Facets of Geometry

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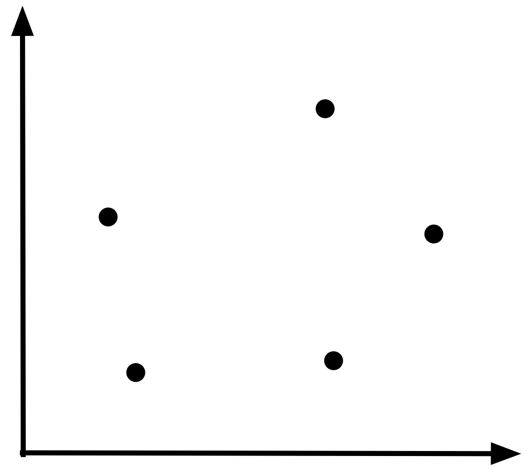
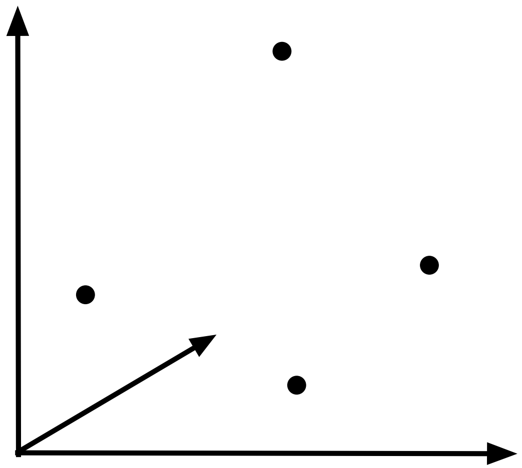
Polytopes

A polytope is the convex hull of a finite point set in $\mathbb{R}^d$:

$$P = \mathrm{conv}(x_1, \ldots, x_n) = \{\sum_{i=1}^{n} \lambda_i x_i \,|\, \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1\}$$

for $x_1, \ldots, x_n$ in $\mathbb{R}^d$.
Define dim P := dim (aff P).





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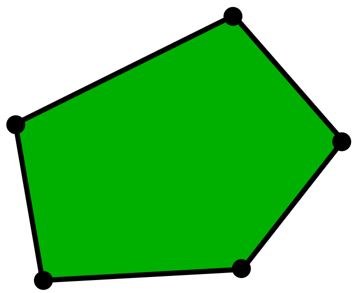
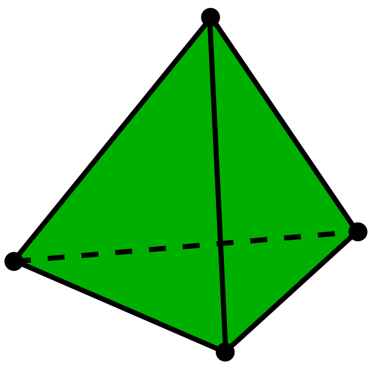
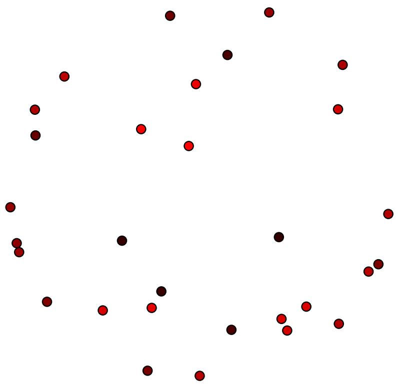
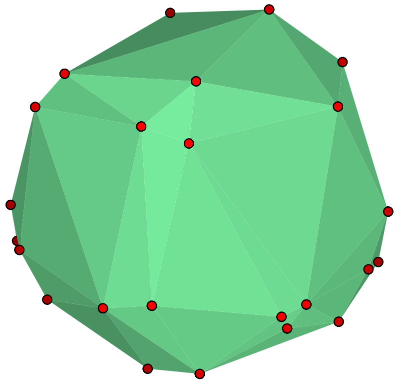


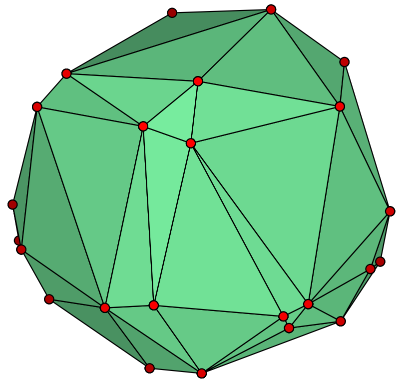


Examples of polytopes

Polytopes can be generated at random . . .





