-2m})$ given by the first m and the last $n - 2m$ columns has maximal rank $n - m$. Note that this is just a $(0, 1)$ matrix. Adding the last

r columns of A^t containing the information about the specific supports A_1, \dots, A_m will increase the rank by r , as a consequence of the fact that the family is essential. Therefore, the rank of the matrix $(A^t, \sigma_1, \dots, \sigma_{n-2m})$ is precisely $n - m + r = n - c$.

We identify $\{1, 2, \dots, n\}$ with the disjoint union of the sets A_1, A_2, \dots, A_m . Thus a generic vector $w \in \mathbb{R}^n$ assigns a height to each point in any of the A_i , and it defines a *tight coherent mixed subdivision* Δ_w of the Minkowski sum $\sum_{i=1}^m A_i$ (cf. [21]). When $c = 1$, the initial form with respect to w of the mixed resultant is described in [21, Theorem 2.1] in terms of the sum of volumes of suitable *mixed cells* of the tight coherent mixed decomposition (TCMD) induced by w . We next generalize this result to resultant varieties of arbitrary codimension c . For a classical study of resultant ideals of dense homogeneous polynomials we refer to [14].

Theorem 6.4. *A codimension c monomial prime $P_\tau = \langle x_{\tau_1}, \dots, x_{\tau_c} \rangle$ is a minimal prime of the monomial ideal $\text{in}_w(X_A^*)$ only if τ consists of one point each from c of the A_j . The multiplicity of P_τ is the total volume of all mixed cells in the tight coherent mixed subdivision Δ_w which use the points of τ in their decomposition.*

Proof. The resultant variety X_A^* is irreducible (by Proposition 6.1), and it has codimension c (by Lemma 6.3). This implies (by [15, Theorem 1]) that every minimal prime of the initial monomial ideal $\text{in}_w(X_A^*)$ has codimension c . Let $P_\tau = \langle x_{\tau_1}, \dots, x_{\tau_c} \rangle$ be such a minimal prime. After relabeling we may assume that each x_{τ_i} in P_τ is a coefficient of one of the last c Laurent polynomials in (6.2).

The proof for the case $c = 1$ is given in [21, §2], and the proof for $c > 1$ uses the same general technique. We write f_i for the i -th Laurent polynomial in (6.2), but with x_i replaced by $x_i \epsilon^{w_i}$. Let \mathbf{K} be the algebraic closure of the field of rational functions over $K = \mathbb{C}\{\{\epsilon\}\}$ in the coefficients of the first r Laurent polynomials f_1, \dots, f_r , let \mathbf{x} denote the vector all coefficients of the last c Laurent polynomials f_{r+1}, \dots, f_m , and consider the polynomial ring $\mathbf{K}[\mathbf{x}]$. Let μ denote the mixed volume of the Newton polytopes of the polynomials f_1, \dots, f_r . Then, by Bernstein's Theorem, the system $f_1 = \dots = f_r = 0$ has μ distinct roots $\mathbf{z}_1(\epsilon), \dots, \mathbf{z}_\mu(\epsilon)$ in $(\mathbf{K}^*)^r$.

For any $j \in \{1, 2, \dots, \mu\}$, the ideal $I_j = \langle f_{r+1}(\mathbf{z}_j(\epsilon)), \dots, f_m(\mathbf{z}_j(\epsilon)) \rangle$ is generated by linear forms in $\mathbf{K}[\mathbf{x}]$. The intersection of these ideals, $I = I_1 \cap I_2 \cap \dots \cap I_\mu$, is an ideal of codimension c and degree μ in $\mathbf{K}[\mathbf{x}]$. Geometrically, we obtain I by embedding the prime ideal of X_A^* into $\mathbf{K}[\mathbf{x}]$ and then replacing x_i by $x_i \epsilon^{w_i}$. This is the higher codimension version of the *product formula for resultants* [21, Eqn. (14)].

The ideal I represents a flat family, and its special fiber $I|_{\epsilon=0}$ at $\epsilon = 0$ coincides with the special fiber of the image of $\text{in}_w(X_A^*)$ in $\mathbf{K}[\mathbf{x}]$. In particular, P_τ is an associated prime of $I|_{\epsilon=0}$, and it contains one of the ideals $I_j|_{\epsilon=0}$. Since the generators of I_j are c linear forms in disjoint groups of unknowns x_ℓ , we see that P_τ contains one unknown from each group. This proves the first statement in Theorem 6.4.

After relabeling we may assume that x_{τ_j} is a coefficient of f_{r+j} for $j = 1, \dots, c$. Each root $\mathbf{z}_j(\epsilon)$ corresponds to a mixed cell C in the TCMD of the small Minkowski sum $A_1 + \dots + A_r$ defined by the restriction of w . By the genericity of w , the mixed cell C corresponds to a unique cell C' in the TCMD Δ_w of the big Minkowski sum $A_1 + \dots + A_r + A_{r+1} + \dots + A_m$, and every mixed cell of Δ_w arises in this manner. The reasoning above implies that the mixed cell C' uses the points of τ in its decomposition if and only if $I_j|_{\epsilon=0} = P_\tau$ in $\mathbf{K}[\mathbf{x}]$. This completes the proof. \square

The first assertion in Theorem 6.4 can also be derived more easily, namely, from the fact that for any $(r + 1)$ -element subset I of $\{1, \dots, m\}$, the mixed resultant of the configurations $A_i, i \in I$, vanishes on X_A^* and only contains unknowns $x_{i,a}$ with $i \in I$. However, for the multiplicity count in the second assertion we need the “product formula” developed above. Theorem 6.4 has the following corollary.

