The HasCasl Prologue: Categorical Syntax and Semantics of the Partial λ-Calculus

Lutz Schröder [1]

BISS, Dept. of Comput. Sci., Universität Bremen, P. O. Box 330 440, 28334 Bremen, Germany

Abstract

We develop the semantic foundations of the specification language HasCasl, which combines algebraic specification and functional programming on the basis of Moggi's partial λ-calculus. Generalizing Lambek's classical equivalence between the simply typed λ-calculus and cartesian closed categories, we establish an equivalence between partial cartesian closed categories (pccc's) and partial λ-theories. Building on these results, we define (set-theoretic) notions of intensional Henkin model and syntactic λ-algebra for Moggi's partial λ-calculus. These models are shown to be equivalent to the originally described categorical models in pccc's via the global element construction. The semantics of HasCasl is defined in terms of syntactic λ-algebras. Correlations between logics and classes of categories facilitate reasoning both on the logical and on the categorical side; as an application, we pinpoint unique choice as the distinctive feature of topos logic (in comparison to intuitionistic higher-order logic of partial functions, which by our results is the logic of pccc's with equality). Finally, we give some applications of the model-theoretic equivalence result to the semantics of HasCasl and its relation to first-order Casl.

Key words: Algebraic specification; categorical logic; partial λ-calculus; Casl.

Introduction

The rigorous development of functional programs calls for a wide-spectrum formalism that supports property-oriented specification, design, and rapid prototyping in an executable sublanguage. Such a framework is provided by the recently developed specification language HasCasl [32,33,36], which extends the standard algebraic specification language Casl [3,23] by a higher-order

[1] Research supported by the DFG project HasCASL (KR 1191/7-1)

logic with type class polymorphism, as well as HOLCF style general recursion [25]. (As a CASL extension, HASCASL has been approved by IFIP WG 1.3 [11].) The executable subset of HASCASL closely corresponds to a large fragment of Haskell. HASCASL has been used as the background formalism for a generic computational logic following the paradigm of side-effect encapsulation via monads [35,34].

Here, we develop the semantic foundations of HASCASL, which is based on the partial λ-calculus [21,22,28]. Since it is one of the objectives of HASCASL to maintain a close relationship with the established language CASL, which comes with a set-theoretic semantics, it is desirable to equip the partial λ-calculus with a set-theoretic semantics. In such a semantics, models would typically be some sort of λ-*algebras* or *Henkin models* in the spirit of Henkin's semantics for classical higher-order logic [16]. By contrast, the semantics for the partial λ-calculus originally given in [21] is defined in terms of models in *partial cartesian closed categories* (pccc's). Here, we show that the two approaches can be reconciled, i.e. that suitably defined Henkin models of partial λ-theories and pccc models are equivalent; this generalizes a corresponding result for the total λ-calculus [4] (similarly, it is shown in [18] that set-theoretic Bruce-Meyer-Mitchell models of the second order λ-calculus are equivalent to Seely's hyperdoctrine models).

For the proof of this result, as well as for the study of the partial λ-calculus in general, it is useful develop a partial version of Lambek's classical results on the total λ-calculus stating that λ-theories are equivalent to cartesian closed categories (ccc's) via a classifying category/internal language correspondence [19]. The categorical basis for partiality are *dominions* in the sense of [28], i.e. classes of admissible domains for partial morphisms. We prove several equivalence results for logics with partiality. In particular, first-order partial equational theories are equivalent to cartesian categories (i.e. categories with finite products) equipped with a dominion, and partial λ-theories are equivalent to pccc's. Such equivalences can be exploited for technical purposes on both sides; i.e. one can reason about categories at a logical level, and about theories at a categorical level. A crucial aspect of the partial λ-calculus is that, in the presence of an internal equality predicate, which categorically corresponds to the requirement that the dominion contains all regular monomorphisms, it allows the definition of a full intuitionistic higher-order logic. As an example of the interplay between logic and category theory, we show that topos logic is distinguished from this logic by the *unique choice* axiom.

Related work includes [8], where a representation of partial conditional equational theories as left exact categories is constructed along partly similar lines as here. In [12], a term construction of classifying *g-monoidal* categories for partial signatures is given. A construction of a classifying *p-category with p-exponentials* for the partial λ-calculus is outlined in [28].

The material is organized as follows. First order partial equational logic is introduced in Section 1, and dominional categories in Section 2. Cartesian dominional categories are shown to be equivalent to partial equational theories in Section 3. In Section 4, internal equality is discussed. In Sections 5 and 6, the partial λ-calculus and its categorical counterpart are introduced. The main results of this work are presented in Sections 7–9: in Section 7, equivalence results are proved for partial λ-theories and pre-pccc's, which generalize pccc's, as well as for partial λ-theories and pccc's. The latter result involves the construction of a pccc completion for pre-pccc's. In Section 8, we introduce Henkin models and prove their equivalence with pccc models. In Section 9, the relationship between the partial λ-calculus with equality and topos logic is discussed. Finally, applications of the equivalence result for Henkin models relevant for the semantics of HasCasl are presented in Section 10. Preliminary versions of results presented here have appeared in [29–31].

1 Partial Equational Logic

We begin by describing a first order partial equational logic with *existentially conditioned equations (ece's)* [6]. We will then show that partial equational theories are equivalent to so-called cartesian dominional categories; this will be established by constructing a classifying category for each partial equational theory and, conversely, an internal language for each cartesian dominional category. None of these results will be regarded as being overly novel or surprising; however, their precise formulation, including the syntax and deduction system of the logic, not only paves the ground for the treatment of partial higher order logic, but also constitutes necessary background material for the model-theoretic equivalence results to be proved in Section 8.

A *(first order) signature* Σ is a pair (S, Ω) consisting of a set S of sorts and a set Ω of *operators* f with given *profiles* (or arities) written $f : \bar{s} \rightharpoonup t$, where t is a sort and $\bar{s} = (s_1, \dots, s_n)$ is a list of sorts, also called a *multi-sort* (we will generally indicate lists by a bar notation, with indexed components implicitly assumed). A context Γ is a list $\Gamma = (\bar{x} : \bar{s}) = (x_1 : s_1, \dots, x_n : s_n)$ of sort assignments for variables. In a context, one can form terms and *multi-terms*, i.e. lists $\bar{\alpha}$ of terms α_i, according to the rules

$$\textbf{(var)}\ \frac{x : s \text{ in } \Gamma}{\Gamma \rhd x : s} \qquad\qquad \textbf{(op)}\ \frac{\Gamma \rhd \bar{\alpha} : \bar{s} \quad f : \bar{s} \rightharpoonup t}{\Gamma \rhd f(\bar{\alpha}) : t}\ .$$

Here, the judgement $\Gamma \rhd \alpha : t$ reads 'term α has sort t in context Γ'; correspondingly, multi-sorts are assigned to multi-terms. The sort t may occasionally be omitted from the notation. The empty multi-term $()$, also denoted $*$, doubles as a term of 'type' (), also denoted 1.

Remark 1 In order to obtain the categorical equivalence results proved below, it is not necessary to introduce a product type constructor, which we therefore, in accordance with [21], omit in the interest of syntactic economy (e.g., this makes structural induction easier, and the deduction system becomes smaller). By the said results, product types can, however, be conservatively added to the language.

A *morphism* between signatures is a pair of maps between the corresponding sets of sorts and operators, respectively, that is compatible with operator profiles; the action of a morphism σ on terms, equations etc. is also denoted by σ. This notion of morphism will be complemented by a notion of derived signature morphism induced by the categorical equivalence proved below.

A *partial equational theory* $\mathcal{T} = (\Sigma, \mathcal{A})$ is a signature Σ together with a set $\mathcal{A}$ of *axioms* that take the form of existentially conditioned equations: an (existential) *equation* in context Γ, written $\alpha_1 \stackrel{e}{=} \alpha_2$, is read '$\alpha_1$ and α_2 are defined and equal', where the α_i are terms of the same sort in context Γ (by contrast, a *strong equation* $\alpha_1 \stackrel{s}{=} \alpha_2$ is read 'α_1 is defined iff α_2 is defined, and in this case, the two are equal'). The equation $\alpha \stackrel{e}{=} \alpha$ is abbreviated as $\operatorname{def} \alpha$; this corresponds to the existence predicate of [39]. Sentences of this form are called *definedness assertions*. Equations between multi-terms of the same length are to be understood as conjunctive sets of equations. In particular, definedness assertions $\operatorname{def} \bar{\alpha}$ for multi-terms are just conjunctions of definedness assertions $\operatorname{def} \alpha_i$. We will use the conjunction symbol $\wedge$ to denote the union of such sets, and $\top$ will denote the empty set of assertions. Sets of definedness assertions will be denoted by $\Phi, \Psi, \dots$, while $\phi, \psi, \dots$ denote equations. Notations such as $\Phi[\beta/x]$ indicate substitution in all assertions of the set. An *existentially conditioned equation (ece)* in context Γ is a sentence of the form $\Gamma \rhd \Phi \Rightarrow \psi$, where Φ is a set of definedness assertions and ψ is an equation in context Γ.

Figure 1 shows a deduction system for existential equality in a theory $(\Sigma, \mathcal{A})$ and a fixed context Γ. Rule (sub) uses subderivations with local assumptions, marked by square brackets, in a locally enlarged context; i.e. its first premise reads 'ψ is deducible in the context extended by $\bar{y} : \bar{t}$ under the additional assumptions Φ'. We write $\Gamma \rhd \Phi \vdash \Psi$ or $\Gamma \rhd \Phi \vdash \psi$ if a set Ψ of definedness assertions or an equation ψ, respectively, can be deduced from a set Φ of definedness assertions in context Γ. An ece $\Gamma \rhd \Phi \Rightarrow \psi$ is a *theorem* if $\Gamma \rhd \Phi \vdash \psi$. (We are not covering actual conditional equations here; for this reason, we have restricted also the notion of entailment to definedness conditions as premises.) A *translation* between theories is a signature morphism that transforms axioms into theorems.

$$
\textbf{(var)}\ \frac{x : s \text{ in } \Gamma}{\operatorname{def} x}
\quad
\textbf{(sym)}\ \frac{\alpha \stackrel{e}{=} \beta}{\beta \stackrel{e}{=} \alpha}
\quad
\textbf{(tr)}\ \frac{\begin{array}{c}\alpha \stackrel{e}{=} \beta \\ \beta \stackrel{e}{=} \gamma\end{array}}{\alpha \stackrel{e}{=} \gamma}
\quad
\textbf{(str)}\ \frac{\operatorname{def} f(\bar{\alpha})}{\operatorname{def} \bar{\alpha}}
\quad
(*)\ \frac{x : 1 \text{ in } \Gamma}{x \stackrel{e}{=} *}
$$

$$
\textbf{(cg)}\ \frac{\begin{array}{c}\bar{\alpha} \stackrel{e}{=} \bar{\beta} \\ \operatorname{def} f(\bar{\alpha})\end{array}}{f(\bar{\alpha}) \stackrel{e}{=} f(\bar{\beta})}
\quad
\textbf{(ax)}\ \frac{\begin{array}{c}(\Delta \rhd \Phi \Rightarrow \psi) \in \mathcal{A} \\ \Gamma \text{ contains } \Delta \\ \Phi\end{array}}{\psi}
\quad
\textbf{(sub)}\ \frac{\begin{array}{ccc}[\bar{y} : \bar{t}; \Phi] & & \\ \vdots & & \\ \psi & \Phi[\bar{\alpha}/\bar{y}] & \operatorname{def} \bar{\alpha}\end{array}}{\psi[\bar{\alpha}/\bar{y}]}
$$

Fig. 1. Deduction rules for partial equational logic

2 Dominions

We briefly review some standard notions relating to the concept of partial maps in a categories. An *($\mathcal{M}$-)partial morphism* from A to B in a category $\mathbf{C}$ is a span

$$
\begin{array}{ccc}
\bullet & \xrightarrow{f} & B \\
{\scriptstyle m}\downarrow & & \\
A & & ,
\end{array}
$$

written $(m, f) : A \rightharpoonup B$, where m is a monomorphism belonging to a class $\mathcal{M}$ of *admissible* subobjects. Two partial morphisms (m_1, f_1) and (m_2, f_2) are regarded as equal if there exists an isomorphism h such that $f_1 h = f_2$ and $m_1 h = m_2$. In order to obtain a category $\mathbf{P}(\mathbf{C}, \mathcal{M})$ of $\mathcal{M}$-partial morphisms containing $\mathbf{C}$ as a subcategory of *total* morphisms, we need to require that $\mathcal{M}$ contains all identities and is closed under composition and pullbacks, the latter in the sense that pullbacks, or inverse images, of $\mathcal{M}$-morphisms along arbitrary morphisms exist and are in $\mathcal{M}$. Following [28], we call such a class $\mathcal{M}$ a *dominion*, and the pair $(\mathbf{C}, \mathcal{M})$ a *dominional category*. The composite of (m, f) and a partial morphism $(n, g) : B \rightharpoonup C$ is $(m \circ f^* n), g \circ \bar{f})$, where

$$
\begin{array}{ccc}
\bullet & \xrightarrow{\bar{f}} & \bullet \\
{\scriptstyle f^* n}\downarrow & & \downarrow{\scriptstyle n} \\
\bullet & \xrightarrow[f]{} & B
\end{array}
$$

is a pullback.

It is easy to see that a dominion $\mathcal{M}$ is also *closed under intersections* (i.e. intersections of $\mathcal{M}$-morphisms exist and are in $\mathcal{M}$) and enjoys a *left cancel-*

lation property: if m is a monomorphism and $mg \in \mathcal{M}$, then $g \in \mathcal{M}$. This implies that $\mathcal{M}$ contains all isomorphisms. (Recall that the *intersection* of two subobjects m, n of A is the subobject of A obtained by pulling back m along n.)

Example 2 In any category with pullbacks, the class of monomorphisms is a dominion, as is the class of regular monomorphisms, provided that the latter is closed under composition. (Recall that a monomorphism is called *regular* if it is the equalizer of some pair of morphisms.) In the category of topological spaces, the classes of open, closed, and clopen embeddings, respectively, are dominions. The class of upclosed sets and the class of downclosed sets are dominions on the category of partial orders. Similary, the Scott open sets [38] form a dominion on the category of complete partial orders (cpo's).