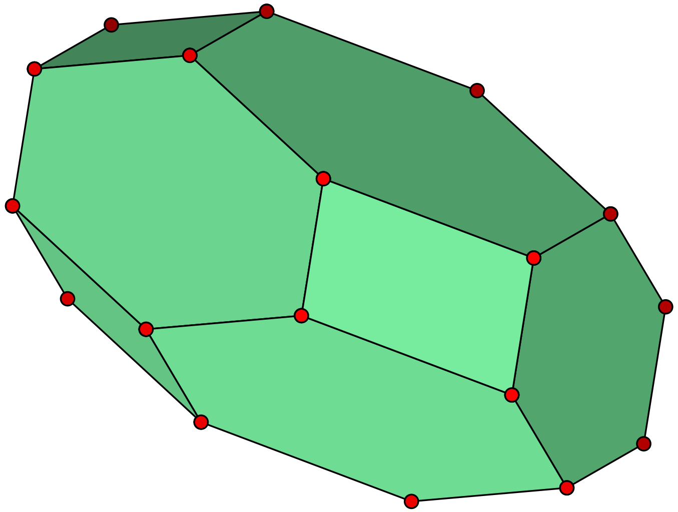


Examples of polytopes

. . . or can be constructed systematically:

$$\Pi_{d-1} \;=\; \mathrm{conv}\{(\sigma(1),\ldots,\sigma(d))\,|\,\sigma\in\mathfrak{S}_d\}\;\subseteq\mathbb{R}^d$$

Permutohedron



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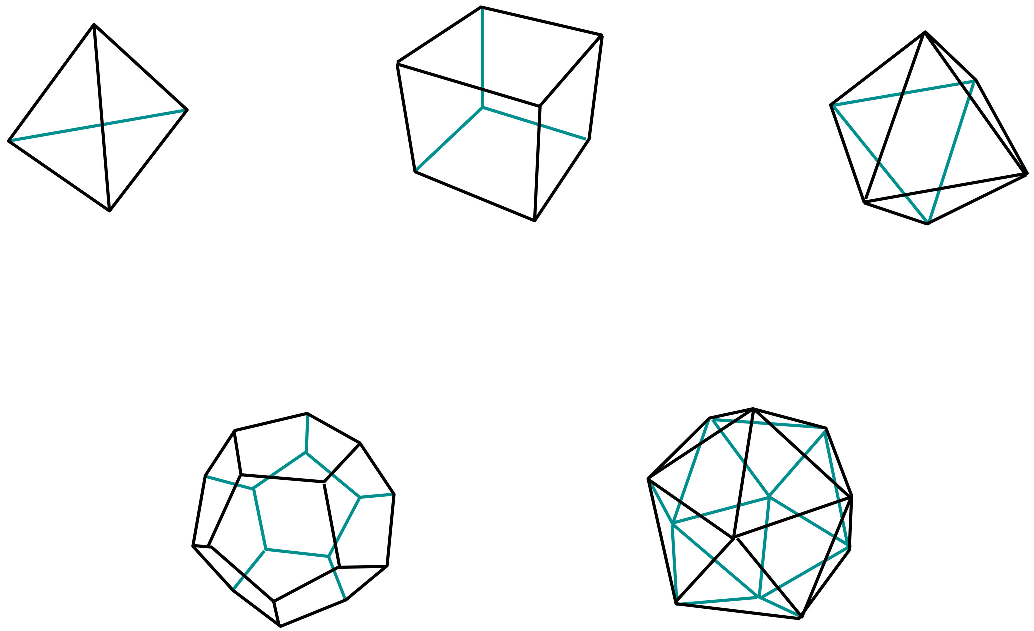
Polytopes: a digression on history

