The Logic of the Partial λ-Calculus With Equality

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Abstract. We investigate the logical aspects of the partial λ-calculus with equality, exploiting an equivalence between partial λ-theories and partial cartesian closed categories (pcccs) established here. The partial λ-calculus with equality provides a full-blown intuitionistic higher order logic, which in a precise sense turns out to be almost the logic of toposes, the distinctive feature of the latter being unique choice. We give a linguistic proof of the generalization of the fundamental theorem of toposes to pcccs with equality; type theoretically, one thus obtains that the partial λ-calculus with equality encompasses a Martin-Löf-style dependent type theory. This work forms part of the semantical foundations for the higher order algebraic specification language HasCasl.

Introduction

Partial functions play an important role in modern algebraic specification, serving to model both non-termination and irregular termination; specification languages featuring partial functions include RSL [8], SPECTRUM [3], and Casl [2, 15]. The natural generalization of the simply typed λ-calculus to partial functions is the partial λ-calculus [13, 14, 18], which forms the basis for the recently introduced wide-spectrum language HasCasl [23, 25]. HasCasl offers a setting for both specification and implementation of higher order functional programs; moreover, it has served as a background formalism for the development of monad-generic computational logics [22, 24]. A central role in all this is played by the fact that the partial λ-calculus *with equality* induces a full intuitionistic higher order logic, corresponding to HasCasl's internal logic [23]. Here, we investigate the character and expressivity of this logic more closely.

The central tool for this investigation is an equivalence between partial λ-theories with equality and *partial cartesian closed categories (pcccs)* with equality proved here. One associates to each pccc a partial λ-theory, its internal language, and conversely to each partial λ-theory a classifying category; the two constructions are essentially mutually inverse. Thus, one can freely move back and forth between logical and categorical formulations and arguments.

It turns out that in a hierarchy of categorical notions comprising in ascending order of strength locally cartesian closed categories, quasitoposes, and toposes,

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pcccs with equality fit between locally cartesian closed categories and quasitoposes. In terms of logic, locally cartesian closed categories correspond to Martin-Löf style dependent type theory [27], and toposes to intuitionistic type theory with power types [11] (the precise logical counterpart of quasitoposes is, to our knowledge, open). In particular, this means that the partial λ-calculus with equality encodes a dependent type theory; more precisely, one even has a partial version of dependent product types, which categorically relates to a novel notion of locally partial cartesian closed category. Moreover, we show that topos logic is characterized within the partial λ-calculus by the axiom of unique choice; differently put, topos logic can be recovered from the partial λ-calculus with equality by giving up the distinction between functions and functional relations.

Related work includes [13], where a semantics for the partial λ-calculus in left exact pcccs is given, as well as [18], where a classifying category construction for the pure partial λ-calculus is described, using however different categorical notions. A fuller exposition of some of the results presented here can be found in [19].

1 The Partial λ-Calculus

The partial λ-calculus [13, 14, 18] is a typed higher-order formalism that explicitly handles partial functions. It is formally similar to the simply typed λ-calculus, the crucial difference being that function types are thought of as types of partial functions. This is reflected both in the semantics [13] and in the deductive system, which has to keep track of definedness of terms.

We now give a brief definition of the syntax and deduction system of the partial λ-calculus, with one modification in comparison to [13]: there are various types of equations between partial terms, two of the more common being *existential* equations, to be read 'both sides are defined and equal, and *strong* equations, to be read 'one side is defined iff the other is, and then the two sides are equal'. While the presentation in [13] is based on strong equations, we focus mainly on existential equations, since these are slightly better suited for our categorical treatment, and give a correspondingly adapted deduction system; the expressivity of both types of equations is the same [13]. We will, moreover, for the purposes of this paper mostly be interested in theories that possess an equality predicate, which has the effect of transforming a simple λ-calculus into a full-blown higher order logic.

A *partial λ-theory* consists of *axioms* over a *signature*. A signature is given by sets of basic *sort* and *operation* symbols, where the latter (thought of as representing partial maps) consist of their *name* and *profile*, written in the form $f : \bar{s} \rightharpoonup t$. Here, t is a *type* and $\bar{s} = (s_1, \dots, s_n)$ is a *multi-type*, i.e. a list of types (the bar notation is used to indicate lists of items throughout). Types are freely generated from the basic sorts by closing them under the formation of *partial function types* written

$$\bar{s} \multimap\!\!\!\rightarrow t,$$

with $\bar{s}$ and t as above (one cannot resort to currying for multi-argument partial functions [13]). Following [13], we assume application operators $(\bar{s} \multimap t)\bar{s} \rightharpoonup t$ in the signature, so that application does not require extra typing or deduction rules. We use the notation $\bar{s} \multimap \bar{t}$ to denote the multi-type with components $\bar{s} \multimap t_i$ (which is *not* the same as a function type into the product of the t_i).

Given a signature, typed *terms* and *multi-terms*, i.e. lists $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$ of terms, in a context $\Gamma = (\bar{x} : \bar{s}) = (x_1 : s_1, \dots, x_n : s_n)$ of distinct variables x_i with assigned types s_i are formed according to the typing rules

$$\frac{x : s \text{ in } \Gamma}{\Gamma \rhd x : s} \qquad \frac{\Gamma \rhd \bar{\alpha} : \bar{t} \quad f : \bar{t} \rightharpoonup u}{\Gamma \rhd f(\bar{\alpha}) : u} \qquad \frac{\Gamma, \bar{y} : \bar{t} \rhd \alpha : u}{\Gamma \rhd \lambda \bar{y} : \bar{t}.\, \alpha : \bar{t} \multimap u},$$

where the judgement $\Gamma \rhd \bar{\alpha} : \bar{t}$ is read '(multi-)term $\bar{\alpha}$ has (multi-)type $\bar{t}$ in context Γ'; here, typing judgements for multi-terms are just collections of typing judgements for the constituent terms. The higher order application operator is denoted by juxtaposition, while term formation using operators from the signature is written, as above, with brackets. Where convenient, terms will be regarded as singleton multi-terms, similarly for types. For convenience, we regard the empty multi-type and the empty multi-term also as a type and a term denoted by 1 and by $*$, respectively.