Definition 3 A *dominional functor* between dominional categories $(\mathbf{C}_1, \mathcal{M}_1)$ and $(\mathbf{C}_2, \mathcal{M}_2)$ is a functor $F : \mathbf{C}_1 \to \mathbf{C}_2$ that preserves admissible subobjects and their pullbacks (here and further below, preservation properties are up to isomorphism rather than on-the-nose). Such an F is called *dominionally full* or a *full dominional functor* if F is full and each $\mathcal{M}_2$-subobject of an object of the form FA has a representative of the form Fm, $m \in \mathcal{M}_1$. A dominionally full equivalence functor is called a *dominional equivalence.* A dominional category $(\mathbf{A}, \mathcal{N})$ is called a *(full) dominional subcategory* of $(\mathbf{C}, \mathcal{M})$ if $\mathbf{A}$ is a (full) subcategory of $\mathbf{C}$ and the inclusion $\mathbf{A} \hookrightarrow \mathbf{C}$ is a (full) dominional functor.

A dominional functor $F : (\mathbf{C}, \mathcal{M}) \to (\mathbf{A}, \mathcal{N})$ induces a functor $\mathbf{P}(\mathbf{C}, \mathcal{M}) \to \mathbf{P}(\mathbf{A}, \mathcal{N})$, which we denote by $\mathbf{P}(F)$.

Lemma 4 *A dominional functor F is dominionally full iff $\mathbf{P}(F)$ is full.* □

Proposition 5 *If $F : (\mathbf{C}, \mathcal{M}) \to (\mathbf{A}, \mathcal{N})$ is a dominional equivalence, and F and $G : \mathbf{A} \to \mathbf{C}$ form an equivalence of categories, then $G : (\mathbf{A}, \mathcal{N}) \to (\mathbf{C}, \mathcal{M})$ is a dominional equivalence.* □

Remark 6 It is shown in [9] that full subcategories of categories of the form $\mathbf{P}(\mathbf{C}, \mathcal{M})$ are characterized as *restriction categories*, in which to each (partial) morphism $f : A \rightharpoonup B$, an idempotent $\bar{f} : A \rightharpoonup A$ is assigned representing the domain of definition of f. A restriction structure on $\mathbf{P}(\mathbf{C}, \mathcal{M})$ is defined by assigning (m, m) to (m, f). Restriction categories arising in this way from categories $\mathbf{P}(\mathbf{C}, \mathcal{M})$ are characterized as those where all restriction idempotents split. These results are in the context of earlier work including [7,13,17,26], which is however based on the notion of partial product, i.e. the monoidal structure on $\mathbf{P}(\mathbf{C}, \mathcal{M})$ arising from a cartesian product on $\mathbf{C}$. (E.g. a *p-category* [26,28] is a category $\mathbf{C}$, equipped with a bifunctor $\times : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ and transformations $\Delta_X : X \to X \times X$, $p_{X,Y} : X \times Y \to X$, and $q_{Y,X} : Y \times X \to X$, natural in X but not necessarily in Y, such that the equations

$$\begin{aligned} p_{X,X}\Delta_X = id_X = q_{X,X}\Delta_X \qquad & (p_{X,Y} \times q_{X,Y})\Delta_{X\times Y} = id_{X\times Y} \\ p_{X,Y}(id_X \times p_{Y,Z}) = p_{X,Y\times Z} \qquad & p_{X,Z}(id_X \times q_{Y,Z}) = p_{X,Y\times Z} \\ q_{X,Y}(p_{X,Y} \times id_Z) = q_{X\times Y,Z} \qquad & q_{X,Z}(q_{X,Y} \times id_Z) = q_{X\times Y,Z} \end{aligned}$$

hold and such that the associativity and commutativity morphisms $\alpha_{X,Y,Z} : X \times (Y \times Z) \to (X \times Y) \times Z$ and $\tau_{X,Y} : X \times Y \to Y \times X$ defined by

$$\begin{aligned} \alpha_{X,Y,Z} &= ((id_X \times p_{Y,Z}) \times (q_{Y,Z}q_{X,Y\times Z}))\Delta_{X\times(Y\times Z)} \\ \tau_{X,Y} &= (q_{X,Y} \times p_{X,Y})\Delta_{X\times Y} \end{aligned}$$

are natural in all variables.)

One can thus move freely between the constructive representation of partial maps in dominional categories and the axiomatic representation in restriction categories and related structures. For the purposes of this work, we found it more convenient to use dominional categories, in particular since this makes for a clearer formulation of the equivalence result for Henkin models.

Dominions can be constructed from suitable classes of representatives:

Definition 7 A *predominion* is a class $\mathcal{M}_0$ of monomorphisms, containing all identities and closed under composition, such that there *exist*, in $\mathcal{M}_0$, pullbacks of $\mathcal{M}_0$-morphisms along arbitrary morphisms.

Lemma 8 *The closure* $\mathcal{M} = \mathcal{M}_0 \circ \mathrm{Iso}$ *of a predominion* $\mathcal{M}_0$ *under right composition* $\_ \circ h$ *with isomorphisms* h *is a dominion.* □

As usual, we call a category (functor, subcategory) *cartesian* if it has (preserves, is closed under) finite products. Throughout, we use the symbol $\times$ to denote cartesian products in the category of *total* morphisms. The terminal object is denoted by 1; the factorizing morphism for morphisms $f_i : B \to A_i$, $i = 1, \dots, n$, called their *tupling*, will be written $\langle f_i \rangle = \langle f_1, \dots, f_n \rangle : B \to A_1 \times \dots \times A_n$. In a cartesian dominional category $(\mathbf{C}, \mathcal{M})$, $\mathcal{M}$ is closed under products (but not in general under pairing).

Cartesian dominional categories allow the interpretation of partial equational theories:

Definition 9 A *model* M of a signature Σ in a cartesian dominional category $(\mathbf{C}, \mathcal{M})$ consists of an assigment of a $\mathbf{C}$-object $M[\![s]\!]$ to each sort s, with $M[\![\bar{s}]\!]$ defined as $M[\![s_1]\!] \times \dots \times M[\![s_n]\!]$ for multi-sorts $\bar{s}$, and a partial map $M[\![f]\!] : M[\![\bar{s}]\!] \rightharpoonup M[\![t]\!]$ to each operator $f : \bar{s} \rightharpoonup t$.

Given a model M, we put $M[\![\Gamma]\!] = M[\![\bar{s}]\!]$ for each context $\Gamma = (\bar{x} : \bar{s})$. We can then interpret terms in context $\Gamma \rhd \alpha : t$ as partial morphisms

$$\begin{array}{ccc} M[\![\Gamma.\,\mathrm{def}\,\alpha]\!] & \xrightarrow{M[\![\Gamma.\,\alpha]\!]} & M[\![t]\!] \\ \big\downarrow & & \\ M[\![\Gamma]\!] & & \end{array} \quad ,$$

correspondingly for multi-terms, by recursion over the term structure: variables are interpreted as projections (in particular, $M[\![\Gamma.\,\mathrm{def}\,x]\!] = M[\![\Gamma]\!]$) and operator application as composition of partial morphisms. The interpretation of a multi-term $\bar{\alpha}$ has as its domain $M[\![\Gamma.\,\mathrm{def}\,\bar{\alpha}]\!]$ the intersection of the $M[\![\Gamma.\,\mathrm{def}\,\alpha_i]\!]$, and its action $M[\![\Gamma.\,\bar{\alpha}]\!]$ is the corresponding restriction of the tupling $\langle M[\![\Gamma.\,\alpha_i]\!]\rangle$. In order to avoid cluttering the notation, we will denote domain-codomain restrictions of $M[\![\Gamma.\,\alpha]\!]$ to admissible subobjects of $M[\![\Gamma.\,\mathrm{def}\,\alpha]\!]$ and $M[\![t]\!]$, respectively, by $M[\![\Gamma.\,\alpha]\!]$ as well; moreover, we shall occasionally use $M[\![\Gamma.\,\alpha]\!]$ to refer to the entire partial morphism shown above.

This interpretation leads to a notion of satisfaction in $\mathbf{C}$:

Definition 10 An existentially conditioned equation $\theta \equiv (\Gamma \rhd \Phi \Rightarrow \beta_1 \overset{e}{=} \beta_2)$ in Σ *holds* in M if $M[\![\Gamma.\,\Phi]\!]$ is contained in $M[\![\Gamma.\,\mathrm{def}\,\beta_i]\!]$, $i = 1, 2$, and the morphisms $M[\![\Gamma.\,\beta_i]\!]$ *agree on* $M[\![\Gamma.\,\Phi]\!]$, i.e. their respective restrictions to $M[\![\Gamma.\,\Phi]\!]$ coincide. In this case, we write $M \models \theta$. We say that M is a *model* of a partial equational theory $(\Sigma, \mathcal{A})$ if $M \models \mathcal{A}$, i.e. $M \models \theta$ for every $\theta \in \mathcal{A}$.

The deduction system of Figure 1 is sound for this semantics:

Theorem 11 (Soundness) *The theorems of a partial equational theory $\mathcal{T}$ hold in all models of $\mathcal{T}$.*

The proof relies on the following lemma, proved by induction over β:

Lemma 12 (Substitution) *Let $\Gamma \rhd \bar{\alpha} : \bar{s}$ be a multi-term, and let $\Delta \rhd \bar{\beta} : \bar{t}$ be a term or multi-term in Σ, where $\Delta = (\bar{y} : \bar{s})$. Then $M[\![\Gamma.\,\bar{\alpha}]\!]$ has a restriction $M[\![\Gamma.\,\mathrm{def}(\bar{\beta}[\bar{\alpha}/\bar{y}], \bar{\alpha})]\!] \to M[\![\Delta.\,\mathrm{def}\,\bar{\beta}]\!]$. In the arising diagram*

$$\begin{array}{ccccc} & & M[\![\Gamma.\,\mathrm{def}(\bar{\beta}[\bar{\alpha}/\bar{y}], \bar{\alpha})]\!] & \hookrightarrow & M[\![\Gamma.\,\mathrm{def}\,\bar{\alpha}]\!] \\ & \swarrow^{M[\![\Gamma.\,\bar{\beta}[\bar{\alpha}/\bar{y}]]\!]} & \big\downarrow{\scriptstyle M[\![\Gamma.\,\bar{\alpha}]\!]} & & \big\downarrow{\scriptstyle M[\![\Gamma.\,\bar{\alpha}]\!]} \\ M[\![\bar{t}]\!] & \xleftarrow[M[\![\Delta.\,\bar{\beta}]\!]]{} & M[\![\Delta.\,\mathrm{def}\,\bar{\beta}]\!] & \hookrightarrow & M[\![\Delta]\!] \end{array} \quad ,$$

the triangle commutes and the square is a pullback.

(This can be phrased in terms of composition of partial morphisms: in $\mathbf{P}(\mathbf{C}, \mathcal{M})$, $M[\![\Delta.\,\bar{\beta}]\!] \circ M[\![\Gamma.\,\bar{\alpha}]\!] = M[\![\Gamma.\,\bar{\beta}[\bar{\alpha}/\bar{y}]]\!] \circ d$, where d is the restriction idempotent for the partial morphism $M[\![\Gamma.\,\bar{\alpha}]\!]$ as in Remark 6.)

PROOF (Theorem 11). We prove only soundness of the substitution rule (sub). In the notation of the rule, let ψ be $\beta_1 = \beta_2$, and put $\Delta = (\bar{y} : \bar{t})$. Let M be a model of $\mathcal{T}$. By the substitution lemma, we have, for $i = 1, 2$, a commutative diagram

$$\begin{array}{ccccc} M[\![\Gamma.\,\Phi[\bar\alpha/\bar y]\wedge \operatorname{def}\bar\alpha]\!] & \hookrightarrow & M[\![\Gamma.\,\operatorname{def}(\beta_i[\bar\alpha/\bar y],\bar\alpha)]\!] & \xrightarrow{M[\![\Gamma.\,\beta_i[\bar\alpha/\bar y]]\!]} & \bullet \\ \downarrow{\scriptstyle M[\![\Gamma,\Delta.\,(\bar x,\bar\alpha)]\!]} & & \downarrow{\scriptstyle M[\![\Gamma,\Delta.\,(\bar x,\bar\alpha)]\!]} & & \\ M[\![\Gamma,\Delta.\,\Phi]\!] & \hookrightarrow & M[\![\Gamma,\Delta.\,\operatorname{def}\beta_i]\!] & \xrightarrow[M[\![\Gamma.\,\beta_i]\!]]{} & \bullet \end{array}$$

where the inclusion at the top is obtained by the pullback assertion of the lemma. By correctness of the subderivation, the β_i are equalized in the bottom row of the diagram and thus also in the top row, as required. □

3 Equivalence of Syntax and Semantics of Partial Equational Logic

We will now define an adjunction between theories and categories, associating to each cartesian dominional category $(\mathbf{C}, \mathcal{M})$ a partial equational theory $\mathsf{L}(\mathbf{C}, \mathcal{M})$, called the *internal language*, and to each partial equational theory $\mathcal{T}$ a *classifying category* $\mathsf{Cl}(\mathcal{T})$. It will turn out that the category of cartesian dominional categories is the Kleisli category for the induced monad, which means that cartesian dominional categories are essentially the same as partial equational theories, equipped with a more general notion of morphism, i.e. translation. (Strictly speaking, this adjunction and the associated notions of monad and Kleisli category live in the realm of 2-categories; however, we shall mostly leave 2-categorical aspects out of consideration here. Also, the collection of cartesian dominional categories fails to form a class, so that one would actually need to speak of quasicategories [1] rather than categories.)

To a cartesian dominional category $(\mathbf{C}, \mathcal{M})$, we associate a signature Σ, whose sorts are the objects of $\mathbf{C}$, and whose operators of profile $\bar{A} \rightharpoonup B$ are the partial morphisms $A_1 \times \cdots \times A_n \rightharpoonup B$ in $(\mathbf{C}, \mathcal{M})$. This signature has an obvious *canonical model* in $(\mathbf{C}, \mathcal{M})$; interpretation in this model is denoted just by $[\![\_]\!]$ (e.g. $[\![A]\!] = A$ etc.). The axioms of $\mathsf{L}(\mathbf{C}, \mathcal{M})$ are defined to be the ece's that hold in the canonical model. The operation L is functorial: given a dominional functor F, we have a signature morphism $\mathsf{L}(F)$ that translates the sort A to FA, and a partial operator (i.e. partial morphism) f to $\mathbf{P}(F)(f)$. The preservation of axioms follows from the fact, proved by induction over (multi-)terms, that we have an equality

$$[\![\mathsf{L}(F)(\Gamma.\,\bar{\alpha})]\!] = \mathbf{P}(F)\,[\![\Gamma.\,\bar{\alpha}]\!]$$

of partial morphisms for each term or multiterm $\Gamma \rhd \bar{\alpha}$.