Platonic solids



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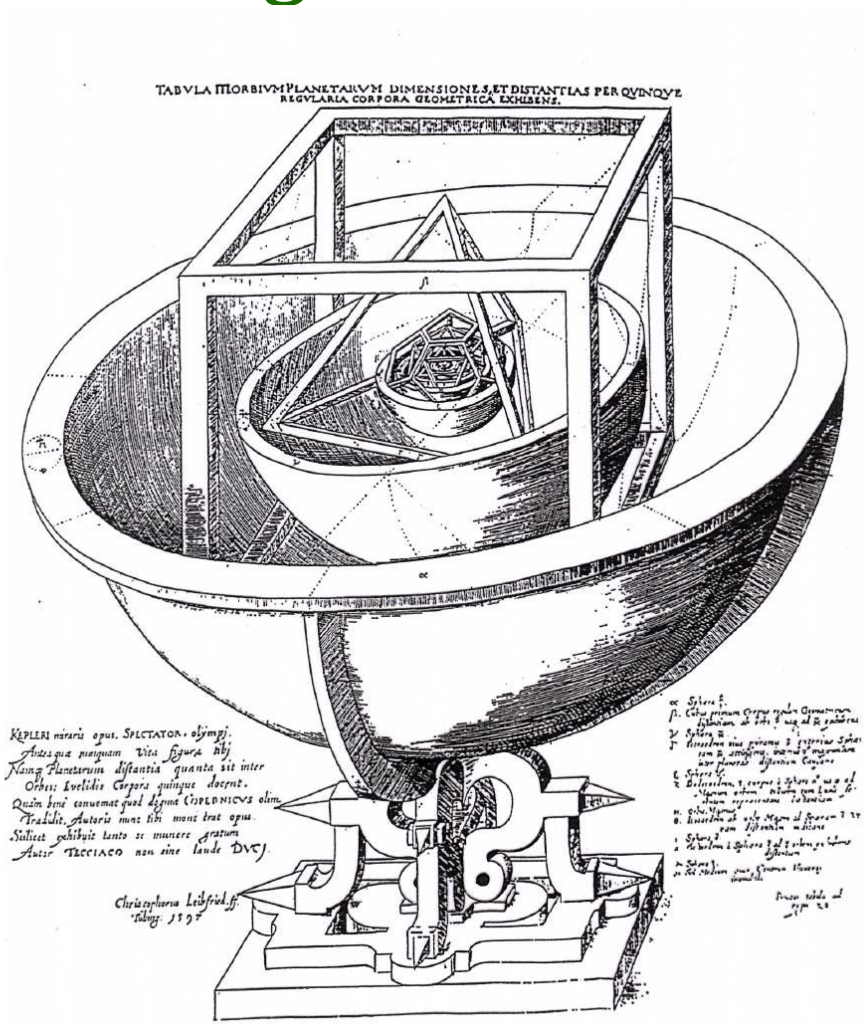
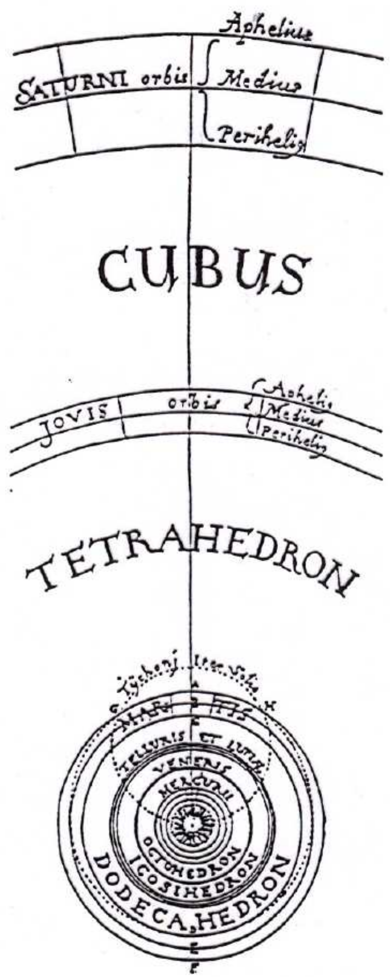
Polytopes: a digression on history