An *existential equation* between two terms in context Γ is written $\Gamma \rhd \alpha \overset{e}{=} \beta$ or $\alpha \overset{e}{=} \beta$, to be understood as indicated above. Equations between multi-terms are regarded as sets of equations between terms; the union of such sets is denoted by $\wedge$, and the empty set of equations by $\top$. Equations of the form $\alpha \overset{e}{=} \alpha$ just state that α is defined; they are abbreviated as $\operatorname{def} \alpha$ (and e.g. $\operatorname{def}(\alpha, \beta)$ codes the same set of equations as $\operatorname{def} \alpha \wedge \operatorname{def} \beta$). An *existentially conditioned equation (ece)* [5] in context Γ is a sentence of the form $\Gamma \rhd \operatorname{def} \bar{\alpha} \Rightarrow \psi$, where $\bar{\alpha}$ is a multi-term and ψ is an existential equation in context Γ. The axioms of a partial λ-theory are given as eces.

The deduction system for the partial λ-calculus is shown in Figure 1. Deduction takes place over a fixed context Γ and in a theory with the set $\mathcal{A}$ of axioms. We write $\Gamma \rhd \operatorname{def} \bar{\alpha} \vdash \phi$ if an equation ϕ can be deduced from $\operatorname{def} \bar{\alpha}$ in context Γ by means of these rules; in this case, $\Gamma \rhd \operatorname{def} \bar{\alpha} \Rightarrow \phi$ is a *theorem*. Subderivations are also denoted in the form $\Delta \rhd \operatorname{def} \bar{\alpha} \vdash \phi$, where the context Δ and the assumption $\operatorname{def} \bar{\alpha}$ are to be understood as extending the ambient context and assumptions. Strong equations $\Delta \rhd \alpha \overset{s}{=} \beta$ are used as abbreviations for subderivations $\Delta \rhd \operatorname{def} \alpha \vdash \operatorname{def} \beta$ and $\Delta \rhd \operatorname{def} \beta \vdash \alpha \overset{e}{=} \beta$.

The usual forms of the β- and η-rules can be derived by means of the substitution rule. Rule (ξ) implies that all λ-terms are defined.

A *morphism* between two signatures is a pair of maps for sorts and operators, respectively, that is compatible with operator profiles. A *translation* between partial λ-theories is a signature morphism which transforms axioms into theorems. Partial λ-theories and translations form a category $\mathbf{p\lambda Th}$.

$$
\textbf{(var)}\ \frac{x : s \text{ in } \Gamma}{\operatorname{def} x} \quad
\textbf{(st)}\ \frac{\operatorname{def} f(\bar\alpha)}{\operatorname{def} \bar\alpha} \quad
\textbf{(unit)}\ \frac{x : 1 \text{ in } \Gamma}{x \overset{e}{=} *} \quad
\textbf{(sym)}\ \frac{\alpha \overset{e}{=} \beta}{\beta \overset{e}{=} \alpha} \quad
\textbf{(tr)}\ \frac{\begin{array}{c}\alpha \overset{e}{=} \beta \\ \beta \overset{e}{=} \gamma\end{array}}{\alpha \overset{e}{=} \gamma}
$$

$$
\textbf{(cg)}\ \frac{\begin{array}{c}\bar\alpha \overset{e}{=} \bar\beta : \bar t \\ f : \bar t \rightharpoonup u\end{array}}{f(\bar\alpha) \overset{s}{=} f(\bar\beta)} \quad
\textbf{(ax)}\ \frac{\begin{array}{c}(\bar y : \bar t \rhd \operatorname{def} \bar\alpha \Rightarrow \phi) \in \mathcal{A} \\ \bar y : \bar t \text{ in } \Gamma \\ \operatorname{def} \bar\alpha\end{array}}{\phi} \quad
\textbf{(sub)}\ \frac{\begin{array}{c}\bar y : \bar t \rhd \operatorname{def} \bar\alpha \vdash \phi \\ \operatorname{def}(\bar\beta, \bar\alpha[\bar\beta/\bar y])\end{array}}{\phi[\bar\beta/\bar y]}
$$

$$
(\eta)\ \frac{x : \bar t \mathrel{\circ\!\!\!\rightarrow} u \text{ in } \Gamma}{(\lambda \bar y : \bar t.\, x\, \bar y) \overset{e}{=} x} \quad
(\beta)\ \frac{\bar y : \bar t \text{ in } \Gamma}{(\lambda \bar y : \bar t.\, \alpha)\, \bar y \overset{s}{=} \alpha} \quad
(\xi)\ \frac{\Delta \rhd \alpha \overset{s}{=} \beta}{(\lambda \Delta.\, \alpha) \overset{e}{=} \lambda \Delta.\, \beta}
$$

Fig. 1. Deduction rules for existential equality in context Γ

We introduce an important shorthand notation: for (multi-)terms $\bar\alpha$, $\bar\beta$, we have *conditioned terms* [5, 14]

$$\bar\alpha \upharpoonright \bar\beta := (\lambda \bar x, \bar y.\, \bar x)(\bar\alpha, \bar\beta)$$

denoting $\bar\alpha$ with its domain restricted to the common domain of $\bar\alpha$ and $\bar\beta$. A first use of conditioned terms is λ-abstraction of multi-terms: $\lambda \bar y : \bar t.\, \bar\alpha$ denotes the multi-term with components $\lambda \bar y : \bar t.\, \alpha_i \upharpoonright \bar\alpha$.

The partial λ-calculus automatically comes with a rudimentary logic: we can regard $\Omega = 1 \mathrel{\circ\!\!\!\rightarrow} 1$ as a type of truth values, $s \mathrel{\circ\!\!\!\rightarrow} 1$ as the type of predicates on s, and (partial) terms $\phi : 1$ as formulas. For such ϕ, we will shortly write ϕ in place of $\operatorname{def} \phi$, and we can turn definedness assertions into formulas by observing that for any term α, $\operatorname{def} \alpha$ is, in this notation, equivalent to $(\lambda x.\, *)\, \alpha$. In the way of connectives, however, one generally does not have more than conjunction and truth, expressed e.g. via $p \wedge q = (\lambda x, y : 1.\, *)(p, q)$ and $\top = *$.