Proposition 13 *Every model of a partial equational theory $\mathcal{T}$ in $(\mathbf{C}, \mathcal{M})$ factors through the canonical model via a unique translation $\mathcal{T} \to \mathsf{L}(\mathbf{C}, \mathcal{M})$.* $\square$

Conversely, we construct the classifying category $\mathsf{Cl}(\mathcal{T})$ for a given partial equational theory $\mathcal{T}$ as follows. The objects of $\mathsf{Cl}(\mathcal{T})$ are pairs $(\Gamma.\,\Phi)$ consisting of a context $\Gamma = (\bar{x} : \bar{s})$ and a finite set $\Gamma \rhd \Phi$ of definedness assertions, with $(\Gamma.\,\top)$ abbreviated as (Γ). Morphisms $(\Gamma.\,\Phi) \to (\bar{y} : \bar{t}.\,\Psi)$ are multiterms $\Gamma \rhd \bar{\alpha} : \bar{t}$ such that

$$\Gamma \rhd \Phi \vdash \Psi[\bar{\alpha}/\bar{y}] \wedge \operatorname{def} \bar{\alpha},$$

taken modulo equality deducible from Φ. The identity on $(\Gamma.\,\Phi)$ is $\bar{x}$, and composition is defined by substitution; it is easy to see that this defines a category. The dominion is defined as the isomorphism closure of the predominion $\mathcal{M}_0$ (cf. Lemma 8) consisting of the morphisms of the form

$$\bar{x} : (\Gamma.\,\Phi) \to (\Gamma.\,\Psi).$$

The pullback of $\bar{x} : (\Gamma.\,\Phi) \to (\Gamma.\,\Psi)$ along $\bar{\alpha} : (\Delta.\,\Xi) \to (\Gamma.\,\Psi)$ is

$$\begin{array}{ccc} (\Delta.\,\Xi \wedge \Phi[\bar{\alpha}/\bar{x}]) & \xrightarrow{\bar{\alpha}} & (\Gamma.\,\Phi) \\ \big\downarrow & & \big\downarrow \bar{x} \\ (\Delta.\,\Xi) & \xrightarrow[\bar{\alpha}]{} & (\Gamma.\,\Psi). \end{array}$$

Since all structure on the classifying categories is given syntactically, a translation $\sigma : \mathcal{T}_1 \to \mathcal{T}_2$ gives rise to a cartesian dominional functor

$$\mathsf{Cl}(\sigma) : \mathsf{Cl}(\mathcal{T}_1) \to \mathsf{Cl}(\mathcal{T}_2).$$

The unit of the envisioned adjunction is the translation

$$\eta : \mathcal{T} \to \mathsf{L}(\mathsf{Cl}(\mathcal{T}))$$

mapping a sort s to $(x_s : s)$ and an operator $f : \bar{s} \rightharpoonup t$ to the operator given by the partial morphism $f(\overline{x_s}) : (\overline{x_s} : \bar{s}) \rightharpoonup \eta(t)$, where $\overline{x_s}$ is the list of the x_{s_i}.

The co-unit at a cartesian dominional category $(\mathbf{C}, \mathcal{M})$ is the functor

$$E_{(\mathbf{C},\mathcal{M})} : \mathsf{Cl}(\mathsf{L}(\mathbf{C}, \mathcal{M})) \to (\mathbf{C}, \mathcal{M})$$

that maps an object $(\Gamma.\,\Phi)$ to $[\![\Gamma.\,\Phi]\!]$ and a morphism $\bar{\alpha} : (\Gamma.\,\Phi) \to (\Delta.\,\Psi)$ to the composite

$$[\![\Gamma.\,\Phi]\!] \hookrightarrow [\![\Gamma.\,\Psi[\bar{\alpha}/\bar{y}] \wedge \operatorname{def} \bar{\alpha}]\!] \xrightarrow{[\![\Gamma.\,\bar{\alpha}]\!]} [\![\Delta.\,\Psi]\!]$$

of the inclusion provided by the soundness theorem and the restriction of $[\![\Gamma.\,\bar{\alpha}]\!]$ according to the substitution lemma.

Theorem 14 *(i) The extension* η *is* conservative*, i.e. if* $\eta(\Gamma \rhd \Phi \Rightarrow \psi)$ *is a theorem in* $\mathsf{L}(\mathsf{Cl}(\mathcal{T}))$*, then* $\Gamma \rhd \Phi \Rightarrow \psi$ *is a theorem in* $\mathcal{T}$*.*

(ii) The functor $E_{(\mathbf{C},\mathcal{M})}$ *is a dominional equivalence.*

(iii) $\mathsf{Cl}(\mathcal{T})$ *is freely generated by* $\mathcal{T}$ *in the sense that any translation* $\sigma : \mathcal{T} \to \mathsf{L}(\mathbf{C},\mathcal{M})$ *factors essentially uniquely as* $\mathsf{L}(\sigma^{\#})\eta$*:*

$$
\begin{array}{ccc}
\mathcal{T} & \xrightarrow{\ \eta\ } & \mathsf{L}(\mathsf{Cl}(\mathcal{T})) \\
 & \searrow^{\sigma} & \Big\downarrow \mathsf{L}(\sigma^{\#}) \\
 & & \mathsf{L}(\mathbf{C},\mathcal{M})
\end{array}
$$

Here, 'essentially' means that $\sigma^{\#}$ is unique up to a unique natural isomorphism that is the identity on $\mathsf{Cl}(\mathcal{T})$-objects of the form $\eta(s)$. Thus, $\mathsf{Cl}(\mathcal{T})$ is determined up to equivalence by this property.

PROOF. *(i):* By soundness, $\eta(\Gamma \rhd \Phi \Rightarrow \psi)$ holds in $\mathsf{Cl}(\mathcal{T})$. By construction of $\mathsf{Cl}(\mathcal{T})$, it follows that $\Gamma \rhd \Phi \Rightarrow \psi$ is a theorem.

(ii): $E_{(\mathbf{C},\mathcal{M})}$ is surjective on objects. We have to show that $E_{(\mathbf{C},\mathcal{M})}$ is a full and faithful dominional functor. By soundness, $E_{(\mathbf{C},\mathcal{M})}$ is well-defined on morphisms. It is clear that $E_{(\mathbf{C},\mathcal{M})}$ preserves identities and admissible subobjects. By the substitution lemma, $E_{(\mathbf{C},\mathcal{M})}$ preserves composition and is dominional. Faithfulness holds by construction of $\mathsf{L}(\mathbf{C},\mathcal{M})$.

By Lemma 4, it remains to show that $\mathbf{P}(E_{(\mathbf{C},\mathcal{M})})$ is full. Let $f : [\![\Gamma.\,\Phi]\!] \rightharpoonup [\![\Delta.\,\Psi]\!]$ be a partial morphism, where $\Gamma = (\bar{x} : \bar{A})$ and $\Delta = (\bar{y} : \bar{B})$. The arising partial morphisms $f_i : [\![\bar{A}]\!] \rightharpoonup B_i$ are operators $\bar{A} \rightharpoonup B_i$ in $\mathsf{L}(\mathbf{C},\mathcal{M})$; let $\bar{\alpha}$ be the multiterm with components $f_i(\bar{x})$. Then $\mathrm{def}\,\bar{\alpha} \vdash \Phi \wedge \Psi[\bar{\alpha}/\bar{y}]$ by construction and by the substitution lemma. Thus, $\bar{\alpha}$ induces a partial morphism $(\Gamma.\,\Phi) \rightharpoonup (\Delta.\,\Psi)$, which by construction is mapped to f under $\mathbf{P}(E_{(\mathbf{C},\mathcal{M})})$.

(iii): Uniqueness is clear; to prove existence, put $\sigma^{\#} := E_{(\mathbf{C},\mathcal{M})} \circ \mathsf{Cl}(\sigma)$. $\square$

In combination with Proposition 13, Theorem 14 (iii) says that models of $\mathcal{T}$ are equivalent to *models* of $\mathsf{Cl}(\mathcal{T})$, i.e. cartesian dominional functors. Moreover, we obtain from (ii) that the cartesian dominional categories form 'essentially' (in a sense made precise in 2-dimensional category theory) the Kleisli category of the 'adjunction' $\mathsf{Cl} \dashv \mathsf{L}$. This category consists of the partial equational theories, with translations $\mathcal{T}_1 \to \mathsf{L}(\mathsf{Cl}(\mathcal{T}_2))$ as morphisms $\mathcal{T}_1 \to \mathcal{T}_2$. These morphisms are *derived translations*: sorts are mapped to domains of multi-terms, and symbols are mapped to *conditioned multi-terms*, i.e. multi-terms $\bar{\alpha} \upharpoonright \bar{\beta}$ denoting $\bar{\alpha}$ with its domain restricted to that of $\bar{\beta}$ (we can regard $\upharpoonright$ as an operator described by the axiom $x \restriction \bar{y} \stackrel{e}{=} x$). Two morphisms of this kind are identified if they map all symbols to provably strongly equal terms. The above can be summed up in the slogan

partial equational theories are equivalent to cartesian dominional categories.

Remark 15 It follows already from the fact that $\mathcal{T}$ has the model η in $\mathsf{Cl}(\mathcal{T})$ that the deduction system of Figure 1 is complete for models in cartesian dominional categories. Since representable functors are cartesian and dominional, we obtain a stronger completeness result from the universality of $\mathsf{Cl}(\mathcal{T})$, namely that the deduction system is complete for models in **Set**. This is shown along the same lines as a similar result in [8], namely by applying the validity of $\Gamma \rhd \Phi \Rightarrow \psi$ to the model $\hom((\Gamma.\,\Phi), \_)$ of $\mathsf{Cl}(\mathcal{T})$.

Remark 16 The above can be easily modified to obtain the simpler result that unary partial equational theories (with single-variable contexts) are equivalent to dominional categories [29].

Remark 17 We sometimes regard terms $\Gamma \rhd \alpha : 1$ as *formulae* or *predicates*, and then write α in place of $\mathrm{def}\,\alpha$. The sentence $\mathrm{def}\,\bar{\beta}$ can be coded as the predicate $* \restriction \bar{\beta}$. Conjunction of predicates is defined by $\phi \wedge \psi :\equiv \mathrm{def}(\phi, \psi)$; the constant true predicate is $\top :\equiv *$.

4 Internal equality

An important special case is that equality is internalized in a partial equational theory as a predicate (see also [13,21]):

Definition 18 A partial equational theory has *(internal) equality* if there exists, for each type s, a binary predicate eq_s (i.e. a term $x, y : s \rhd eq_s(x, y) : 1$, cf. Remark 17) such that

$$x, y : s \rhd eq_s(x, y) \Rightarrow x \stackrel{e}{=} y \quad \text{and} \quad x : s \rhd eq_s(x, x)$$

Such a predicate allows coding conditional equations as ece's. Note that the axioms for eq_s are themselves ece's. Equality has a simple categorical correlate:

Definition 19 A cartesian dominional category $(\mathbf{C}, \mathcal{M})$ has *(internal) equality* if $\mathcal{M}$ contains all diagonals $A \to A \times A$, equivalently all regular monomorphisms.

This condition implies that $\mathbf{C}$ has equalizers (hence is finitely complete, shortly: *left exact* or *lex*) and that $\mathcal{M}$ is closed under pairing [1].

Lemma 20 *Let* $\mathbf{C}$ *be a lex category in which regular monomorphisms compose, so that* $(\mathbf{C}, \mathrm{RegMono}(\mathbf{C}))$ *is a cartesian dominional category, and let* $(\mathbf{A}, \mathcal{N})$ *be a cartesian dominional category with equality. A functor* $F : \mathbf{C} \to \mathbf{A}$ *is a cartesian dominional functor* $(\mathbf{C}, \mathrm{RegMono}(\mathbf{C})) \to (\mathbf{A}, \mathcal{N})$ *iff* F *is lex.* □

Theorem 21 *A partial equational theory* $\mathcal{T}$ *has equality iff* $\mathsf{Cl}(\mathcal{T})$ *does.* □

(By Theorem 14, it follows that a cartesian dominional category $(\mathbf{C}, \mathcal{M})$ has equality iff $\mathsf{L}(\mathbf{C}, \mathcal{M})$ does.) Thus, by the results of Section 3,

partial equational theories with equality are equivalent to
cartesian dominional categories with equality.

Example 22 The dominions formed by the open (closed, clopen) subspaces of topological spaces and the upclosed (downclosed) subsets of partially ordered sets, respectively, fail to have equality, as does the dominion of Scott open sets on the category of cpo's. The dominion of closed embeddings does have equality when restricted to the category of Hausdorff spaces; generally, the dominions formed by all (regular) monomorphisms in a category with pullbacks (with composition stable regular monos) have equality. An example of a dominion with equality that contains more than just the regular monomorphisms is the dominion of subspace embeddings on the category of Hausdorff spaces.

Remark 23 Conditional equations in general require, on the categorical side, left exactness, *without* however restricting the dominion. Partial conditional equational theories can be *represented* as lex dominional categories by a construction similar to one given in [8] (cf. [29] for a proof sketch). I.e. one can define classifying categories satisfying an analogue of Theorem 14 (iii), while the analogue of Theorem 14 (ii) fails, even for classifying categories of partial conditional equational theories. It is an open problem to find a categorical *equivalent* of partial conditional equational theories, possibly along the lines of corresponding results for the total case [2].

5 The Partial λ-Calculus

The natural generalization of the simply typed λ-calculus to the setting of partial functions is the partial λ-calculus [21,22,28], which extends partial equational logic (cf. Section 1) as follows: from the set of sorts, the set of *types* is inductively generated by the formation of *partial function types*

$$\bar{s} \rightharpoonup t,$$

with t a type and $\bar{s}$ a *multi-type*, i.e. a list of types (currying partial functions requires total function types [21]). Operators have profiles $f : \bar{s} \rightharpoonup t$ with t a type and $\bar{s}$ a multi-type. Following [21], we assume application operators with profile $(\bar{s} \multimap t)\bar{s} \rightharpoonup t$ in the signature, so that application does not require extra typing or deduction rules. We denote these operators, i.e. application of *functional values*, by juxtaposition, but continue to write *operator* application with brackets. These data constitute a *(higher-order) signature*. *Signature morphisms* are defined as before as mapping *sorts* and operators, preserving application operators.