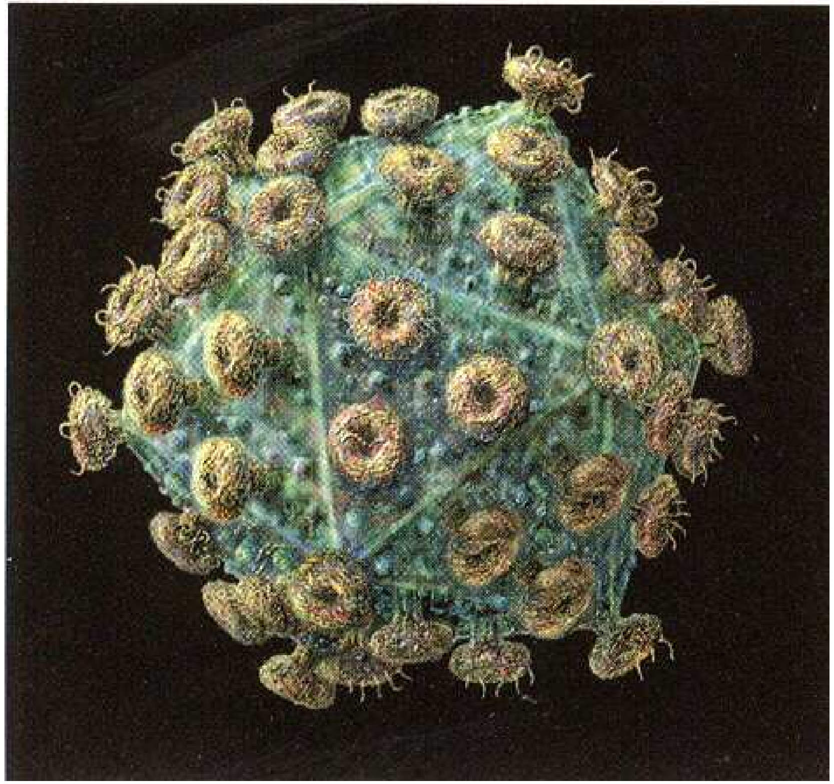
Johannes Kepler (1571–1630)
Mysterium Cosmographicum, Tübingen, 1596.

Polytopes: a digression on history



Johannes Kepler (1571–1630)
Harmonices Mundi, Linz, 1619.

Polytopes: a digression on history



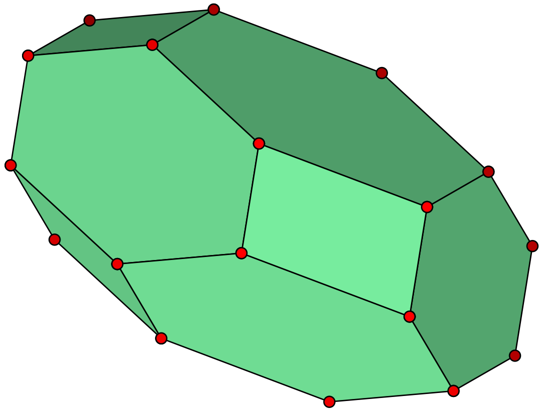
Sabine Yerly, et al.,
Antiviral Therapy 2004; 3: 375-384.

Faces of polytopes

Let P be a polytope in $\mathbb{R}^d$, $c \in (\mathbb{R}^d)^*$, $c_0 \in \mathbb{R}$, such that $cx \le c_0$ for all $x \in P$. Then

$$F = P \cap \{x \in \mathbb{R}^d \,|\, cx = c_0\}$$

is called a face of P.



$\dim F = \dim(\mathrm{aff}\, F)$, and according to dimension we talk about vertices, edges, faces and facets.

Face numbers of polytopes

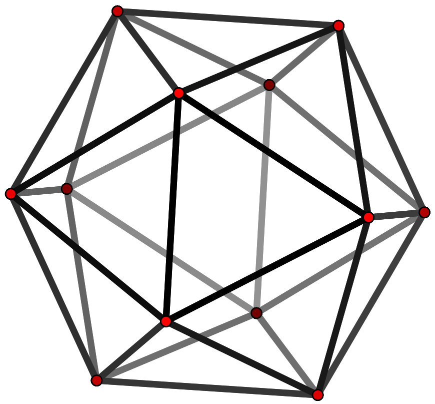
Let P be a d-polytope. Define

f_i:= number of i-dimensional faces of P for $i = -1, \ldots, d$.

$f(P)$:= $(f_{-1}, \ldots, f_d)$ f-vector of P.

Example:

$$f(P) = (1, 12, 30, 20, 1)$$



Question: Which $(d+2)$-tuples of natural numbers occur as f-vectors of polytopes?