The picture changes completely in the presence of an equality predicate:

Definition 1. A partial λ-theory *has equality* if there exists, for each type s, a (defined) closed term $eq_s : ss \mathrel{\circ\!\!\!\rightarrow} 1$ (i.e. a binary predicate on s) such that

$$x, y : s \rhd eq_s\ (x, y) \Rightarrow x \overset{e}{=} y \quad \text{and} \quad x : s \rhd eq_s\ (x, x)$$

(See also [6, 13].) Note that the axioms for eq_s are eces. When there is no danger of confusion, we shall write $\alpha \overset{e}{=} \beta$ in place of $eq\ (\alpha, \beta)$. Equality gives rise to a full-fledged intuitionistic logic, along much the same lines as in [6, 11]: letting p

and q range over (partial) terms of type 1, we can put

$$
\begin{aligned}
p \Rightarrow q &:= ((\lambda.\, p) \stackrel{e}{=} \lambda.\, p \wedge q), \\
\forall \bar{y} : \bar{t}.\, p &:= ((\lambda \bar{y} : \bar{t}.\, p) \stackrel{e}{=} \lambda \bar{y} : \bar{t}.\, \top), \\
\bot &:= \forall a : \Omega.\, a *, \\
\neg p &:= p \Rightarrow \bot, \\
p \vee q &:= \forall a : \Omega.\, ((p \Rightarrow a *) \wedge (q \Rightarrow a *)) \Rightarrow a *, \text{ and} \\
\exists \bar{y} : \bar{t}.\, p &:= \forall a : \Omega.\, (\forall \bar{y} : \bar{t}.\, p \Rightarrow a *) \Rightarrow a *,
\end{aligned}
$$

where we omit unused variables of type 1 from λ-abstractions (note that all right hand sides are partial terms of type 1). The usual deduction rules of intuitionistic higher order logic are obtained as lemmas. The main topic of this paper is the closer investigation of this logic.

2 Partial Cartesian Closed Categories

We now give a brief outline of the categorical setting for the semantics, and indeed the syntax, of the partial λ-calculus.

Given a category whose morphisms are thought of as total functions, partial functions $A \rightharpoonup B$ correspond to *partial morphisms*, i.e. spans (m, f) of the form

$$
\begin{array}{ccc}
\bullet & \xrightarrow{f} & B \\
{\scriptstyle m}\downarrow & & \\
A & & ,
\end{array}
$$

where m is a monomorphism of a restricted class $\mathcal{M}$ representing the domain of definition, taken modulo isomorphism in the obvious sense. The composite of (m, f) and a partial morphism (n, g) from B to C is defined as $(mf^{-1}(n), gf^{*})$, where

$$
\begin{array}{ccc}
\bullet & \xrightarrow{f^{*}} & \bullet \\
{\scriptstyle f^{-1}(n)}\downarrow & & \downarrow{\scriptstyle n} \\
\bullet & \xrightarrow[f]{} & B
\end{array}
$$

is a pullback. In order for this to be possible, we have to require a few closure properties of $\mathcal{M}$:

Definition 2. A class of monomorphisms in a category $\mathbf{C}$ is called a *dominion* [18] if it contains all identities and is closed under composition and pullbacks, i.e. pullbacks of $\mathcal{M}$-morphisms (along arbitrary morphisms) exist and are in $\mathcal{M}$. A *dominional category* is a pair $(\mathbf{C}, \mathcal{M})$, where $\mathcal{M}$ is a dominion on $\mathbf{C}$; an *admissible* subobject is an element of $\mathcal{M}$. A functor between dominional categories is called *dominional* if it preserves admissible subobjects and their pullbacks. An equivalence functor between dominional categories is called a *dominional equivalence* if it preserves and reflects admissible subobjects; it is then automatically a dominional functor.

A dominion $\mathcal{M}$ is closed under intersections. If m is a monomorphism and $mg \in \mathcal{M}$, then $g \in \mathcal{M}$. In particular, $\mathcal{M}$ contains all isomorphisms. For a dominional category $(\mathbf{C}, \mathcal{M})$, the partial morphisms form a category $\mathbf{P}(\mathbf{C}, \mathcal{M})$, which contains $\mathbf{C}$ as a (non-full) subcategory [17, 18].

As usual, we call a category (functor, subcategory) *cartesian* if it has (preserves, is closed under) finite products; the terminal object is denoted by 1. In a cartesian dominional category $(\mathbf{C}, \mathcal{M})$, $\mathcal{M}$ is closed under products (but not under pairing). Cartesian dominional categories are equivalent to first order partial equational theories [20].

Definition 3. A cartesian dominional category $(\mathbf{C}, \mathcal{M})$ *has equality* if $\mathcal{M}$ contains all diagonals $A \to A \times A$.

If $(\mathbf{C}, \mathcal{M})$ has equality, then $\mathbf{C}$ has equalizers (hence is finitely complete, shortly: left exact or *lex*), and $\mathcal{M}$ contains all regular monomorphisms and is closed under pairing.

The semantics of the partial λ-calculus has been given in terms of a class of dominional categories called partial cartesian closed categories (pcccs) [13]. The crucial feature of a pccc is that it admits the interpretation of partial function types as *partial function spaces* $A \rightharpoonup B$, which are defined by the property that partial morphisms from $C \times A$ to B are in bijective correspondence with total morphisms $C \to (A \rightharpoonup B)$. More formally,

Definition 4. A cartesian dominional category $(\mathbf{C}, \mathcal{M})$ is called a *partial cartesian closed category (pccc)* if the composite functor

$$\mathbf{C} \xrightarrow{-\times A} \mathbf{C} \hookrightarrow \mathbf{P}(\mathbf{C}, \mathcal{M})$$

has a right adjoint for each object A in $\mathbf{C}$.