Anonymous partial functions are formed by the additional typing rule

$$\textbf{(abs)}\quad \frac{\Gamma, \bar{y} : \bar{t} \rhd \alpha : u}{\Gamma \rhd \lambda \bar{y} : \bar{t}.\, \alpha : \bar{t} \multimap u}.$$

The types of the bound variables, as well as unused variables of type 1, will occasionally be omitted. For $\bar{t} = (t_1, \ldots, t_m)$, $\bar{s} \multimap \bar{t}$ denotes the multi-type $(\bar{s} \multimap t_1, \ldots, \bar{s} \multimap t_m)$, not to be confused with the (non-existent) 'type' $\bar{s} \multimap t_1 \times \cdots \times t_m$. We can then define projections, conditioned terms, and conjunction (cf. Remark 17) by

$$\begin{aligned} \mathit{fst} &:= \lambda x_1, \ldots, x_n.\, x_1 \text{ etc.}, \\ \alpha \restriction \bar{\beta} &:= \mathit{fst}(\alpha, \bar{\beta}) \\ p \wedge q &:= (\lambda x, y : 1.\, *)(p, q). \end{aligned}$$

In particular, definedness assertions for multi-terms can now be replaced by definedness assertions for terms. The abstraction $\lambda \bar{y} : \bar{t}.\, \bar{\alpha}$ of a multi-term $\bar{\alpha}$ is defined as the multi-term consisting of the terms $\lambda \bar{y} : \bar{t}.\, (\alpha_i \restriction \bar{\alpha})$, taking into account that a partial function into a product type is a tuple of partial functions *with the same domain*. In view of Remark 17, we can regard $\Omega = 1 \multimap 1$ as a type of truth values.

A *partial λ-theory* $\mathcal{T}$ consists of a higher-order signature and a set of ece's, its *axioms*. Figure 2 shows a set of higher-order proof rules that complement the first-order rules of Figure 1. This deduction system is closely related to the one presented in [21], but deals with existential rather than strong equations. We use strong equations $\Delta \rhd \alpha \stackrel{s}{=} \beta$, or just $\alpha \stackrel{s}{=} \beta$, as abbreviations for subderivations '$\Delta \rhd \operatorname{def} \alpha \vdash \operatorname{def} \beta$ and $\Delta \rhd \operatorname{def} \beta \vdash \alpha \stackrel{e}{=} \beta$'. In particular, rule (β) is really two rules. Rule (ξ) implies that all λ-terms are defined.

Remark 24 Due to λ-abstraction, ece's, existential equations and strong equations all have the same expressive power; in particular, the axioms of a partial λ-theory could have been restricted to existential equations. However, in view of the first-order case and the general nature of deduction, it seems more natural to continue working with ece's.

$$(\eta)\ \frac{x : \bar{t} \rightharpoonup\!\!\!\!\circ\!\!\rightarrow u \text{ in } \Gamma}{(\lambda \bar{y} : \bar{t}.\, x\, \bar{y}) \stackrel{e}{=} x} \qquad (\beta)\ \frac{\bar{y} : \bar{t} \text{ in } \Gamma}{(\lambda \bar{y} : \bar{t}.\, \alpha)\, \bar{y} \stackrel{s}{=} \alpha} \qquad (\xi)\ \frac{\bar{y} : \bar{t} \rhd \alpha \stackrel{s}{=} \beta}{\lambda \bar{y} : \bar{t}.\, \alpha \stackrel{e}{=} \lambda \bar{y} : \bar{t}.\, \beta}$$

Fig. 2. Additional deduction rules for the partial λ-calculus

We say that a partial λ-theory has *(internal) equality* if each *type* has an equality predicate, denoted from now on by $\stackrel{e}{=}$, as in Definition 18. Equality gives rise to a full-fledged intuitionistic logic, the *internal logic*, along the same lines as in [13,19]: Letting p and q range over (partial) terms of type 1, we put

$$\begin{aligned} p \Rightarrow q &:= ((\lambda.\, p) \stackrel{e}{=} \lambda.\, p \wedge q), \\ \forall \bar{y} : \bar{t}.\, p &:= ((\lambda \bar{y} : \bar{t}.\, p) \stackrel{e}{=} \lambda \bar{y} : \bar{t}.\, \top), \\ \bot &:= \forall a : \Omega.\, a\, *, \\ \neg p &:= p \Rightarrow \bot, \\ p \vee q &:= \forall a : \Omega.\, ((p \Rightarrow a\, *) \wedge (q \Rightarrow a\, *)) \Rightarrow a\, *, \text{ and} \\ \exists \bar{y} : \bar{t}.\, p &:= \forall a : \Omega.\, (\forall \bar{y} : \bar{t}.\, p \Rightarrow a\, *) \Rightarrow a\, * \end{aligned}$$

(note that all right hand sides have type 1). The usual deduction rules of intuitionistic higher-order logic are obtained as lemmas. This logic plays a prominent role in HasCasl [33].

6 Partial Cartesian Closed Categories

The categorical correlate for partial λ-theories are *partial cartesian closed categories (pccc's)* [21], i.e. cartesian dominional categories in which partial morphisms $A \rightharpoonup B$ are represented in partial function spaces $A \rightharpoonup\!\!\!\!\circ\!\!\rightarrow B$.

Definition 25 A cartesian dominional category $(\mathbf{C}, \mathcal{M})$ is called a *partial cartesian closed category (pccc)* if the composite functor

$$\mathbf{C} \xrightarrow{- \times A} \mathbf{C} \hookrightarrow \mathbf{P}(\mathbf{C}, \mathcal{M})$$

has a right adjoint for each object A in $\mathbf{C}$.

(As is by now standard [14], we do not require finite completeness as in [21].) Explicitly, a pccc $(\mathbf{C}, \mathcal{M})$ has *partial function spaces* $A \rightharpoonup\!\!\!\!\circ\!\!\rightarrow B$ with partial *evaluation morphisms* $ev : (A \rightharpoonup\!\!\!\!\circ\!\!\rightarrow B) \times A \rightharpoonup B$ such that every partial morphism $f : C \times A \rightharpoonup B$ factors uniquely as $ev \circ (\hat{f} \times A)$ by a total morphism $\hat{f}$ called its *abstraction*. The following statement relating abstractions with restriction will be useful in proving the higher-order soundness theorem.

Lemma 26 *Let f be a partial morphism $C \times A \rightharpoonup B$ with abstraction $\hat{f}$, and let E be an admissible subobject of C. Then the abstraction of the restriction of f to $E \times A$ is the restriction of $\hat{f}$ to E.* $\square$

For a pccc $(\mathbf{C}, \mathcal{M})$, the embedding $\mathbf{C} \hookrightarrow \mathbf{P}(\mathbf{C}, \mathcal{M})$ has a right adjoint, being isomorphic to $\_ \times 1$. Explicitly, each A in $\mathbf{C}$ has a *lifting*, i.e. an $\mathcal{M}$-extension $\rho_A : A \to \tilde{A} \cong 1 \multimap A$ that *classifies* $\mathcal{M}$-partial morphisms (m, f) into A: for each such (m, f), there exists a unique total f^* such that

$$\begin{array}{ccc} \bullet & \xrightarrow{m} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^*} \\ A & \xrightarrow[\rho_A]{} & \tilde{A} \end{array}$$

is a pullback. A special case is the object of truth values $\Omega = \tilde{1}$, with ρ_1 denoted by $\top$. This object classifies $\mathcal{M}$-subobjects, i.e. each $\mathcal{M}$-subobject m of A has a unique *characteristic map* $\chi_m : A \to \Omega$ such that

$$\begin{array}{ccc} \bullet & \xrightarrow{m} & A \\ \downarrow & & \downarrow{\scriptstyle \chi_m} \\ 1 & \xrightarrow[\top]{} & \Omega \end{array}$$

is a pullback. By consequence, $\mathcal{M}$ *consists of regular monomorphisms*, so that $\mathcal{M} = \mathrm{RegMono}(\mathbf{C})$ whenever $\mathbf{C}$ has equality. Thus, we can *omit the mention of $\mathcal{M}$ for pccc's with equality.*

Remark 27 The functor taking A to $\tilde{A}$ induces a monad on $\mathbf{C}$, the so-called *lifting monad*. There has been a lot of recent interest in this structure, which may be treated independently of function types; in particular, categories of partial morphisms may in the presence of partial map classifiers be treated (and axiomatized) as Kleisli categories for the lifting monad [5,10,15].

Example 28 Every (quasi-)topos is a pccc with equality (but not every pccc with equality is a quasitopos; cf. Section 9). Typical examples of pccc's without equality are the category of partial orders, equipped with the dominion of upclosed (downclosed) sets, and the category of cpo's, equipped with the dominion of Scott open sets. The latter example motivates the axiomatic treatment in [14,15].

Definition 29 A cartesian dominional functor between two pccc's is called *partial cartesian closed (pcc)* if it preserves partial function spaces and evaluation morphisms. A dominional subcategory $(\mathbf{A}, \mathcal{N})$ of a pccc $(\mathbf{C}, \mathcal{M})$ is a *sub-pccc* if $(\mathbf{A}, \mathcal{N})$ is a pccc and $(\mathbf{A}, \mathcal{N}) \hookrightarrow (\mathbf{C}, \mathcal{M})$ is a pcc functor.

More generally, we will need to consider the case that partial function spaces exist only for certain objects:

Definition 30 A *pre-pccc* $(\mathbf{C}, \mathcal{M}, \mathfrak{A})$ is a cartesian dominional category $(\mathbf{C}, \mathcal{M})$, equipped with a class $\mathfrak{A} \subseteq \operatorname{Ob} \mathbf{C}$ of objects called *type objects* such that $\mathfrak{A}$ contains 1 and has partial function spaces in the sense that $(A_1 \times \cdots \times A_n \rightharpoonup B)$ exists and is in $\mathfrak{A}$ for all $A_1, \dots, A_n, B \in \mathfrak{A}$, $n \geq 0$, and such that $\mathfrak{A}$ generates $(\mathbf{C}, \mathcal{M})$ as a cartesian dominional category (explicitly: every object of $\mathbf{C}$ is an admissible subobject of a finite product of $\mathfrak{A}$-objects). Pccc's are regarded as pre-pccc's with $\mathfrak{A} = \operatorname{Ob} \mathbf{C}$. A cartesian dominional functor between pre-pccc's is called *pre-pcc* if it preserves type objects and their partial function spaces and evaluation morphisms.

The notions of pre-pccc and pccc coincide in important special cases, e.g. in the total case (i.e. when $\mathcal{M} = \mathrm{Iso}$, the reason being that the total function space into a product is the product of the function spaces into the components), and when idempotents split in $\mathbf{C}$ (cf. Section 7), which includes the case with equality.

The standard technique of producing a domininonal category from a category of partial maps, namely, to split all restriction idempotents, yields in general only a pre-pccc when applied to typical direct models of the partial λ-calculus such as p-categories with p-exponentials [28] or pccc's in the sense of [13]. (By definition, a p-category $\mathbf{C}$, cf. Remark 6, *has p-exponentials* if for each object X, the functor $\_ \times X : \mathbf{C}_t \to \mathbf{C}$ has a right adjoint, where $\mathbf{C}_t$ denotes the subcategory of *total* maps consisting of all morphisms $f : X \to Y$ in $\mathbf{C}$ such that $p_{X,Y}(id_X \times f)\Delta_X = id_X$.)

Proper pre-pccc's exist:

Example 31 Let $(\mathbf{Pos}, \mathcal{M})$ be the pccc of partially ordered sets and down-closed subsets. Therein, let $(\mathbf{C}, \mathcal{N})$ be the full dominional subcategory of partial orders with bottom element $\bot$ such that every element $x \neq \bot$ is above an *atom*, i.e. a minimal element of $X - \{\bot\}$. Then $\mathbf{C}$ is closed under products in $\mathbf{Pos}$. Let $\mathfrak{A}$ be the class of $\mathbf{C}$-objects that additionally have a top element $\top$. Then $(\mathbf{C}, \mathcal{N}, \mathfrak{A})$ is a pre-pccc, because $\mathfrak{A}$ is closed under partial function spaces in $(\mathbf{Pos}, \mathcal{M})$. To see this, note that for $X, Y \in \mathfrak{A}$, the atoms in $X \rightharpoonup Y$ are the functions f such that $f\ \top$ is an atom and $f\ x = \bot$ for $x \neq \top$. However, $(\mathbf{C}, \mathcal{N})$ fails to be closed under partial function spaces in $(\mathbf{Pos}, \mathcal{M})$ and hence fails to be a pccc: $X = (\mathbb{N}, \leq)$ is in $\mathbf{C}$, but $X \rightharpoonup 1$ (ordered by '$\supseteq$'!) is atomless.

A slight variation of this example, where a pre-pccc is generated in $(\mathbf{Pos}, \mathcal{M})$ by taking as type objects the $\mathbf{C}$-objects with at most one atom, shows that the type objects need not be closed (more precisely, closable) under products.

Pre-pccc's serve as semantic domains for the partial λ-calculus:

Definition 32 A *model* M of a higher-order signature Σ in a pre-pccc $(\mathbf{C}, \mathcal{M}, \mathfrak{A})$ consists of an assigment of a type object to each sort, inducing a semantics $M[\![\_]\!]$ for types and multi-types in the obvious way (with types interpreted as type objects and multi-types as objects), and a partial map $M[\![f]\!] : M[\![\bar{s}]\!] \rightharpoonup M[\![t]\!]$ to each operator $f : \bar{s} \rightharpoonup t$.

An interpretation $M[\![\_]\!]$ for contexts, terms, multi-terms, and definedness assertions is then defined as in Section 2, with an additional clause for λ-abstraction:

$$M[\![\Gamma.\, \lambda\bar{y} : \bar{t}.\, \alpha]\!] : M[\![\Gamma]\!] \to M[\![\bar{t} \multimapdotboth u]\!]$$

is defined as the abstraction of $M[\![\Gamma, \bar{y} : \bar{t}.\, \alpha]\!] : M[\![\Gamma]\!] \times M[\![\bar{y} : \bar{t}]\!] \rightharpoonup M[\![u]\!]$. Satisfaction of ece's in M is defined as in Section 2.

Theorem 33 (Soundness) *The theorems of a partial λ-theory $\mathcal{T}$ hold in all models of $\mathcal{T}$ in pre-pccc's.*

The proof needs a higher-order version of the *substitution lemma*, whose formulation is otherwise literally the same as in Section 2; the proof of the substitution lemma involves Lemma 26.