Face numbers of polytopes

dim $P = 3$, Euler's formula:

$$f_0 - f_1 + f_2 = 2$$

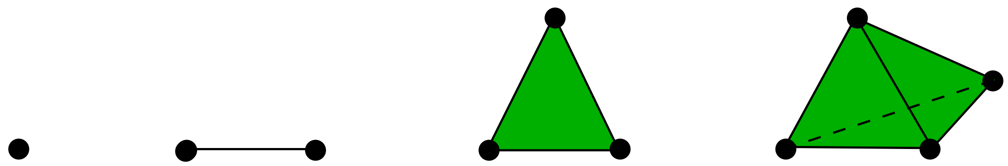
Leonard Euler
(1707-1783)

dim $P = d \geq 3$, Euler-Poincaré formula:

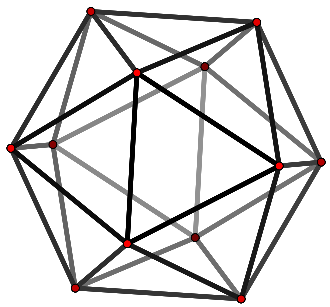
$$\sum_{i=-1}^{d} (-1)^i f_i = 0$$

Simplicial polytopes

The convex hull of $d+1$ affinely independent points in $\mathbb{R}^d$ is called a d-simplex.



A polytope is called simplicial if all its proper faces are simplices.



Question: Which $(d+2)$-tuples of natural numbers occur as f-vectors of simplicial polytopes?

The h-vector of simplicial polytopes

Let P be a simplicial d-polytope. Define

$$h_k := \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{d-k} f_{i-1} \qquad \text{for } k = 0, \dots, d\,.$$

$h(P) := (h_0, \dots, h_d)$ h-vector of P.

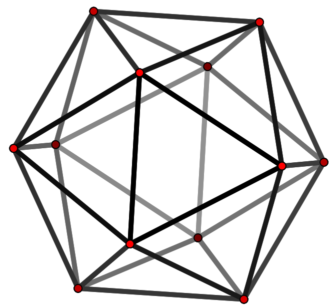
A variation of Pascal's triangle:

$$\begin{array}{ccccccccccc}
 & & & & & f_{-1} & & & & & \\
 & & & & 1 & & f_0 & & & & \\
 & & & 1 & & & & f_1 & & & \\
 & & \cdot^{\cdot^{\cdot}} & & & & & & \ddots & & \\
 & & & & a & & b & & & & \\
 & 1 & & & & b-a & & & & & f_{d-1} \\
\hline
h_0 & & h_1 & & & \dots & & & & h_d &
\end{array}$$

The h-vector of simplicial polytopes

Example: The h-vector of the icosahedron

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 12 & & & \\
 & & 1 & & 11 & & 30 & & \\
 & 1 & & 10 & & 19 & & 20 & \\
\hline
1 & & 9 & & 9 & & 1 & &
\end{array}
$$



Dehn-Sommerville equations:

Let $(h_0, \ldots, h_d)$ be the h-vector of a simplicial polytope, then

$$h_k = h_{d-k} \qquad \text{for } k = 0, \ldots, d\,.$$

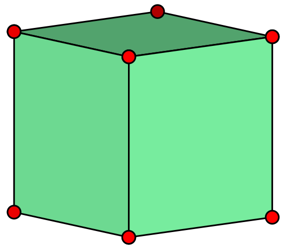
Dual polytopes

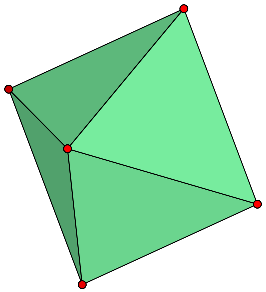
Let P be a d-dimensional polytope, $0 \in \mathrm{int}(P)$. Then

$$P^\diamond \;:=\; \{c \in (\mathbb{R}^d)^* \,|\, cx \le 1 \text{ for any } x \in P\}$$

is the dual polytope of P.
Observe that $f_i(P^\diamond) = f_{d-i-1}(P)$.