(This definition is weaker than the one given in [13] in that we do not require left exactness.) Every lex pccc is cartesian closed [6].

Partial function spaces $A \rightharpoonup B$ in a pccc come with a co-universal partial *evaluation morphism ev* from $(A \rightharpoonup B) \times A$ to B. Explicitly, every partial morphism f from $C \times A$ to B factors uniquely as $ev \circ (\hat{f} \times A)$ in $\mathbf{P}(\mathbf{C}, \mathcal{M})$ by a total morphism $\hat{f} : C \to (A \rightharpoonup B)$ called its *abstraction*.

For a pccc $(\mathbf{C}, \mathcal{M})$, the embedding $\mathbf{C} \hookrightarrow \mathbf{P}(\mathbf{C}, \mathcal{M})$ is left adjoint, being isomorphic to $\_\times 1$. Spelling this out yields that $\mathcal{M}$-partial morphisms in $(\mathbf{C}, \mathcal{M})$ are representable [1, 18], with the partial morphisms into A represented by $1 \rightharpoonup A$. In particular, $\Omega = 1 \rightharpoonup 1$ classifies $\mathcal{M}$-subobjects. By consequence, *every map in $\mathcal{M}$ is a regular monomorphism.*

Of particular interest is the case that the pccc $(\mathbf{C}, \mathcal{M})$ has equality. In this case, $\mathcal{M} = \mathrm{RegMono}(\mathbf{C})$, so that we can *omit the mention of $\mathcal{M}$*. In fact, pcccs with equality can be succinctly characterized as cartesian closed categories with representable regular partial morphisms in which regular monos are stable under composition; a further characterization will be given in Section 5. In particular, every quasi-topos is a pccc with equality (but not conversely [1]). A typical example of a pccc without equality is the category of cpos and continuous functions with Scott open sets [26] as admissible subobjects.

Definition 5. A cartesian dominional functor between two pcccs is called *partial cartesian closed (pcc)* if it preserves partial function spaces. This defines the category $\mathbf{PCCC}$ of pcccs.

Remark 6. There is a large number of axiomatizations of categories where partial morphisms are directly treated as arrows. Essentially, these axiomatizations characterize full subcategories of $\mathbf{P}(\mathbf{C}, \mathcal{M})$ for some cartesian dominional category $(\mathbf{C}, \mathcal{M})$ or, in the higher order case equivalently, of Kleisli categories arising from representations of partial morphisms, i.e. from the adjunction between $(\mathbf{C}, \mathcal{M})$ and $\mathbf{P}(\mathbf{C}, \mathcal{M})$ for some $(\mathbf{C}, \mathcal{M})$ [4, 6, 17]. In these approaches, categories of the form $\mathbf{P}(\mathbf{C}, \mathcal{M})$ are typically distinguished by a splitting condition for subfunctions of the identity which ensures that domains of partial functions are actually objects. For the purposes of this paper, as well as the (logically posterior) paper on Henkin models [21], it appears to be more convenient to work directly with the underlying dominional categories. In particular, this makes the relation of pcccs with toposes and locally cartesian closed categories more immediate; moreover, certain categorical techniques such as in particular the use of representable functors (which plays a crucial role in [21]) are more directly available.

3 The Internal Language and its Interpretation

We now establish an equivalence between partial λ-theories with equality and pcccs with equality, proceeding as follows: we associate to each pccc $(\mathbf{C}, \mathcal{M})$ an *internal language* $\mathsf{L}(\mathbf{C}, \mathcal{M})$, thus obtaining a functor

$$\mathsf{L} : \mathbf{PCCC} \to \mathbf{p\lambda Th}.$$

In this process, we will introduce an interpretation of $\mathsf{L}(\mathbf{C}, \mathcal{M})$ in $(\mathbf{C}, \mathcal{M})$, for which we prove a soundness theorem. We will then construct classifying categories for partial λ-theories with equality, i.e. free objects w.r.t. the functor L. It will turn out that every pccc with equality is equivalent to the classifying category of its internal language, and that the internal logic of the classifying category of a partial λ-theory is a conservative extension, so that pcccs with equality are essentially the same as partial λ-theories with equality.

To begin, we associate a signature Σ to a pccc $(\mathbf{C}, \mathcal{M})$. The sorts in Σ are the objects of $\mathbf{C}$. An interpretation $[\![\_]\!]$ in $\mathbf{C}$ for types and multi-types is defined recursively in the obvious way using products and partial function spaces. The operators of profile $\bar{s} \to t$ in Σ are the partial morphisms from $[\![\bar{s}]\!]$ to $[\![t]\!]$ in $(\mathbf{C}, \mathcal{M})$, with evaluation morphisms (cf. Section 2) as application operators.

The interpretation $[\![\_]\!]$ is then extended to contexts, terms, multi-terms, and definedness conditions: for a context $\Gamma = (\bar{x} : \bar{s})$, $[\![\Gamma]\!] = [\![\bar{s}]\!]$. Given a term or multi-term $\Gamma \rhd \bar{\alpha} : \bar{t}$, we define a partial morphism denoted

$$[\![\Gamma]\!] \hookleftarrow [\![\Gamma.\ \mathrm{def}\ \bar{\alpha}]\!] \xrightarrow{[\![\Gamma.\,\bar{\alpha}]\!]} [\![\bar{t}]\!]$$

by recursion over the term structure: variables are interpreted as (total) product projections, and operator application as composition of partial morphisms. Multi-terms are modelled by intersecting the domains of the components and tupling the resulting restrictions. Finally, $[\![\Gamma.\,\lambda\bar{y}:\bar{u}.\,\beta]\!]$ is defined as the abstraction of $[\![\Gamma,\bar{y}:\bar{u}.\,\beta]\!]$ (cf. Section 2). We will denote any existing domain-codomain restrictions of $[\![\Gamma.\,\bar{\alpha}]\!]$ to subobjects of $[\![\Gamma]\!]$ and $[\![\bar{t}]\!]$, respectively, by $[\![\Gamma.\,\bar{\alpha}]\!]$ as well.