PROOF (Theorem 33). We prove only the correctness of rule (ξ). Let M be a model of $\mathcal{T}$. The premise of the rule says that the restrictions of the partial morphisms $M[\![\Gamma, \bar{y} : \bar{t}.\, \alpha]\!]$ and $M[\![\Gamma, \bar{y} : \bar{t}.\, \beta]\!]$ to the subobject $M[\![\Gamma.\, \Phi]\!] \times M[\![\bar{y} : \bar{t}]\!]$ determined by the local assumptions are equal. Hence, so are their abstractions, and by Lemma 26, these abstractions are the restrictions of $M[\![\Gamma.\, \lambda\bar{y} : \bar{t}.\, \alpha]\!]$ and $M[\![\Gamma.\, \lambda\bar{y} : \bar{t}.\, \beta]\!]$, respectively, to $M[\![\Gamma.\, \Phi]\!]$. $\square$

7 Equivalence of Higher-Order Syntax and Semantics

In analogy with the first-order case, we now establish adjoint equivalences between partial λ-theories on the one hand and pre-pccc's and pccc's, respectively, on the other hand. These results extend Lambek's classical result that λ-theories are equivalent to cartesian closed categories.

To begin, we define the *internal language* $\mathsf{L}(\mathbf{C}, \mathcal{M}, \mathfrak{A})$ of a pre-pccc $(\mathbf{C}, \mathcal{M}, \mathfrak{A})$; this covers also pccc's as a special case. The higher-order signature Σ associated to $(\mathbf{C}, \mathcal{M}, \mathfrak{A})$ has the type objects as sorts; for a type t and a multi-type $\bar{s}$, the operators of profile $\bar{s} \rightharpoonup t$ are the partial morphisms $[\![\bar{s}]\!] \rightharpoonup [\![t]\!]$, with evaluation morphisms as application operators. This signature has an obvious *canonical model* $[\![\_]\!]$ in $(\mathbf{C}, \mathcal{M}, \mathfrak{A})$. The axioms of $\mathsf{L}(\mathbf{C}, \mathcal{M}, \mathfrak{A})$ are the ece's that hold in this model, which enjoys a couniversal property analogous to Proposition 13. Every pcc functor $F : (\mathbf{C}, \mathcal{M}, \mathfrak{A}) \to (\mathbf{A}, \mathcal{N}, \mathfrak{B})$ induces a translation $\mathsf{L}(F) : \mathsf{L}(\mathbf{C}, \mathcal{M}, \mathfrak{A}) \to \mathsf{L}(\mathbf{A}, \mathcal{N}, \mathfrak{B})$. Whenever there is a risk of confusion, we shall refer to the first-order internal language of $(\mathbf{C}, \mathcal{M})$ according to Section 3 as $\mathsf{L}_{\mathsf{fo}}(\mathbf{C}, \mathcal{M})$ and to the higher-order internal language as $\mathsf{L}_{\mathsf{ho}}(\mathbf{C}, \mathcal{M}, \mathfrak{A})$.

Conversely, we construct the *classifying pre-pccc* of a partial λ-theory $\mathcal{T}$, denoted by $\mathsf{Sy}(\mathcal{T})$ for the sake of distinction from the classifying pccc $\mathsf{Cl}(\mathcal{T})$ to be described below, as follows: objects and morphisms are definedness assertions in context and multi-terms, respectively, as in the first-order case (Section 3), the only difference being the presence of higher types and λ-abstraction; one thus obtains a cartesian dominional category. The type objects are the objects of the form $(x : t)$. This defines a pre-pccc: the partial function space $(x_1 : s_1) \times \cdots \times (x_n : s_n) \multimap (y : t)$ is $(z : s_1 \dots s_n \multimap t)$, and the type objects generate $\mathsf{Sy}(\mathcal{T})$ by construction. We have a unit translation $\eta : \mathcal{T} \to \mathsf{L}(\mathsf{Sy}(\mathcal{T}))$ and a co-unit $E_{(\mathbf{C},\mathcal{M},\mathfrak{A})} : \mathsf{Sy}(\mathsf{L}(\mathbf{C}, \mathcal{M}, \mathfrak{A})) \to (\mathbf{C}, \mathcal{M}, \mathfrak{A})$, defined as in the first-order case, as well as, for each translation $\sigma : \mathcal{T}_1 \to \mathcal{T}_2$, an induced pre-pcc functor $\mathsf{Sy}(\sigma) : \mathsf{Sy}(\mathcal{T}_1) \to \mathsf{Sy}(\mathcal{T}_2)$.

Theorem 34 *(i) The extension η is conservative.*
(ii) $E_{(\mathbf{C},\mathcal{M},\mathfrak{A})}$ is a dominional equivalence (in particular a pre-pcc functor).
(iii) $\mathsf{Sy}(\mathcal{T})$ is the free pre-pccc over $\mathcal{T}$.

PROOF. Analogous to Theorem 14, noting that the generation condition for $(\mathbf{C}, \mathcal{M}, \mathfrak{A})$ implies that $E_{(\mathbf{C},\mathcal{M},\mathfrak{A})}$ is up to isomorphism surjective on objects.□

We thus obtain that *partial λ-theories are equivalent to pre-pccc's.* Since there is a second equivalence of this kind with pre-pccc's replaced by pccc's, we stress that the induced notion of derived translations is the same as in the first-order case; in particular, sorts are mapped to domains of multi-terms.

Remark 35 By construction, the ece's that hold in $\mathsf{Sy}(\mathcal{T})$ are precisely the theorems of $\mathcal{T}$, so that *deduction is complete for models in pre-pccc's.*

Theorem 34 (ii) implies in particular that $\mathsf{Sy}(\mathcal{T})$ is in general only a pre-pccc. In order to obtain the classifying pccc $\mathsf{Cl}(\mathcal{T})$, we need to construct the free pccc over a pre-pccc. The key to this construction is the following observation:

Theorem 36 *If $(\mathbf{C}, \mathcal{M}, \mathfrak{A})$ is a cauchy complete pre-pccc, then $(\mathbf{C}, \mathcal{M})$ is a pccc.*

Recall here that a category $\mathbf{C}$ is called *cauchy complete* if each idempotent a in $\mathbf{C}$ *splits*, i.e. there exist r and s such that $rs = id$ and $sr = a$. In this situation, we refer to the (essentially unique) pair (r, s), as well as to the domain A of

s and the triple (A, r, s), as the *splitting* of a. If (A, r_a, s_a) and (B, r_b, s_b) are the splittings of idempotents a and b on objects X and Y, respectively, then (partial) morphisms $A \to B$ are in one-to-one correspondence with (partial) morphisms $f : X \to Y$ such that $bfa = f$; in this case, we say that f *induces* the (partial) morphism $r_b f s_a : A \to B$.

PROOF (Theorem 36). We show that partial function spaces exist in $\mathsf{Sy}(\mathsf{L}(\mathbf{C}, \mathcal{M}, \mathfrak{A})) \cong (\mathbf{C}, \mathcal{M}, \mathfrak{A})$. The partial function space $(\bar{x} : \bar{A}.\, \phi) \rightharpoonup\!\!\!\!\to (\bar{y} : \bar{B}.\psi)$ is the splitting of the idempotent

$$\lambda \bar{x} : \bar{A}.\, (\bar{z}\, \bar{x} \restriction (\phi \wedge \psi[\bar{z}\, \bar{x}/\bar{y}]))$$

on $(\bar{z} : \bar{A} \rightharpoonup\!\!\!\!\to \bar{B})$ (recall that abstraction of multi-terms is encoded via restriction, cf. Section 5), with the evaluation morphism induced by the partial morphism $\bar{z}\, \bar{x} \restriction (\phi \wedge \psi[\bar{z}\, \bar{x}/\bar{y}]) : (\bar{z} : \bar{A} \rightharpoonup\!\!\!\!\to \bar{B}) \times (\bar{x} : \bar{A}) \rightharpoonup \bar{B}$. The abstraction of a partial morphism $\bar{\alpha} : (\bar{y} : \bar{C}.\, \chi) \times (\bar{x} : \bar{A}.\, \phi) \rightharpoonup (\bar{y} : \bar{B}.\psi)$ is induced by $\lambda \bar{y}.\, \bar{\alpha}$, assuming w.l.o.g. that $\operatorname{def} \alpha \Rightarrow \chi \wedge \phi$. □

Every category $\mathbf{C}$ has a free cauchy complete extension $\mathbf{K}(\mathbf{C})$, its *Karoubi envelope*, whose objects are the idempotents of $\mathbf{C}$; morphisms $f : a \to b$ between idempotents a, b are $\mathbf{C}$-morphisms f such that $bfa = f$, equivalently: $bf = f$ and $fa = f$. The embedding $\mathbf{C} \hookrightarrow \mathbf{K}(\mathbf{C})$ takes an object A to the idempotent id_A. The category $\mathbf{K}(\mathbf{C})$ is determined uniquely up to equivalence as a cauchy complete full extension of $\mathbf{C}$ in which every object is a splitting of an idempotent in $\mathbf{C}$. Every functor $F : \mathbf{A} \to \mathbf{C}$ extends to a functor $\mathbf{K}(F) : \mathbf{K}(\mathbf{A}) \to \mathbf{K}(\mathbf{C})$ which takes $f : a \to b$ to $Ff : Fa \to Fb$. If $\mathbf{C}$ is cartesian, then $\mathbf{K}(\mathbf{C})$ is the free cauchy complete cartesian category over $\mathbf{C}$, with the product of idempotents a and b being the idempotent $a \times b$.

We extend the Karoubi construction to dominional categories as follows.

Theorem 37 *Let $(\mathbf{C}, \mathcal{M})$ be a dominional category, and let*

$$\mathcal{M}_0^K = \{am : b \to a \mid m \in \mathcal{M}, a, b \in \operatorname{Ob} \mathbf{K}(\mathbf{C}), am = mb\}.$$

(i) *$\mathcal{M}_0^K$ is a predominion on $\mathbf{K}(\mathbf{C})$.*
(ii) *$(\mathbf{C}, \mathcal{M})$ is a full dominional subcategory of $\mathbf{K}(\mathbf{C}, \mathcal{M}) = (\mathbf{K}(\mathbf{C}), \mathcal{M}^K)$, where $\mathcal{M}^K$ is the isomorphism closure of $\mathcal{M}_0^K$.*
(iii) *For $\mathbf{C}$ cauchy complete, $(\mathbf{C}, \mathcal{M})$ is dominionally equivalent to $\mathbf{K}(\mathbf{C}, \mathcal{M})$.*
(iv) *$\mathbf{K}(\mathbf{C}, \mathcal{M})$ is the free cauchy complete dominional category over $(\mathbf{C}, \mathcal{M})$.*

(The definition of $\mathcal{M}^K$ says that admissible subobjects of an idempotent a on X are restrictions of a to admissible subobjects of X.)

PROOF. *(i):* It is clear that $\mathcal{M}_0^K$ contains all identities and is closed under composition. It remains to construct the pullback of $am : b \to a$ in $\mathcal{M}_0^K$ along a morphism $f : c \to a$. Let

$$\begin{array}{ccc} \bullet & \xrightarrow{\bar{f}} & \bullet \\ {\scriptstyle \bar{m}}\downarrow & & \downarrow{\scriptstyle m} \\ \bullet & \xrightarrow[f]{} & \bullet \end{array}$$

be a pullback in $\mathbf{C}$. Then $\bar{m} \in \mathcal{M}$. Since $f(c\bar{m}) = m\bar{f}$, we obtain d such that $\bar{m}d = c\bar{m}$ (hence d is idempotent) and $\bar{f}d = \bar{f}$. The pullback in $\mathbf{K}(\mathbf{C})$ is then

$$\begin{array}{ccc} d & \xrightarrow{\bar{f}} & b \\ {\scriptstyle c\bar{m}}\downarrow & & \downarrow{\scriptstyle am} \\ c & \xrightarrow[f]{} & a \end{array} \quad .$$

(ii): It is, by the above construction of $\mathcal{M}^K$-pullbacks, clear that $(\mathbf{C}, \mathcal{M})$ is a dominional subcategory of $\mathbf{K}(\mathbf{C}, \mathcal{M})$. Dominional fullness is just the fact that restrictions of identities to admissible subobjects are again identities.

(iii): By (ii), the equivalence $\mathbf{C} \cong \mathbf{K}(\mathbf{C})$ is dominional.

(iv): Let $F : (\mathbf{C}, \mathcal{M}) \to (\mathbf{A}, \mathcal{N})$ be a dominional functor, with $\mathbf{A}$ cauchy complete. The essentially unique extension of F to a functor $F^\# : \mathbf{K}(\mathbf{C}) \to \mathbf{A}$ is the composite of $\mathbf{K}(F)$ with the equivalence $\mathbf{K}(\mathbf{A}) \to \mathbf{A}$. It is easily seen that $\mathbf{K}(F) : \mathbf{K}(\mathbf{C}, \mathcal{M}) \to \mathbf{K}(\mathbf{A}, \mathcal{N})$ is a dominional functor; by (iii), it follows that $F^\# : \mathbf{K}(\mathbf{C}, \mathcal{M}) \to (\mathbf{A}, \mathcal{N})$ is dominional. □

Corollary 38 *If* $(\mathbf{C}, \mathcal{M})$ *is a cartesian dominional category, then* $\mathbf{K}(\mathbf{C}, \mathcal{M})$ *is the free cauchy complete cartesian dominional category over* $(\mathbf{C}, \mathcal{M})$*.*

Theorem 39 *Let* $(\mathbf{C}, \mathcal{M}, \mathfrak{A})$ *be a pre-pccc. Then* $\mathbf{K}(\mathbf{C}, \mathcal{M})$ *is a pccc, the free cauchy complete pccc over* $(\mathbf{C}, \mathcal{M}, \mathfrak{A})$*.*

This result is partly related to a result on restriction categories with liftings for partial morphisms proved in [10].

PROOF. We have already seen that $\mathbf{K}(\mathbf{C}, \mathcal{M})$ is a cartesian dominional category. It is shown as in the proof of Theorem 36 that partial function spaces for objects of $\mathbf{C}$ exist in $\mathbf{K}(\mathbf{C}, \mathcal{M})$; the only point to note here is that by Theorem 37, $(\mathbf{C}, \mathcal{M})$ is a full dominional subcategory of $\mathbf{K}(\mathbf{C}, \mathcal{M})$, so that partial morphisms in $\mathbf{K}(\mathbf{C}, \mathcal{M})$ are induced by partial morphisms in $(\mathbf{C}, \mathcal{M})$.