Example:





Simplicial versus simple polytopes

P simplicial	$\leftrightarrow$	$P^\diamond$ simple
any facet F in P has d facets	$\leftrightarrow$	any vertex v in $P^\diamond$ lies on d edges
any k such facets intersect in a $(d-k-1)$-face of P	$\leftrightarrow$	any k of these edges span a k-face of $P^\diamond$

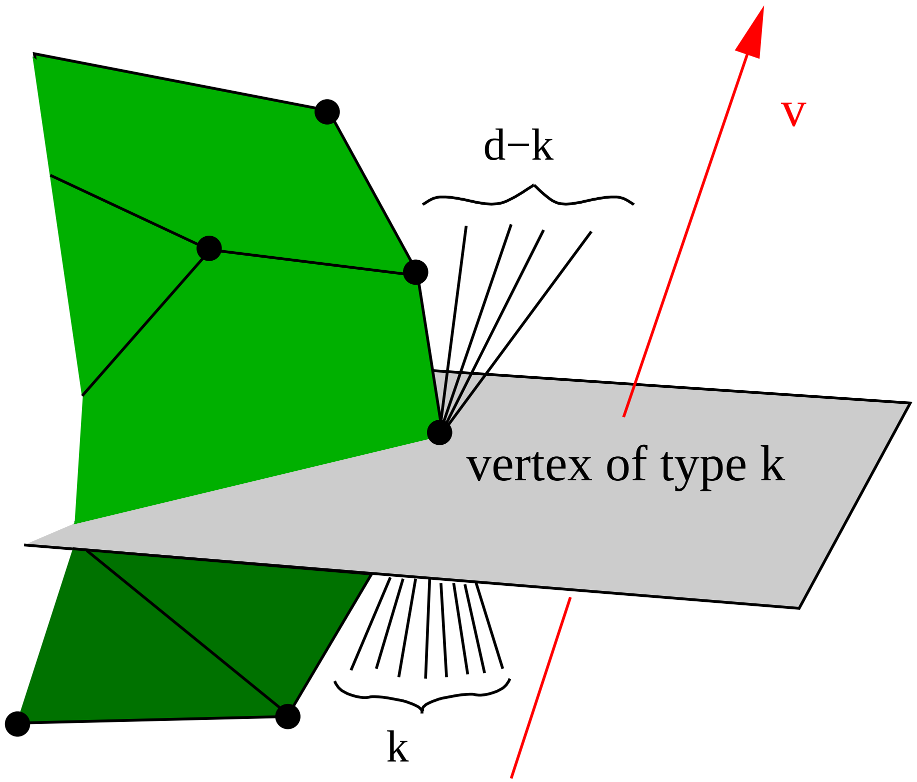
h- and f-polynomials of simplicial polytopes:

$$f(P,t) = \sum_{i=0}^{d} f_{d-i-1}(P)t^i \qquad h(P,t) = \sum_{i=0}^{d} h_{d-i}(P)t^i$$

By definition,

$$f(P,t) = h(P,t+1)\,.$$

Proof of the Dehn-Sommerville equations



$\longrightarrow$ Count faces of $P^\diamond$ along v.

Proof of the Dehn-Sommerville equations

Contribution to

$$f(P,t) = \sum_{i=0}^{d} f_i(P^\diamond)t^i = \sum_{i=0}^{d} h_{d-i}(P)(t+1)^i = h(P,t+1)$$

at a vertex of $P^\diamond$ of type k:

$$t^0 + kt^1 + \binom{k}{2}t^2 + \ldots + \binom{k}{k}t^k = (t+1)^k .$$

$$\begin{aligned} \Longrightarrow \quad h_{d-k} &= \text{number of vertices in } P^\diamond \text{ of type } k \text{ w.r.t. } v \\ &= \text{number of vertices in } P^\diamond \text{ of type } d{-}k \text{ w.r.t. } -v \\ &= h_k \end{aligned}$$

Face numbers of simplicial polytopes

g-Theorem:

[Billera & Lee 1980, Stanley 1980]

$(h_0, \ldots, h_d) \in \mathbb{N}^{d+1}$ is the h-vector of a simplicial d-polytope if and only if

- $h_k = h_{d-k}$ for $k = 0, \ldots, d$
- $g(P) := (h_0, h_1 - h_0, \ldots, h_{\lfloor d/2 \rfloor} - h_{\lfloor d/2 \rfloor - 1})$ is a Macaulay sequence.