This interpretation leads to a notion of satisfaction in $\mathbf{C}$:

Definition 7. An ece $\Gamma \rhd \mathrm{def}\,\bar{\alpha} \Rightarrow \beta_1 \stackrel{e}{=} \beta_2$ in Σ *holds* in $(\mathbf{C},\mathcal{M})$ if $[\![\Gamma.\,\mathrm{def}\,\bar{\alpha}]\!]$ is contained in $[\![\Gamma.\,\mathrm{def}(\beta_1,\beta_2)]\!]$ and the restrictions of the $[\![\Gamma.\,\beta_i]\!]$ to $[\![\Gamma.\,\mathrm{def}\,\bar{\alpha}]\!]$ coincide.

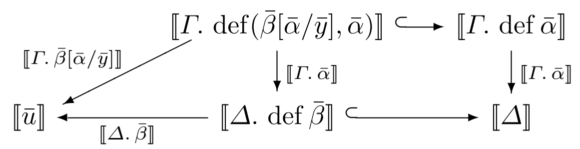
The definition of $\mathsf{L}(\mathbf{C},\mathcal{M})$ is completed by taking the eces that hold in $(\mathbf{C},\mathcal{M})$ as the axioms of $\mathsf{L}(\mathbf{C},\mathcal{M})$. The theory $\mathsf{L}(\mathbf{C},\mathcal{M})$ has equality iff $(\mathbf{C},\mathcal{M})$ does.

The deduction system of Figure 1 is sound for this interpretation:

Theorem 8 (Soundness). *All theorems of* $\mathsf{L}(\mathbf{C},\mathcal{M})$ *hold in* $(\mathbf{C},\mathcal{M})$.

The proof hinges on the following lemma:

Lemma 9 (Substitution). *Let* $\Gamma \rhd \bar{\alpha}:\bar{t}$ *and* $\Delta \rhd \bar{\beta}:\bar{u}$ *be multi-terms in* $\mathsf{L}(\mathbf{C},\mathcal{M})$*, where* $\Delta = (\bar{y}:\bar{t})$*. Then* $[\![\Gamma.\,\bar{\alpha}]\!]$ *has a restriction* $[\![\Gamma.\,\mathrm{def}(\bar{\beta}[\bar{\alpha}/\bar{y}],\bar{\alpha})]\!] \to [\![\Delta.\,\mathrm{def}\,\bar{\beta}]\!]$*. In the arising diagram*



the triangle commutes and the square is a pullback.

4 Classifying Categories

Thanks to the internal logic, the effort required for the construction of a classifying category for a partial λ-theory with equality is essentially no greater than in the first order case as carried out in [20]: in that case, objects of the classifying category are pairs $(\Gamma.\,\phi)$ consisting of a context Γ and a definedness assertion ϕ in that context. This construction can be copied literally for partial λ-theories with equality; the point is that partial function spaces $(\Gamma.\,\phi) \multimap (\Delta.\,\psi)$ may safely be regarded as subobjects of the partial function space $\Gamma \multimap \Delta$, so that no additional objects are required to obtain partial cartesian closedness.

In general, given a partial λ-theory $\mathcal{T}$, we construct a *syntactic category* $\mathsf{Sy}(\mathcal{T})$ as follows: the objects are definedness-assertions-in-context $(\Gamma.\,\phi)$ as indicated above (i.e. $\phi \equiv \mathrm{def}\,\bar{\beta}$ for some multiterm $\bar{\beta}$). Morphisms $(\Gamma.\,\phi) \to (\Delta.\,\psi)$, where $\Gamma = (\bar{x}:\bar{s})$ and $\Delta = (\bar{y}:\bar{t})$, are multi-terms $\Gamma \rhd \bar{\alpha}:\bar{t}$ such that

$$\Gamma \rhd \phi \vdash \psi[\bar{\alpha}/\bar{y}] \wedge \mathrm{def}\,\bar{\alpha},$$

taken modulo equality deducible from ϕ. The identity on $(\Gamma.\,\phi)$ is $\bar{x}$; composition is substitution. A subobject is admissible (regular) in $\mathsf{Sy}(\mathcal{T})$ iff it has a representative of the form

$$\bar{x} : (\Gamma.\,\phi) \hookrightarrow (\Gamma.\,\psi).$$

It is shown as in the first order case [20] that $\mathsf{Sy}(\mathcal{T})$ is a cartesian dominional category, with products given by concatenation of contexts and conjunction of definedness assertions, $1 = (())$, and the pullback of $(\Gamma.\,\phi) \hookrightarrow (\Gamma.\,\psi)$ along a morphism $\bar{\alpha} : (\Delta.\,\chi) \to (\Gamma.\,\psi)$ being $(\Delta.\,\chi \wedge \phi[\bar{\alpha}/\bar{x}])$.

Theorem 10. *If $\mathcal{T}$ has equality, then $\mathsf{Sy}(\mathcal{T})$ is a pccc.*

Proof. Recall the logic defined using equality as described in Section 1. Given objects $(\Gamma.\,\phi)$ and $(\Delta.\,\psi)$ as above, and $\bar{z} : \bar{s} \multimap\!\!\to \bar{t}$ (cf. Section 1), let $dc(\bar{z},\bar{x})$ abbreviate the formula

$$(\mathrm{def}\; z_1\,\bar{x} \Rightarrow \mathrm{def}\; z_2\,\bar{x}) \wedge \ldots \wedge (\mathrm{def}\; z_m\,\bar{x} \Rightarrow \mathrm{def}\; z_1\,\bar{x})$$

ensuring that all components of $\bar{z}$ have the same domain of definition, and write $\bar{z}(\bar{x})$ for $(z_1(\bar{x}),\ldots,z_m(\bar{x}))$. Then the object

$$\Big(\bar{z} : \bar{s} \multimap\!\!\to \bar{t}.\ \forall\bar{x}.\, dc(\bar{z},\bar{x}) \wedge \big(\,\mathrm{def}\;\bar{z}\,\bar{x} \Rightarrow (\phi \wedge \psi[\bar{z}\,\bar{x}/\bar{y}])\big)\Big)$$

is the partial function space $(\Gamma.\,\phi) \multimap\!\!\to (\Delta.\,\psi)$, with $\bar{z}(\bar{x})$ as evaluation map. □

A translation $\sigma : \mathcal{T}_1 \to \mathcal{T}_2$ between partial λ-theories induces a functor

$$\mathsf{Sy}(\sigma) : \mathsf{Sy}(\mathcal{T}_1) \to \mathsf{Sy}(\mathcal{T}_2)$$

which, in the case with equality, preserves the pccc structure since this structure has the syntactical description given above.