Now given idempotents $a_1, \ldots, a_n$ and b on objects $A_1, \ldots, A_n$ and B, respectively, of $\mathbf{C}$, the partial function space $a_1 \times \cdots \times a_n \rightarrow\!\!\!\!\circ\!\!\rightarrow b$ is the splitting of the idempotent c given by the term

$$b \circ z \circ (a_1 \times \cdots \times a_n)$$

on $(z : A_1 \times \cdots \times A_n \rightarrow\!\!\!\!\circ\!\!\rightarrow B)$, where $\circ$ denotes the composition operator for partial functions (which exists in $\mathbf{K}(\mathbf{C}, \mathcal{M})$ because the involved partial function spaces exist). The evaluation map for $a_1 \times \cdots \times a_n \rightarrow\!\!\!\!\circ\!\!\rightarrow b$ is induced by $ev \circ (c \times id_{A_1 \times \cdots \times A_n})$, where ev is the evaluation morphism for $A_1 \times \cdots \times A_n \rightarrow\!\!\!\!\circ\!\!\rightarrow B$. The abstraction of a partial morphism $f : d \times a_1 \times \cdots \times a_n \rightharpoonup b$ in $\mathbf{K}(\mathbf{C}, \mathcal{M})$, where d is an idempotent on $D \in \mathrm{Ob}\, \mathbf{C}$, is induced by the abstraction of the partial morphism $f : D \times A_1 \times \cdots \times A_n \rightharpoonup B$.

The universal property of $\mathbf{K}(\mathbf{C}, \mathcal{M})$ is established as in the proof of Theorem 37. Given a cauchy-complete pccc $(\mathbf{A}, \mathcal{N})$ and a pcc functor $F : (\mathbf{C}, \mathcal{M}, \mathfrak{A}) \to (\mathbf{A}, \mathcal{N})$, we have to show additionally that $\mathbf{K}(F) : \mathbf{K}(\mathbf{C}, \mathcal{M}) \to \mathbf{K}(\mathbf{A}, \mathcal{N})$ preserves partial function spaces. This is clear by the construction of partial function spaces described above. □

Remark 40 It follows from Theorem 39 and Remark 35 that our deduction system (like the system given in [21]) is *complete for models in pccc's.*

Since pre-pccc's with equality are left exact, Theorem 39 implies

Corollary 41 *For every pre-pccc* $(\mathbf{C}, \mathfrak{A})$ *with equality,* $\mathbf{C}$ *is a pccc with equality, and every pre-pcc functor with domain* $(\mathbf{C}, \mathfrak{A})$ *is a pcc functor (i.e. preserves* all *partial function spaces).*

Combining this with Theorem 34 (iii), we arrive at

Corollary 42 *If* $\mathcal{T}$ *has equality, then* $\mathsf{Sy}(\mathcal{T})$ *is the free pccc* $\mathsf{Cl}(\mathcal{T})$ *over* $\mathcal{T}$*.*

We thus have a restriction of the equivalence between partial λ-theories and pre-pccc's:

partial λ-theories with equality are equivalent to pccc's with equality.

Remark 43 The pccc structure on $\mathsf{Sy}(\mathcal{T})$, for $\mathcal{T}$ with equality, can also be described by means of the internal logic. E.g., the partial function space $(\bar{x} : \bar{s}.\, \phi) \rightarrow\!\!\!\!\circ\!\!\rightarrow (y : t.\, \psi)$ is the object

$$(z : \bar{s} \rightarrow\!\!\!\!\circ\!\!\rightarrow t.\, (\forall \bar{x} : \bar{s}.\ \mathrm{def}\, z\, \bar{x} \Rightarrow \phi \wedge \psi[z\, \bar{x}/y])).$$

Partial function spaces into objects $(\bar{y} : \bar{t}.\, \psi)$ need an additional condition to ensure that all components have the same domain.

Since the partial morphisms between splittings are precisely the induced partial morphisms, the sub-pccc $(\bar{\mathbf{C}}, \bar{\mathcal{M}})$ of $\mathbf{K}(\mathbf{C}, \mathcal{M})$ generated by a pre-pccc $(\mathbf{C}, \mathcal{M}, \mathfrak{A})$ is a full dominional subcategory.

Corollary 44 $(\bar{\mathbf{C}}, \bar{\mathcal{M}})$ *is the pccc reflection of* $(\mathbf{C}, \mathcal{M}, \mathfrak{A})$.

PROOF. Let $F : (\mathbf{C}, \mathcal{M}, \mathfrak{A}) \to (\mathbf{A}, \mathcal{N})$ be a pre-pcc functor into a pccc. By Theorem 39, $\mathbf{K}(F) : \mathbf{K}(\mathbf{C}, \mathcal{M}) \to \mathbf{K}(\mathbf{A}, \mathcal{N})$ is a pcc functor. The preimage of $(\mathbf{A}, \mathcal{N})$ under $\mathbf{K}(F)$ is a sub-pccc, hence contains $(\bar{\mathbf{C}}, \bar{\mathcal{M}})$; thus, $\mathbf{K}(F)$ restricts to the required extension $(\bar{\mathbf{C}}, \bar{\mathcal{M}}) \to (\mathbf{A}, \mathcal{N})$. Uniqueness is clear. □

Thus, pccc's are (2-categorically) reflective in pre-pccc's, with the reflective arrows being both monic and epic. Now let $\mathsf{Cl}(\mathcal{T})$ be the pccc reflection of $\mathsf{Sy}(\mathcal{T})$. Again, we have a unit translation $\eta : \mathcal{T} \to \mathsf{L}(\mathsf{Cl}(\mathcal{T}))$, and a co-unit $E_{(\mathbf{C},\mathcal{M})} : \mathsf{Cl}(\mathsf{L}(\mathbf{C}, \mathcal{M})) \to (\mathbf{C}, \mathcal{M})$ obtained as the extension of the co-unit for $\mathsf{Sy}(\mathsf{L}(\mathbf{C}, \mathcal{M}))$.

Theorem 45 *(i) The extension* η *is conservative.*
(ii) $E_{(\mathbf{C},\mathcal{M})}$ *is a dominional equivalence.*
(iii) $\mathsf{Cl}(\mathcal{T})$ *is the free pccc over* $\mathcal{T}$*.*

PROOF. *(i):* If a $\mathcal{T}$-formula is derivable in $\mathsf{L}(\mathsf{Cl}(\mathcal{T}))$, then it holds in $\mathsf{Cl}(\mathcal{T})$ and thus, by universality, in all pccc-models, hence is derivable in $\mathcal{T}$ by completeness (Remark 40).

(ii): By Theorem 34 (ii), $\mathsf{Sy}(\mathsf{L}(\mathbf{C}, \mathcal{M}))$ is a pccc. Hence, $\mathsf{Sy}(\mathsf{L}(\mathbf{C}, \mathcal{M})) \hookrightarrow \mathsf{Cl}(\mathsf{L}(\mathbf{C}, \mathcal{M}))$ is a dominional equivalence.

(iii): By Theorem 34 and Corollary 44. □

Thus we have arrived at showing that

partial λ-theories are equivalent to pccc's.

This equivalence gives rise to a more involved notion of derived translation than the equivalence of partial λ-theories with pre-pccc's: a derived translation $\mathcal{T}_1 \to \mathcal{T}_2$ maps sorts in $\mathcal{T}_1$ to objects of $\mathsf{Cl}(\mathcal{T}_2)$, which can be described inductively as a class of objects in $\mathbf{K}(\mathsf{Sy}(\mathcal{T}_2))$, similarly for operators.

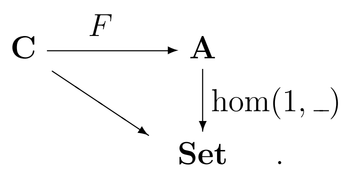
8 Henkin Models

In [4], it is stated that intensional Henkin models of a (typed) total λ-theory are equivalent to models in cartesian closed categories; here, a Henkin model is a set-theoretic model where function types need not be interpreted by full function sets (this is the original idea of [16]), and the word intensional is meant to indicate that two inhabitants of a function type may be distinct although they produce the same output on all inputs. We now proceed to establish a corresponding result for the partial λ-calculus.

In the total case, the equivalence can be stated as follows. Recall that (total) λ-theories are equivalent to cartesian closed categories (ccc's) [19].

Definition 46 A *model* of a ccc $\mathbf{C}$ in a ccc $\mathbf{A}$ is a cartesian closed (cc) functor $F : \mathbf{C} \to \mathbf{A}$. An (intensional) *Henkin model* of $\mathbf{C}$ is a cartesian functor $\mathbf{C} \to \mathbf{Set}$.

This definition is in accordance with the intuition that Henkin models demote higher-order to first-order structure. Henkin models as just defined are easily seen to be the same as (intensional) Henkin models in the original sense (called *syntactic λ-algebras* in [4]). Every ccc model gives rise to a Henkin model by composition with $\hom(1, \_)$, since $\hom(1, \_)$ is a cartesian functor:



Conversely, every Henkin model $G : \mathbf{C} \to \mathbf{Set}$ arises in this way: Because the ccc structure (composition, abstraction etc.) is internalized in $\mathbf{C}$ as an equational theory, one can construct from G a ccc model $G^* : \mathbf{C} \to \mathbf{A}$ by taking the objects of $\mathbf{A}$ to be those of $\mathbf{C}$, and $\hom_{\mathbf{A}}(A, B) = G(A \to B)$, where $A \to B$ denotes the function space. The functor G^* takes a morphism $f : A \to B$ to the element of $G(A \to B)$ determined by its *name* $1 \to (A \to B)$. The property $\hom(1, \_) \circ G^* \cong G$ determines G^* uniquely up to equivalence of the full subcategory spanned by its image.

Establishing a correspodence between set-theoretic models and pccc models via postcomposition with $\hom(1, \_)$ will also be the program for the partial case. The analogy to the total case suggests the following definitions:

Definition 47 A *pccc model* of a pccc $(\mathbf{C}, \mathcal{M})$ is a pcc functor F from $(\mathbf{C}, \mathcal{M})$ into some pccc $(\mathbf{A}, \mathcal{N})$. The model F is called *generated* if it does not factor through a non-equivalent full sub-pccc of $(\mathbf{A}, \mathcal{N})$. An *(intensional) Henkin model* of $(\mathbf{C}, \mathcal{M})$ is a cartesian dominional functor $(\mathbf{C}, \mathcal{M}) \to \mathbf{Set}$. A model in either sense of a partial λ-theory $\mathcal{T}$ is a model of $\mathsf{Cl}(\mathcal{T})$. *Morphisms* of Henkin models are natural transformations.

Remark 48 By Theorem 39, every pcc functor into a pre-pccc gives rise to a pccc model. In the definition of models of a partial λ-theory $\mathcal{T}$, $\mathsf{Cl}(\mathcal{T})$ may, up to equivalence of model categories, be replaced by $\mathsf{Sy}(\mathcal{T})$ — for pccc models, this is just the fact that $\mathsf{Cl}(\mathcal{T})$ is the free pccc over $\mathsf{Sy}(\mathcal{T})$, and for Henkin models, this follows from Corollary 38, since $\mathsf{Cl}(\mathcal{T})$ lives in $\mathbf{K}(\mathsf{Sy}(\mathcal{T}))$.

Henkin models may be described as syntactic λ-algebras modeled on the corresponding notion defined for the total λ-calculus in [4]:

Definition 49 A (partial) *syntactic λ-algebra* M for a partial λ-theory $\mathcal{T}$ consists of

- a set $M[\![s]\!]$ for each *type* s of $\mathcal{T}$, with $M[\![\bar{x} : \bar{s}]\!] := M[\![\bar{s}]\!] := M[\![s_1]\!] \times \cdots \times [\![s_n]\!]$
- a partial function
$$M[\![\Gamma.\,\alpha]\!] : M[\![\Gamma]\!] \to M[\![t]\!]$$
for each term $\Gamma \rhd \alpha : t$ in $\mathcal{T}$, with $M[\![(\alpha_1, \dots, \alpha_n)]\!](\bar{x})$ defined as $(M[\![\alpha_1]\!](\bar{x}), \dots, M[\![\alpha_n]\!](\bar{x}))$.

The interpretation of terms is subject to the following conditions:

(i) $M[\![\Gamma.\,x_i]\!]$, where $\Gamma = (\bar{x} : \bar{s})$, is the i-th projection;
(ii) $M[\![\Delta.\,\gamma]\!] \circ M[\![\Gamma.\,\bar{\beta}]\!] = M[\![\Gamma.\,(\lambda\Delta.\,\gamma)\,\bar{\beta}]\!]$;
(iii) whenever $\Gamma \rhd \operatorname{def} \alpha \vdash \beta \stackrel{e}{=} \gamma$ in $\mathcal{T}$ and $M[\![\Gamma.\,\alpha]\!](\bar{x})$ is defined, then $M[\![\Gamma.\,\beta]\!](\bar{x}) = M[\![\Gamma.\,\gamma]\!](\bar{x})$ are defined.

A *morphism* $h : M \to N$ of syntactic λ-algebras is given by a family of maps

$$h_s : M[\![s]\!] \to N[\![s]\!],$$

indexed over types s, with $h_{(\bar{x}:\bar{s})} := h_{\bar{s}} := h_{s_1} \times \cdots \times h_{s_n}$. These data are subject to the condition that for every term $\Gamma \rhd \alpha : t$,

$$h_t(M[\![\Gamma.\,\alpha]\!](\bar{x})) = N[\![\Gamma.\,\alpha]\!](h_\Gamma(\bar{x}))$$

whenever $M[\![\Gamma.\,\alpha]\!](\bar{x})$ is defined.

This is the definition of model used in the semantics of HasCasl given in [33].

Remark 50 It is implicit in [22] that syntactic λ-algebras are equivalent to the combinatorically defined λ_p-*algebras* considered there.

Remark 51 Most of the structure of a syntactic λ-algebra is implicit in Condition (iii) of Definition 49, which implies e.g. that models respect β etc.

Theorem 52 *The category of Henkin models of a partial λ-theory $\mathcal{T}$ is equivalent to the category of syntactic λ-algebras for $\mathcal{T}$.*

PROOF. We associate to $\mathcal{T}$ a partial equational theory (cf. Section 3) $\mathsf{fo}(\mathcal{T})$ which has the types of $\mathcal{T}$ as sorts, and for every term $\Gamma \rhd \alpha : t$ in $\mathcal{T}$ an operator $(\Gamma.\,\alpha) : \bar{s} \rightharpoonup t$, where $\Gamma = (\bar{x} : \bar{s})$. Every term β in $\mathsf{fo}(\mathcal{T})$ can be collapsed into a term $\mu(\beta)$ in $\mathcal{T}$ by replacing operators $(\Gamma.\,\alpha)$ with applications of $\lambda\Gamma.\,\alpha$. The axioms of $\mathsf{fo}(\mathcal{T})$ are the sentences that collapse into theorems of $\mathcal{T}$.