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Macaulay sequences

An algebraic characterization:

$(g_0, \ldots, g_{\lfloor d/2 \rfloor}) \in \mathbb{Z}^{\lfloor d/2 \rfloor + 1}$ is a Macaulay sequence if and only if there exists a commutative, associative, graded algebra over $\mathbb{Q}$,

$$A \;=\; \bigoplus_{i \geq 0} A_i \,,$$

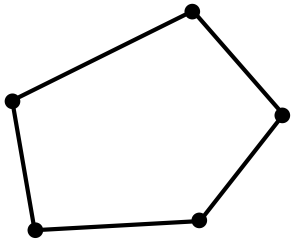
generated in degree 1, and

$$\dim A_i \;=\; g_i \qquad \text{for all } \; i \geq 0 \,.$$

Adding tools: Toric varieties

Let P be a d-polytope with rational vertex coordinates.

X_P the toric variety associated with P:



$$X_P \;=\; (T^d \times P^\diamond)/\sim$$

$$\pi : X_P \longrightarrow P^\diamond$$

$$\pi^{-1}(x) \cong \underbrace{S^1 \times \ldots \times S^1}_{k \text{ times}} \quad \text{for } x \in \operatorname{int} F^\diamond,\ \dim F^\diamond = k\,.$$

Proof of the g-Theorem "$\Longrightarrow$"

[Stanley 1980]

P simplicial, thus realizable with rational vertex coordinates

X_P $\mathbb{Q}$-homology manifold

- $H^*(X_P, \mathbb{Q}) \;=\; H^0(X_P, \mathbb{Q}) \oplus H^2(X_P, \mathbb{Q}) \oplus \ldots \oplus H^{2d}(X_P, \mathbb{Q})$
- dim $H^{2i}(X_P, \mathbb{Q}) = h_i$ for $i = 0, \ldots, d$ *[Danilov 1978]*
- $\omega^{d-2i} \,:\, H^{2i}(X_P, \mathbb{Q}) \xrightarrow{\cong} H^{2(d-i)}(X_P, \mathbb{Q})$ for $i < d/2$,
  $\omega = [Y]$, Y a generic hyperplane section
- witness algebra:

$$H^*(X_P, \mathbb{Q})/\langle \omega \rangle$$

Beyond simplicial polytopes

Can we extend the g-theorem beyond simplicial polytopes?

- In dimension ≥ 4 there exist polytopes that cannot be realized with only rational vertex coordinates.

  [Perles 1965]

- P an arbitrary d-polytope
  $\tilde{h}(P) := (\tilde{h}_0, \ldots, \tilde{h}_d) \in \mathbb{Z}^{d+1}$ toric h-vector
  $\tilde{h}_k = \tilde{h}_{d-k}$ for $k = 0, \ldots, d$ *[Stanley 1987]*

Face numbers of non-simplicial polytopes

P a non-simplicial, but rational d-polytope
X_P the associated toric variety
$IH^*(X_P, \mathbb{Q})$ intersection cohomology of X_P

[Goresky & MacPherson 1980]

- $IH^*(X_P, \mathbb{Q}) \ = \ \bigoplus_{i=0}^{d} IH^{2i}(X_P, \mathbb{Q})$
- dim $IH^{2i}(X_P, \mathbb{Q}) = \tilde{h}_i$ for $i = 0, \ldots, d$ *[MacPherson 1987, Fieseler 1991]*
- $\omega^{d-2i} \ : IH^{2i}(X_P, \mathbb{Q}) \xrightarrow{\cong} IH^{2(d-i)}(X_P, \mathbb{Q})$ for $i < d/2$,
  $\omega = [Y] \in H^2(X_P, \mathbb{Q})$, Y a generic hyperplane section
- no ring structure!

Face numbers of non-rational polytopes

P a non-rational d-polytope,
$IH^*(P, \mathbb{Q})$ intersection cohomology of the normal fan of P

[Barthel, Brasselet, Fieseler, Kaup 2002]

[Bressler, Lunts 2003]

- $IH^*(P, \mathbb{Q}) \;=\; \bigoplus_{i=0}^{d} IH^{2i}(P, \mathbb{Q})$
- $\ell^{d-2i} \;: IH^{2i}(P, \mathbb{Q}) \stackrel{\cong}{\longrightarrow} IH^{2(d-i)}(P, \mathbb{Q})$ for $i < d/2$,
  ℓ a conewise linear strictly convex function on $\Sigma(P)$

  [Karu 2003]
- dim $IH^{2i}(P, \mathbb{Q}) = \tilde{h}_i$ for $i = 0, \ldots, d$ *[Bressler, Lunts 2003]*