We will now show that $\mathsf{Sy}(\mathcal{T})$ is, for $\mathcal{T}$ with equality, free over $\mathcal{T}$ w.r.t. L. The corresponding unit is the translation

$$\eta_{\mathcal{T}} : \mathcal{T} \to \mathsf{L}(\mathsf{Sy}(\mathcal{T}))$$

that maps a sort s to $(x : s)$ and an operator $f : \bar{s} \to t$ to the operator in $\mathsf{L}(\mathsf{Sy}(\mathcal{T}))$ given by the partial morphism $(\bar{x} : \bar{s}) \hookleftarrow (\bar{x} : \bar{s}.\ \mathrm{def}\, f(\bar{x})) \xrightarrow{f(\bar{x})} (x : t)$. By the soundness theorem (for $\mathsf{L}(\mathsf{Sy}(\mathcal{T}))$), *$\eta$ is a conservative extension.*

Given a pccc $(\mathbf{C},\mathcal{M})$, the co-unit

$$E_{(\mathbf{C},\mathcal{M})} : \mathsf{Sy}(\mathsf{L}(\mathbf{C},\mathcal{M})) \to (\mathbf{C},\mathcal{M})$$

of the adjunction maps an object $(\Gamma.\,\phi)$ in $\mathsf{Sy}(\mathsf{L}(\mathbf{C}))$ to $[\![\Gamma.\,\phi]\!]$ and a morphism $\bar{\alpha} : (\Gamma.\,\phi) \to (\Delta.\,\psi)$ to the composite

$$[\![\Gamma.\,\phi]\!] \hookrightarrow [\![\Gamma.\,\psi[\bar{\alpha}/\bar{y}] \wedge \mathrm{def}\;\bar{\alpha}]\!] \xrightarrow{[\![\Gamma.\,\bar{\alpha}]\!]} [\![\Delta.\,\psi]\!]$$

of the inclusion provided by soundness theorem and the restriction of $[\![\Gamma.\,\bar{\alpha}]\!]$ according to the substitution lemma. Using the soundness theorem and the substitution lemma, it is shown that this defines a dominional *equivalence*. In particular, $\mathsf{Sy}(\mathsf{L}(\mathbf{C},\mathcal{M}))$ is a pccc, and $E_{(\mathbf{C},\mathcal{M})}$ is a pcc functor.

This is all we need in order to prove

Theorem 11. *If $\mathcal{T}$ has equality, then $\mathsf{Sy}(\mathcal{T})$ is the free pccc over $\mathcal{T}$ in the sense that any translation $\sigma : \mathcal{T} \to \mathsf{L}(\mathbf{C},\mathcal{M})$, where $(\mathbf{C},\mathcal{M})$ is a pccc, factors essentially uniquely as $\mathsf{L}(\sigma^{\#})\eta_{\mathcal{T}}$, where $\sigma^{\#} : \mathsf{Sy}(\mathcal{T}) \to (\mathbf{C},\mathcal{M})$.*

Here, 'essentially' means that $\sigma^{\#}$ is unique up to a unique natural isomorphism. Thus, $\mathsf{Sy}(\mathcal{T})$ is determined up to equivalence by this property.

Proof. The uniqueness statement is clear. To prove existence, just note that $\sigma^{\#} := E_{(\mathbf{C},\mathcal{M})} \circ \mathsf{Sy}(\sigma)$ has the required properties. ☐

This theorem justifies calling $\mathsf{Sy}(\mathcal{T})$ the *classifying category* of $\mathcal{T}$, denoted from now on by $\mathsf{Cl}(\mathcal{T})$. (For partial λ-theories without equality, $\mathsf{Sy}(\mathcal{T})$ and $\mathsf{Cl}(\mathcal{T})$ will in general be different.) Since $E_{(\mathbf{C},\mathcal{M})}$ is an equivalence, the category of pcccs with equality is essentially (i.e. up to 2-dimensional equivalence) the Kleisli category of the (2-)adjunction $\mathsf{Cl} \dashv \mathsf{L}$. The objects of this category are the partial λ-theories with equality; the morphisms from $\mathcal{T}_1$ to $\mathcal{T}_2$ are the translations from $\mathcal{T}_1$ to $\mathsf{L}(\mathsf{Cl}(\mathcal{T}_2))$. These morphisms are naturally generalized translations: sorts are mapped to 'types', i.e. domains of multi-terms, and symbols are mapped to multi-terms (all partial morphisms in $\mathsf{Sy}(\mathcal{T}_2)$ may be written in the form $(\Gamma.\,\phi) \hookleftarrow (\Gamma.\ \mathrm{def}\ \bar{\alpha}) \xrightarrow{\bar{\alpha}} \bullet$); moreover, two morphisms of this kind are identified if they map all symbols to strongly equal multi-terms. We have established that

pcccs with equality are equivalent to partial λ-theories with equality.

Remark 12. Without equality, the construction of the classifying category becomes more complex, since it is no longer possible to define partial function spaces $(\Gamma.\,\phi) \multimap (\Delta.\,\psi)$ as subspaces of $\Gamma \multimap \Delta$. (The obvious idea of using the Yoneda extension is probably not the right one, for reasons laid out in [19].) In [19], this problem is solved by moving to an extended theory with equality and a dominance [18]; the classifying category of the original theory is then obtained as a subcategory of the classifying category of the extended theory, the latter being constructed as above. This establishes an equivalence between partial λ-theories and pcccs.