A syntactic λ-algebra M is, then, just a model $\bar{M}$ of $\mathsf{fo}(\mathcal{T})$: validity of the axioms of $\mathsf{fo}(\mathcal{T})$ follows from Condition (iii) in Definition 49 and the fact, proved by induction over α using Conditions (i) and (ii), that

$$\bar{M}[\![\Gamma.\,\alpha]\!] = M[\![\Gamma.\,\mu(\alpha)]\!].$$

By Theorem 14, this means that syntactic λ-algebras correspond to cartesian dominional functors $\mathsf{Cl}(\mathsf{fo}(\mathcal{T})) \to \mathbf{Set}$, where Cl denotes the classifying cartesian dominional category. Now the functor $G : \mathsf{Cl}(\mathsf{fo}(\mathcal{T})) \to \mathsf{Sy}(\mathcal{T})$ that maps objects $(\Gamma.\ \mathrm{def}\,\bar{\alpha})$ to $(\Gamma.\ \mathrm{def}\,\mu(\bar{\alpha}))$ and morphisms $\bar{\alpha}$ to $\mu(\bar{\alpha})$ is an equivalence; well-definedness of G follows from the fact, shown by induction over derivations, that every theorem of $\mathsf{fo}(\mathcal{T})$ collapses into a theorem of $\mathcal{T}$. By Remark 48, composition with G maps syntactic λ-algebras to Henkin models; it is clear how this extends to an equivalence of categories. □

In analogy with the total case, a pccc model $F : (\mathbf{C}, \mathcal{M}) \to (\mathbf{A}, \mathcal{N})$ induces a Henkin model $\hom(1, \_) \circ F$, since $\hom(1, \_) : (\mathbf{A}, \mathcal{N}) \to \mathbf{Set}$ is cartesian and dominional. Conversely, we will construct a model F^* of $(\mathbf{C}, \mathcal{M})$ in a pre-pccc $\mathsf{ho}(F)$ from a Henkin model F in a similar manner as $\mathsf{Sy}(\_)$.

We have a number of operations on F induced by morphisms in $(\mathbf{C}, \mathcal{M})$, exploiting that $(\mathbf{C}, \mathcal{M})$ is equivalent to $\mathsf{Cl}(\mathsf{L}(\mathbf{C}, \mathcal{M}))$. All equations claimed below are proved according to the same pattern: computation in $\mathsf{L}(\mathbf{C}, \mathcal{M})$ shows that the equation holds in $(\mathbf{C}, \mathcal{M})$; since F is cartesian, it then follows that the equation holds in F. We use the following notation: Every provably defined closed term α of sort A in $\mathsf{L}(\mathbf{C}, \mathcal{M})$ determines a morphism $f : 1 \to A$; the map Ff defines an element of FA which we denote by $F\alpha$.

- For objects A, B, C, the morphism $f, g \rhd \lambda x.\, g\,(f\,x)$ in $\mathbf{C}$ is mapped by F to an associative *composition* map

  $$\_ \circ_p \_ : F(B \multimap C) \times F(A \multimap B) \to F(A \multimap C),$$

  with *identities* given by $id_A = F(\lambda x.x) \in F(A \multimap A)$.

- From the morphism $f \triangleright \lambda x.\, x \upharpoonright f\, x$, we obtain a map

$$res : F(A \multimap B) \to F(A \multimap A)$$

(corresponding to a restriction structure, cf. Remark 6). We put $f \upharpoonright g := f \circ_p res(g)$. This structure satisfies the equations $res(f) \circ_p g = g \upharpoonright (f \circ_p g)$ and $res(f) \upharpoonright g = res(f \upharpoonright g)$. The restriction $\sqcap$ of $\upharpoonright$ to $F(A \multimap 1)$ makes $F(A \multimap 1)$ into a meet semilattice with top element $! = F(\lambda x.\, *)$; the induced order is denoted by $\sqsubseteq$. We put $dom(f) = ! \upharpoonright f$. Right composition $\_ \circ_p f$ preserves $\sqcap$ and hence is monotonic w.r.t. $\sqsubseteq$.
- For objects A, B, we have *projections* $fst = F(\lambda x, y.\, x) \in F(A \times B \multimap A)$, $snd = F(\lambda x, y.\, y) \in F(A \times B \multimap B)$, as well as a *pairing* function

$$\langle \_, \_ \rangle : F(C \multimap A) \times F(C \multimap B) \to F(C \multimap A \times B)$$

arising from the morphism $f, g \triangleright \lambda x.\, (f\, x, g\, x)$. For $f \in F(A \multimap B)$ and $g \in F(C \multimap D)$, we put $f \times g = \langle f \circ_p fst, g \circ_p snd \rangle$. Projections and pairing are interrelated by the equations $\langle fst, snd \rangle = id$, $\langle f, g \rangle \circ_p h = \langle f \circ_p h, g \circ_p h \rangle$, and $fst \circ_p \langle f, g \rangle = f \upharpoonright g$, correspondingly for snd. Moreover, $res\langle f, g \rangle = res(f) \circ_p res(g)$, and hence $dom\langle f, g \rangle = dom(f) \sqcap dom(g)$.
- For objects A, B, we have an *application operator* $ap = F(\lambda f, x.\, f\, x) \in F((A \multimap B) \times A \multimap B)$. The morphism $f \triangleright \lambda x.\, \lambda y.\, f\,(x, y)$ induces an *abstraction* map

$$\Lambda : F(A \times B \multimap C) \to F(A \multimap B \multimap C).$$

$\Lambda(f)$ is total, i.e. $dom(\Lambda(f)) = !$. Application and abstraction are interrelated by the equations

$$ap \circ_p (\Lambda(f) \times id_A) = f \quad \text{and}$$
$$\Lambda(ap \circ_p (f \times id_A)) \upharpoonright f = f.$$

Note that the equations appearing above are strongly related to equational presentations of partial structure given e.g. in [9,13]. We make no attempt here to distinguish a minimal set of equational axioms.

Using these operations, we construct $F^* : (\mathbf{C}, \mathcal{M}) \to \mathsf{ho}(F)$ as follows: The objects of $\mathsf{ho}(F)$ are pairs $(A.\, m)$, where A is a $\mathbf{C}$-object, and $m \in F(A \multimap 1)$. Morphisms $(A.\, m) \to (B.\, n)$ are elements $f \in F(A \multimap B)$ such that

$$m \sqsubseteq n \circ_p f,$$

taken modulo $f_1 \sim f_2 \iff f_1 \upharpoonright m = f_2 \upharpoonright m$. It follows that $m \sqsubseteq dom(f)$. Thus, $f \sim f \upharpoonright m$, so that we can assume $m = dom(f)$ when convenient. Composition is via $\circ_p$, and the identity on $(A.\, m)$ is id_A. The dominion on $\mathsf{ho}(F)$ is generated as in Lemma 8 by the predominion consisting of all morphisms

$$id_A : (A.\, m) \to (A.\, n).$$

The type objects are the objects of the form $(A.!)$, written shortly as (A). We put $F^*(A) = (A)$ and $F^*(f) = F(\lambda x.\, f(x))$.

Proposition 53 *The above defines a pre-pccc model* $F^* : (\mathbf{C}, \mathcal{M}) \to \mathsf{ho}(F)$*.*

PROOF. Using properties listed above, it is straightforward to show that $\mathsf{ho}(F)$ is a category. The pullback of an admissible subobject $(B.\,p) \hookrightarrow (B.\,n)$ along $f : (A.\,m) \to (B.\,n)$ is $(A.\,(p \circ_p f) \sqcap m)$. The terminal object is (1). The product of $(A.\,m)$ and $(B.\,n)$ is $(A \times B.\,(m \circ_p \mathit{fst}) \sqcap (n \circ_p \mathit{snd}))$; note that the class of type objects is closed under cartesian products. The partial function space for type objects (A) and (B) is $(A \mathrel{-\!\!\circ\!\!\rightarrow} B)$, evaluation being the partial morphism given by ap. The abstraction of a partial morphism $(C.\,m) \times (A) \hookleftarrow (C \times A.\,\mathit{dom}(f)) \xrightarrow{f} (B)$ is $\Lambda(f)$. The preservation of the structure of $(\mathbf{C}, \mathcal{M})$ by F^* amounts to equations in F involving only closed terms in $\mathsf{L}(\mathbf{C}, \mathcal{M})$; all these equations hold in $(\mathbf{C}, \mathcal{M})$ and are preserved by the cartesian functor F. □

By Remark 48, we thus obtain a pccc model of $(\mathbf{C}, \mathcal{M})$, which we also denote by F^*. So far, we have used only that F is cartesian. Establishing that F^* actually reproduces F requires for the first time that F is dominional.

Theorem 54 *For the model* F^* *constructed above,*

$$\hom(1, \_) \circ F^* \cong F.$$

This determines F^* *essentially uniquely as a generated pccc model.*

In other words,

Henkin models are equivalent to generated pccc-models.

PROOF (Theorem 54). For a $\mathbf{C}$-object A, the pullback

$$\begin{array}{ccc}
(x : A) \cong (x : 1 \mathrel{-\!\!\circ\!\!\rightarrow} A.\ \mathrm{def}\, x\, *) & \hookrightarrow & (x : 1 \mathrel{-\!\!\circ\!\!\rightarrow} A) \\
\downarrow & & \downarrow{\scriptstyle \lambda.\ \mathrm{def}\, x\, *} \\
1 & \xrightarrow[\top]{} & \Omega
\end{array}$$

in $(\mathbf{C}, \mathcal{M})$ is preserved by F since F is dominional. Since we have equations $F(\lambda.\ \mathrm{def}\, x\, *)(y) = {!} \circ_p y$ and $(F\top)\, * = {!} \in F(1 \mathrel{-\!\!\circ\!\!\rightarrow} 1)$, this means that

$$FA \cong \{y \in F(1 \mathrel{-\!\!\circ\!\!\rightarrow} A) \mid {!} \circ_p y = {!}\} = \hom(1, F^*A).$$

Uniqueness follows immediately from the fact that the set $\hom(1, F(A \mathrel{-\!\!\circ\!\!\rightarrow} B))$ determines the partial morphisms from F^*A to F^*B. □

Remark 55 The equivalence between Henkin models and pccc models can be extended to morphisms. To begin, this raises the question of what a morphism of pccc models really is. The most natural guess for a notion of morphism between pccc models $F : (\mathbf{C}, \mathcal{M}) \to (\mathbf{A}, \mathcal{N})$ and $G : (\mathbf{C}, \mathcal{M}) \to (\mathbf{B}, \mathcal{P})$ is to take pairs (Φ, μ) consisting of a pcc functor $\Phi : (\mathbf{A}, \mathcal{N}) \to (\mathbf{B}, \mathcal{P})$ and a natural transformation $\mu : \Phi \circ F \to G$. When F and G are generated, this notion can be subjected to Occam's razor: generally, (Φ, μ) can be factorized into $(\Phi, id) : F \to \Phi \circ F$ and $(id, \mu) : \Phi \circ F \to G$. But morphisms of the form $(id, \mu) : \Phi \circ F \to G$ are easily seen to be in one-to-one correspondence with morphisms of the form (Ψ, id), essentially because partial morphisms $A \rightharpoonup B$ are in bijection with global elements of $A \mathrel{-\!\!\ominus\!\!\to} B$. We shall thus define a morphism $F \to G$ to be just a pcc functor Φ such that $\Phi \circ F = G$.

With this definition, we extend the correspondence between generated pccc models and Henkin models to a functorial equivalence: a morphism Φ as above is taken to the natural transformation $\hom(1, F\_) \to \hom(1, \Phi F\_)$ given by the action of Φ on morphisms (note that Φ preserves the terminal object). Conversely, from a morphism $\nu : H \to K$ of Henkin models, we obtain a morphism $\Phi : H^* \to K^*$ which takes objects $(A.\, m)$ of $\mathsf{ho}(F)$ to $(A.\, \nu_{A \mathrel{-\!\ominus\!\to} 1}(m))$, and morphisms $f : (A.\, m) \to (B.\, n)$ to $\nu_{A \mathrel{-\!\ominus\!\to} B}(f)$.

9 Unique Choice

We now investigate the internal logic of partial λ-theories with equality, exploiting the equivalence with pccc's. This section summarizes material from [31], where these matters are treated more thoroughly. Note that by the term internal logic, we always refer to the logic defined in Section 5; in categorical terms, this means that formulae are interpreted in the hyperdoctrine of *regular* subobjects rather than in the hyperdoctrine of arbitrary subobjects.

It is at first sight slightly puzzling that pccc's with equality are equivalent to intuitionistic HOL, although the latter is more commonly associated with toposes; see e.g. [19], where toposes are constructed from type theories that can be translated into the partial λ-calculus with equality. Pccc's with equality are weaker than quasitoposes [1], which in turn are far more general than toposes — e.g., there are non-trivial quasitoposes in topology [41], such as the category of pseudotopological spaces, while the only topos which is at the same time a topological category over **Set** is **Set** itself. It turns out that the crucial point here is unique choice.

For the remainder of this section, let $\mathbf{C}$ be a pccc with equality. In the following, we shall drop the distinction between $\mathbf{C}$ and $\mathsf{Cl}(\mathsf{L}(\mathbf{C}))$ altogether, omitting in particular semantic brackets $[\![\_]\!]$. To begin, we note that a morphism in $\mathbf{C}$

is a monomorphism (epimorphism) iff it is internally injective (surjective):

Lemma 56 *A morphism* $f : A \to B$ *in* **C** *is*

(i) monomorphic iff $\forall x, y : A.\, f(x) \stackrel{e}{=} f(y) \Rightarrow x \stackrel{e}{=} y$*, and*
(ii) epimorphic iff $\forall y : B.\, \exists x : A.\, f(x) \stackrel{e}{=} y$*.*

(Where by stating the formulae we mean that they hold in **C** as formulae of the internal logic.)

PROOF. *(i):* Sufficiency: given morphisms g, h such that $fg = fh$, i.e. $y \rhd f(g(y) \stackrel{e}{=} f(h(y)$, the formula allows us to conclude $y \rhd g(x) \stackrel{e}{=} h(x)$. Necessity: apply monomorphicity to the pair of morphisms $x, y : (x, y : A.\, f(x) \stackrel{e}{=} f(y)) \to A$.