Corollary 6.5. *The degree of the resultant variety X_A^* is the sum of the mixed volumes $MV(A_{i_1}, \dots, A_{i_r})$ as $\{i_1, \dots, i_r\}$ runs over all r -element subsets of $\{1, \dots, n\}$.*

We present two examples to illustrate Theorem 6.4 and Corollary 6.5.

Example 6.6. Let $m = 3, r = 1$ and $A_1 = A_2 = A_3 = \{0, 1, 2\}$ as in Example 5.6, and choose $w \in \mathbb{R}^9$ which represents the *reverse lexicographic term order*. Then

$$\text{in}_w(X_A^*) = \langle x_3x_5x_7, x_6^2x_7^2, x_3x_6x_7^2, x_3^2x_7^2, x_3x_4x_6x_7, x_3^2x_4x_7, x_3^2x_4^2, x_2x_4x_6^2x_7 \rangle.$$

This ideal has seven associated primes, of which three are minimal: $\langle x_3, x_6 \rangle$, $\langle x_3, x_7 \rangle$, and $\langle x_4, x_7 \rangle$. They correspond to the three mixed cells $(3, 6, \{7, 8, 9\})$, $(3, \{4, 5, 6\}, 7)$ and $(\{1, 2, 3\}, 4, 7)$ of the TCMD Δ_w of $A_1 + A_2 + A_3 = \{0, 1, \dots, 6\}$. Each mixed cell has volume two, which implies that the degree of X_A^* is $2 + 2 + 2 = 6$.

Example 6.7. Let $r = 2$ and $m = 4$ and take the A_i to be the four subtriangles of the square with vertices $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$. Here (6.2) is a system of four equations in two unknowns z_1 and z_2 which can be written in matrix form as

$$(6.3) \quad \begin{pmatrix} x_1 & x_2 & x_3 & 0 \\ x_4 & x_5 & 0 & x_6 \\ x_7 & 0 & x_8 & x_9 \\ 0 & x_{10} & x_{11} & x_{12} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ z_1 \\ z_2 \\ z_1z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The resultant variety $X_A^* \subset \mathbb{CP}^{11}$ has codimension 2 and degree $12 = \binom{4}{2} \cdot 2$. This is the sum over the mixed areas of the $\binom{4}{2}$ pairs of triangles. The mixed area is 2 for each such pair. Using computer algebra, we find that the prime ideal of the resultant variety X_A^* is generated by the 4×4 determinant in (6.3) together with ten additional polynomials of degree six in the x_i . Theorem 6.4 gives a combinatorial recipe for constructing all the initial monomial ideals of this prime ideal.

We conclude with a brief discussion of the *tropical resultant* $\tau(X_A^*)$. The results in Section 5 characterize this polyhedral fan in terms of regular subdivisions of A , and we will now rephrase this characterization in terms of coherent mixed subdivisions of (A_1, \dots, A_m) . Theorem 5.1 in [21] implies that every regular subdivision Π_w of A corresponds uniquely to a coherent mixed subdivision (CMD), which we denote by Δ_w . Note that Π_w is a polyhedral cell complex of dimension $r + m - 1$ while Δ_w has only dimension r , and Π_w is a triangulation if and only if Δ_w is a TCMD.

Every cell F of a CMD Δ_w decomposes uniquely as a Minkowski sum $F = F_1 + \dots + F_m$, where $F_i \subset A_i$ for all i . We write \overline{F} for the corresponding cell of Δ_w . We say that the cell F is *fully mixed* if each F_i has affine dimension at least one.

Proposition 6.8. *The tropical resultant $\tau(X_A^*)$ equals the set*

$$(6.4) \quad \{w \in \mathbb{R}^n : \Delta_w \text{ has a maximal cell } F \text{ which is fully mixed}\}.$$

Proof. This can be derived from Lemma 5.1 as follows. If F_i does not have affine dimension at least one, then F_i consists of a single point in A_i . That point is a co-loop in \overline{F} . The extended cell \overline{F}^* contains no other points of A which arise from A_i . Hence F_i remains a co-loop in \overline{F}^* , meaning that F_i represents a strong co-loop of \overline{F} . All co-loops of a full-dimensional subconfiguration of A arise in this way. This holds because every such subconfiguration is a Cayley configuration as well. \square

Specializing to the classical case $c = 1$, when X_A^* is the hypersurface defined by the mixed resultant, we can now recover the combinatorial results in [21, §5] from the more general results in Section 5 above. Specifically, the characterization of R-equivalence for TCMDs of (A_1, \dots, A_m) given in [21, Theorem 5.2] follows from our characterization of Δ -equivalence of regular triangulations of A in Theorem 1.3.

The positive formula for the extreme monomials of the sparse resultant given in [21, Theorem 2.1] can now be recovered as a special case of Theorem 1.2. Thus, one way to look at Sections 1–5 in the present paper is that these extend all the work in [21] from essential Cayley configurations with $m = r + 1$ to arbitrary essential configurations A . A perspective on how this relates to the results in [13] is given by the points (a),(b),(c),(d) found at the end of the introduction in [21, p. 208–209].

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