5 Unique Choice

We now proceed with the investigation of the higher order logic induced by a partial λ-theory with equality, exploiting the equivalence result proved above. In particular, we make use of the fact that every pccc $\mathbf{C}$ with equality is equivalent to $\mathsf{Cl}(\mathsf{L}(\mathbf{C}))$; in fact, in the following we shall not distinguish between these two categories at all.

It is at first sight slightly puzzling that pcccs with equality are equivalent to intuitionistic HOL, although the latter is more commonly associated with toposes; see e.g. [11], where in fact toposes are constructed from type theories that can be translated into the partial λ-calculus with equality. We stress that pcccs with equality are substantially weaker than even quasitoposes [1], which in turn are way more general than toposes — e.g., there are many non-trivial quasitoposes in topology [28], while the only topos which is at the same time a topological category over **Set** is **Set** itself. It turns out that the crucial point here is unique choice.

For the remainder of this section, let $\mathbf{C}$ be a pccc with equality. It can be shown that a morphism in $\mathbf{C}$ is a monomorphism (epimorphism) iff the obvious internal formula expressing injectivity (surjectivity) holds in $\mathbf{C}$. In particular, given a morphism $f : A \to B$, its factorization through the subobject

$$(b : B.\, \exists a : A.\, f(a) \overset{e}{=} b)$$

of B is an (Epi, Regular Mono)-factorization, i.e. (Epi, Regular Mono) is a factorization structure on $\mathbf{C}$ (thus, $\mathbf{C}$ automatically satisfies condition 19.1.1. in [28]). Hence, all extremal monomorphisms in $\mathbf{C}$ are regular; thus, we obtain a further characterization of pcccs with equality as *cartesian closed categories with representable extremal partial morphisms.* In particular, quasitoposes are precisely the finitely cocomplete pcccs with equality.

For each object A in $\mathbf{C}$, we have a type

$$Sg(A) := (x : A \multimap 1.\, \exists! a : A.\, x\, a)$$

of singleton subsets (predicates) of A. A morphism $sg_A : A \to Sg(A)$ is given by the term $(a : A) \rhd \lambda b : A.\, b \overset{e}{=} a$.

Proposition 13. *Let A be an object in $\mathbf{C}$. Then the following are equivalent:*

(i) sg_A *is an isomorphism.*
(ii) Every partial functional relation with codomain A is a partial function; i.e. in context $R : BA \multimap 1$, with B a further $\mathbf{C}$-object, the formula

$$(\forall x : B, y, z : A.\, R\,(x, y) \wedge R\,(x, z) \Rightarrow y \overset{e}{=} z) \implies (\exists f : B \multimap A.\, \forall x : B, y : A.\, f\, x \overset{e}{=} y \;\Leftrightarrow R\,(x, y))$$

holds.
(iii) Every monomorphism with domain A is extremal.

An object that satisfies the equivalent conditions in the above proposition is called *coarse*, following [16] and [28], where Conditions (i) and (iii) are used. Condition (ii) is often referred to as *unique choice* (although it is usually formulated in terms of total functions). An inverse f of sg_A can be regarded as a partial morphism from $A \multimap 1$ to A. Thus, we can define the *unique description operator* by

$$\iota a : A.\, \phi := f(\lambda a : A.\, \phi)$$

for a formula ϕ in context $a : A$ — i.e. $\iota a : A.\, \phi$ is the unique element of A satisfying ϕ, if such an element indeed exists uniquely, and is otherwise undefined.

An immediate consequence of the above proposition is that we can axiomatize toposes in the partial λ-calculus:

Theorem 14. *The classifying category of a partial λ-theory $\mathcal{T}$ with equality is a topos iff $\mathcal{T}$ implies the unique choice axiom for all types.*

Thus, the initial question of what class of categories intuitionistic HOL really corresponds to may the resolved as follows:

Pcccs with equality 'are' intuitionistic HOL,
toposes are intuitionistic HOL with unique description.

In the construction of the classifying topos given in [11], unique choice is implicit in that morphisms *are* functional relations; in the same way, one can construct a topos from $\mathbf{C}$ (i.e. from a partial λ-theory with equality). An alternative way of obtaining an equivalent topos is the following observation.

Theorem 15. *The subcategory* $\mathbf{Ind}(\mathbf{C})$ *of coarse objects is a bireflective subtopos of* $\mathbf{C}$ *(i.e. the reflective arrows are bimorphisms), with reflective arrows* sg_A*. Moreover,* $\mathbf{Ind}(\mathbf{C})$ *is a topos coreflection of* $\mathbf{C}$ *in the sense that every pcc functor* $\mathbf{E} \to \mathbf{C}$*, with* $\mathbf{E}$ *a topos, factors through* $\mathbf{Ind}(\mathbf{C})$.

(The first clause of this theorem slightly generalizes results of [16, 28].) In particular, $\mathbf{Ind}(\mathbf{C})$ contains Ω and all objects of the form $Sg(A)$, since for these objects, the unique choice function can actually be written as a term. $\mathbf{Ind}(\mathbf{C})$ is equivalent to the topos of functional relations over $\mathbf{C}$ because sg_A becomes an isomorphism when regarded as a functional relation.

An important consequence of Theorem 14 is that results obtained using the interplay of partial λ-theories and pcccs apply also to toposes. This includes in particular the equivalence result for Henkin models of pcccs [21], which in its thus obtained specialized form states that, given a topos $\mathbf{E}$, models (logical morphisms) of $\mathbf{E}$ in toposes are essentially equivalent to Henkin models of $\mathbf{E}$, i.e. lex functors $\mathbf{E} \to \mathbf{Set}$.

Remark 16. A further point regarding the relationship of these results to [11] that requires clarification is the following. In loc. cit., it is claimed (correctly) that the extension of a type theory to the internal language of its classifying topos, constructed as the topos of functional relations, is conservative. The type theory used in loc. cit. can be regarded as a sublanguage of the partial λ-calculus with internal equality; the fact that the latter does not prove unique choice, which however holds in all toposes, appears at first sight to contradict the mentioned conservativity result. However, this is resolved by noting that the type theory of [11] (like most other versions of topos logic including the original Mitchell-Bénabou language [12]) in fact cannot *express* unique choice, since it does not have actual function types. In other words, the logic of pcccs with internal equality differs from topos logic in that it takes functions rather than subsets as the primitive notion and then distinguishes between 'maps' (functional relations) and 'morphisms' (functions).

6 Dependent Types

An important aspect of toposes is that they admit a Martin-Löf style dependent type theory; categorically, this means that every topos $\mathbf{E}$ is *locally cartesian closed* [9, 27], i.e. every slice $\mathbf{E}/A$ is cartesian closed — this is the non-trivial part of the *fundamental theorem of toposes* [10]. It is known that this theorem holds also for quasitoposes [16, 28], and a proof that the statement generalizes to pcccs with equality can be extracted from [28]. This implies that the partial λ-calculus with equality already includes dependent type theory, in particular has dependent product types. We now give a simple linguistic proof of the fundamental theorem; moreover, we present a novel notion of locally partial cartesian closed category.

The intuition behind the correspondence between local cartesian closedness and dependent types is the following. Let $\mathbf{C}$ be locally cartesian closed. A type C depending on a variable $y : B$ is regarded as a *bundle*, i.e. a morphism $g : C \to B$, with $C(y)$ being the fibre of g over y. Dependent sum types are then defined simply by composition: if the type $B = B(x)$ depends on a variable $x : A$, i.e. is a bundle $f : B \to A$, then $\sum y : B(x).\, C(y) = \sum_f g$ is just the composite $fg : C \to A$. Dependent product types, on the other hand, arise by exponentiation in the slice category. The point here is that local cartesian closedness is equivalent to the existence of right adjoints Π_f for all pullback functors $f^* : \mathbf{C}/A \to \mathbf{C}/B$, $f : B \to A$ [7]. Intuitively, for types $B(x)$, $D(x)$ depending on $x : A$, i.e. bundles $f : B \to A$, $h : D \to A$, the fibre over $x : A$ of the function space $f \to h$ in $\mathbf{C}/A$ is the function space $B(x) \to D(x)$. For $g : C \to B$ and $f : B \to A$ as above, the fibre over $x : A$ of $\Pi y : B.\, C(y) = \Pi_f g$ is the subspace of sections of g in the fibre of $f \to fg$; this fibre may be thought of as $B(x) \to \sum y : B(x).\, C(y)$.

The mentioned characterization of local cartesian closedness can be generalized to the partial setting: for an object A in a dominional category $(\mathbf{C}, \mathcal{M})$, the $\mathcal{M}$-carried morphisms form a dominion on $\mathbf{C}/A$, also denoted $\mathcal{M}$. For $f : B \to A$, $h : D \to A$, the morphisms $f \to h$ in $\mathbf{P}(\mathbf{C}/A, \mathcal{M})$ are *lax* commutative triangles, i.e. partial morphisms $k : B \rightharpoonup D$ such that $hk = f$ holds *on the domain of* k.

Theorem and Definition 17. *Let $(\mathbf{C}, \mathcal{M})$ be a lex dominional category. Then $(\mathbf{C}/A, \mathcal{M})$ is a pccc for each object A iff the pullback functor*

$$f^* : \mathbf{C}/A \to \mathbf{C}/B \hookrightarrow \mathbf{P}(\mathbf{C}/B, \mathcal{M})$$

has a right adjoint Π^p_f for each morphism $f : B \to A$. In this case, $(\mathbf{C}, \mathcal{M})$ is called locally partial cartesian closed.

The elements of $\Pi^p_f g$ are *partial* sections of g, i.e. partial functions h such that $gh = id$ on the domain of h. In particular, partial function spaces $A \multimap B$ in $(\mathbf{C}, \mathcal{M})$ may be recovered as $\Pi^p x : A.\, B$.

In this terminology, the fundamental theorem reads as follows.

Theorem 18. *Every pccc with equality is locally partial cartesian closed.*

As mentioned above, a categorical proof can essentially be found in [28]. The equivalence result of Section 4 allows a simple and transparent linguistic proof:

Proof. Let $\mathbf{C}$ be a pccc with equality. The partial function space $f \multimap\!\!\rightarrow g$ for two objects $f : B \to A$, $g : C \to A$ of $\mathbf{C}/A$, expressed in $\mathsf{Cl}(\mathsf{L}(\mathbf{C}))$, is

$$(x : A, y : B \multimap\!\!\rightarrow C.\ \ \forall z : B.\ \mathrm{def}\, y\, z \Rightarrow (f(z) \stackrel{e}{=} x \wedge g(y\, z) \stackrel{e}{=} x)) \qquad \square$$

Conclusions and Future Work

We have identified the logic of the partial λ-calculus with equality as the internal logic of partial cartesian closed categories (pcccs) with equality. Building on this result, we have clarified the relationship of this logic with various other higher order logics that have categorical counterparts. In particular, a partial λ-theory with equality is the internal language of a topos iff it satisfies the unique choice axiom, and the partial λ-calculus with equality encodes a dependent type theory with partial dependent products — a known generalization of the fundamental theorem of toposes to pcccs with equality [16, 28], for which we give a transparent linguistic proof. An open problem that remains is to find, somewhere between topos logic and the partial λ-calculus, the precise internal logic of quasitoposes.

This work forms part of the semantical foundations of HasCasl. The equivalence of pcccs and partial λ-theories is needed to prove the equivalence between the semantics of the partial λ-calculus in pcccs on the one hand and a Henkin-style set-theoretic model theory on the other hand [21]. The relevance of unique choice, universally or for certain types, has become apparent e.g. in [22]. The implications of the fact that dependent types are encodable in the partial λ-calculus w.r.t. the specification methodology of HasCasl are under investigation.

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