(ii): Sufficiency: similarly as above. Necessity: apply epimorphicity to the pair of morphisms $\top, (\exists x : A.\, f(x) \stackrel{e}{=} y) : (y : B) \to \Omega$. □

In particular, the factorization of $f : A \to B$ through its *image*

$$\operatorname{Im} f := (b : B.\, \exists a : A.\, f(a) = b)$$

is an (epi, regular mono)-factorization (the inclusion $\operatorname{Im} f \hookrightarrow B$ is a regular mono, being an admissible subobject). Hence, *all extremal monomorphisms in* **C** *are regular* (a mono m is *extremal* if $m = ge$, where e is epi, implies that e is iso). This implies that **C** automatically satisfies condition 19.1.1. in [41]. Moreover, we can reprove a result of [27] for the special case of pccc's with equality (the original result requires only left exactness):

Proposition 57 **C** *is cartesian closed.*

PROOF. The function space $A \to B$ is $(f : A \multimap B.\, \forall x : A.\ \mathrm{def}\, f(x))$. □

Thus, pcccs with equality are characterized as *cartesian closed categories with classifiers for extremal partial morphisms.* In particular, quasitoposes are precisely the finitely cocomplete pcccs with equality.

For each object A, let $sg_A : A \to Sg(A)$ denote the image factorization of the morphism $\lambda b : A.\, b \stackrel{e}{=} a : (a : A) \to (A \multimap 1)$. The object $Sg(A)$ is the type of singleton subsets of A.

Proposition 58 *Let* A *be an object in* **C**. *Then the following are equivalent:*

(i) sg_A *is an isomorphism.*

(ii) Every partial functional relation with codomain A *is a partial function; i.e. for each* B *in* **C***,* **C** *satisfies*

$$\forall R : BA \multimap 1.\, (\forall x : B, y, z : A.\, R(x,y) \wedge R(x,z) \Rightarrow y \overset{e}{=} z) \implies (\exists f : B \multimap A.\, \forall x : B, y : A.\, f(x) \overset{e}{=} y \;\Leftrightarrow R(x,y)).$$

(iii) Every monomorphism with domain A *is extremal.*

An object that satisfies the equivalent conditions above is called *coarse*, following [24,41], where Conditions (i) and (iii) are used. Condition (ii) (usually formulated in terms of total functions) is often referred to as *unique choice*. An inverse of sg_A can be regarded as a partial morphism $(A \multimap 1) \rightharpoonup A$. Thus, we can define the *unique description operator* by

$$\iota a : A.\, \phi := sg_A^{-1}(\lambda a : A.\, \phi)$$

for a formula $a : A \rhd \phi$ — i.e. $\iota a : A.\, \phi$ is the unique element of A satisfying ϕ, if such an element exists uniquely, and is otherwise undefined.

PROOF. *(i)* $\Rightarrow$ *(ii):* The required f is $\lambda x : B.\, \iota y : A.\, R(x,y)$.

(ii) $\Rightarrow$ *(iii):* Let $f : A \to B$ be a monomorphism. By Lemma 56, the relation

$$\lambda x : B, y : A.\, x \overset{e}{=} f(y)$$

is a partial functional relation. The associated partial function $g : B \rightharpoonup A$ induces an inverse of $f : A \to \mathrm{Im}\, f$.

(iii) $\Rightarrow$ *(i):* sg_A is mono and epi. □

By Proposition 58 (iii), we have

Proposition 59 *Toposes are precisely the pccc's with equality in which every object is coarse.* □

Lemma 60 *The class of coarse objects is closed under finite products and admissible subobjects in* **C***.* □

Toposes can be axiomatized in the partial λ-calculus:

Theorem 61 *For a partial* λ*-theory* $\mathcal{T}$ *with equality,* $\mathsf{Cl}(\mathcal{T})$ *is a topos iff* $\mathcal{T}$ *implies unique choice for all types.*

PROOF. By Proposition 59 and Lemma 60, $\mathsf{Cl}(\mathcal{T})$ is a topos iff the object $(x : s)$ is coarse for each type s. Necessity is thereby immediate; sufficiency follows by Proposition 58 (i), which needs unique choice only on types. $\square$

Thus, the question of the categorical correlate of intuitionistic HOL (with equality) may be resolved as follows:

Pccc's with equality are intuitionistic HOL;
toposes are intuitionistic HOL with unique description.

In other words, the logic of pccc's with equality differs from topos logic in taking functions rather than subsets as the primitive notion. In the topos construction of [19], unique choice is implicit: the morphisms in the classifying topos *are* functional relations; in the same way, one can construct a topos from $\mathbf{C}$ (i.e. from a partial λ-theory with equality). An alternative way of obtaining an equivalent topos is the following observation.

Theorem 62 *The subcategory* $\mathbf{Ind}(\mathbf{C})$ *of coarse objects is reflective in* $\mathbf{C}$*, with reflective arrows* sg_A *(in particular, the reflective arrows are both monic and epic). Moreover,* $\mathbf{Ind}(\mathbf{C})$ *is sub-pccc of* $\mathbf{C}$*, hence a topos; as such, it is the topos coreflection of* $\mathbf{C}$*, i.e. every pcc functor* $\mathbf{E} \to \mathbf{C}$*, with* $\mathbf{E}$ *a topos, factors through* $\mathbf{Ind}(\mathbf{C})$*.*

(The first part of this theorem, up to the fact that $\mathbf{Ind}(\mathbf{C})$ is a topos, slightly generalizes results of [24,41].)

PROOF. By Lemma 60, $\mathbf{Ind}(\mathbf{C})$ is a full cartesian dominional subcategory. Objects $Sg(A)$ are coarse: the inverse of $sg_{Sg(A)}$ is the intersection operator

$$x : (A \multimap 1) \multimap 1 \rhd \lambda y : A.\, \forall z : Sg(A).\, x\, z \Rightarrow z\, y.$$

The arrow $sg_A : A \to Sg(A)$ is reflective: the unique extension $Sg(A) \to B$ of a morphism $f : A \to B$, with B coarse, is given by the term

$$y : A \multimap 1 \rhd \iota b : B.\, \forall a : A.\, x\, a \Rightarrow f(a) \stackrel{e}{=} b.$$

The reflector Sg is easily seen to be cartesian, so that $\mathbf{Ind}(\mathbf{C})$ is a cartesian closed subcategory [1]. Finally, Ω is shown to be coarse analogously as $Sg(A)$. Thus, $\mathbf{Ind}(\mathbf{C})$ is a topos. The embedding $\mathbf{Ind}(\mathbf{C}) \hookrightarrow \mathbf{C}$ is a coreflective arrow because pcc functors preserve the internal logic, hence also the type constructor Sg and thus, by Proposition 58 (i), coarse objects. $\square$

The topos $\mathbf{Ind}(\mathbf{C})$ is equivalent to the topos of functional relations over $\mathbf{C}$ because sg_A becomes an isomorphism when regarded as a functional relation.

Remark 63 The reflector $Sg : \mathbf{C} \to \mathbf{Ind}(\mathbf{C})$ can moreover be shown to be a (faithful) fibration. It may be hoped that this property and similar ones will eventually lead to a classification theorem for pccc's with equality/quasitoposes as concrete categories over toposes.

Remark 64 In [19] it is stated that the extension of a type theory to the internal language of its classifying topos, constructed as the topos of functional relations, is conservative. The type theory used in loc. cit. can be regarded as a sublanguage of the partial λ-calculus with equality. The fact that the latter does not prove unique choice does not contradict the mentioned conservativity result; this means merely that in a partial λ-theory with equality, types of functional relations need not coincide with partial function types.

A consequence of Theorem 61 is that results based on the interplay of partial λ-theories and pccc's apply also to toposes. This includes the results of Section 8, which imply that topos models (logical morphisms) of a topos $\mathbf{E}$ are equivalent to Henkin models, i.e. lex functors $\mathbf{E} \to \mathbf{Set}$.

10 Applications

The semantics of HasCasl is defined in terms of syntactic λ-algebras, i.e. Henkin models. By the results of Section 8, this semantics profits from the 'best of both worlds'. Henkin models, on the one hand, provide a tight connection with set-theoretic models as used in typical first-order specification languages such as Casl. E.g., any partial equational theory can be regarded as a partial λ-theory. Under this identification, we have

Theorem 65 *Every* **Set***-valued model of a partial equational theory $\mathcal{T}$ has a persistent free extension to a Henkin model of $\mathcal{T}$ qua partial λ-theory.*

This implies that every Casl model extends canonically to a HasCasl model. The proof uses the following fact, which is just a special case of the well-known corresponding statement for positive Horn theories:

Proposition 66 **Set***-valued models have free extensions along translations of partial equational theories.*

This applies in particular to translations $\mathsf{fo}(\sigma) : \mathsf{fo}(\mathcal{T}_1) \to \mathsf{fo}(\mathcal{T}_2)$ arising from a translation $\sigma : \mathcal{T}_1 \to \mathcal{T}_2$ between partial λ-theories, where $\mathsf{fo}(\mathcal{T})$ is the first-order language associated to $\mathcal{T}$ as in the proof of Theorem 52; i.e. *Henkin models have free extensions along translations of partial λ-theories.*

PROOF (Theorem 65). The existence of the free extension $\sigma^{\#}$ of a model σ follows from Proposition 66, applied to the extension

$$\mathcal{T} \to \mathsf{L}_{\mathsf{fo}}(\mathsf{Cl}(\mathcal{T}))$$

(where $\mathsf{Cl}(\mathcal{T})$ is the classifying pccc, and L_{fo} is the first-order internal language defined in Section 3). Persistence amounts to invertibility of the unit. The unit is injective because σ embeds into a standard model, i.e. a **Set**-valued pccc model of $\mathsf{Cl}(\mathcal{T})$, obtained by observing that $\mathsf{Cl}(\mathcal{T})$ is the free pccc over the *first-order* (i.e. cartesian dominional) classifying category of $\mathcal{T}$.

To prove surjectivity, we have to show that every higher-order term $\Gamma \rhd \alpha : s$ in $\mathcal{T}$, with s and the types appearing in Γ being *sorts*, is equivalent to a first-order term. We can assume that α is β-normal, regarding conditioned terms $\_ \upharpoonright \_$ as a separate syntactic entity (cf. Appendix A). If α is a variable, a conditioned term, or an application of a first-order operator, we are done by induction. Since s is a sort, the only remaining case is that α is an application of a functional value. However, because all β-normal terms of functional type in context Γ are λ-abstractions, this would contradict β-normality. $\square$

A further pleasant aspect of Henkin models is that by Lemma 20, the category $\mathbf{Mod}(\mathbf{C})$ of Henkin models of a pccc $\mathbf{C}$ with equality is the category of lex **Set**-valued functors on $\mathbf{C}$. In particular, $\mathbf{Mod}(\mathbf{C})$ is locally finitely presentable, with the finitely presentable objects being precisely the representable functors [20]; one consequence of this is that the (contravariant) model functor $\lambda\mathbf{C}.\,\mathbf{Mod}(\mathbf{C})$ reflects equivalences [37]. Moreover, $\mathbf{Mod}(\mathbf{C})$ has the *weak amalgamation property*, i.e. takes pushouts of theories to weak pullbacks of model categories. This follows from Proposition 66 and the fact that the model functor for lex categories takes pushouts to actual pullbacks.

On the other hand, we can use categorical machinery for reasoning about, and in particular for constructing models. E.g., the domain equation

$$L \cong L \multimap L \qquad (*)$$

axiomatizing the untyped partial λ-calculus can be solved in a suitable category of domains, and this solution induces a Henkin model. Nontrivial pccc's solving $(*)$ necessarily fail to have equality, since the arising internal logic would otherwise lead to a Russell-type paradox. The domain equation $L \cong L \to L$ axiomatizing the untyped *total* λ-calculus can be solved in the presence of equality, e.g. in a presheaf topos [40].

11 Conclusion

We have proved the equivalence of the categorical semantics of the partial λ-calculus given in terms of models in partial cartesian closed categories (pccc's) and a set-theoretic semantics based on Henkin models and syntactic λ-algebras, respectively. This work forms the backbone of the HasCasl semantics. Applications to the model theory of HasCasl include a persistent extension result for the embedding of Casl into HasCasl (i.e. of first-order theories into higher-order theories). Moreover, this result reconciles the mostly set-theoretic viewpoint of mainstream logic and algebraic specification with categorical logic and semantics.

The equivalence result for Henkin models builds on equivalence results between partial theories on the one hand and dominional categories on the other hand. In particular, partial equational theories are equivalent to cartesian dominional categories, and partial λ-theories are equivalent to pccc's. The latter results are of interest in their own right in that they are useful both in the analysis of the involved categories and in the study of the interrelations of logical theories. In particular, a categorical representation of signatures and theories is helpful in connection with colimit constructions for purposes of structured specification; for first-order Casl, this has been illustrated by the solution for the amalgamation problem given in [37]. Here, we have applied the equivalence result for partial λ-theories with equality, which encode a full intuitionistic higher-order logic, to prove that toposes are logically characterized among the pccc's with equality by the unique choice axiom (see [35] for an application of local unique choice). Giving a similar description of the logic of quasitoposes is the subject of future research.

Acknowledgements

The author wishes to thank Till Mossakowski, Christian Maeder, and Kathrin Hoffmann for collaboration on HasCasl, Christoph Lüth for useful discussions, and the anonymous referees for helpful comments. In particular, the second referee suggested an improved construction of the free pccc over a pre-pccc, notably by pointing out Theorem 36.

References

[1] J. Adámek, H. Herrlich, and G. E. Strecker, *Abstract and concrete categories*, Wiley Interscience, 1990.

[2] J. Adámek and H.-E. Porst, *Algebraic theories of quasivarieties*, J. Algebra **208** (1998), 379–398.

[3] M. Bidoit and P. D. Mosses, Casl *user manual*, Lect. Notes Comput. Sci., vol. 2900, Springer, 2003.

[4] V. Breazu-Tannen and A. R. Meyer, *Lambda calculus with constrained types*, Logic of Programs (R. Parikh, ed.), Lect. Notes Comput. Sci., vol. 193, Springer, 1985, pp. 23–40.

[5] A. Bucalo, C. Führmann, and A. Simpson, *Equational lifting monads*, Theoret. Comput. Sci. **294** (2003), 31–60.

[6] P. Burmeister, *Partial algebras — survey of a unifying approach towards a two-valued model theory for partial algebras*, Algebra Universalis **15** (1982), 306–358.

[7] A. Carboni, *Bicategories of partial maps*, Cahier Topol. Géom. Diff. **28** (1987), 111–126.

[8] I. Claßen, M. Große-Rhode, and U. Wolter, *Categorical concepts for parametrized partial specifications*, Math. Struct. Comput. Sci. **5** (1995), 153–188.

[9] R. Cockett and S. Lack, *Restriction categories I: categories of partial maps*, Theoret. Comput. Sci. **270** (2002), 223–259.

[10] ______, *Restriction categories II: partial map classification*, Theoret. Comput. Sci. **294** (2003), 61–102.

[11] The Common Framework Initiative (CoFI), http://www.cofi.info.

[12] A. Corradini and F. Gadducci, *A functorial semantics for multi-algebras and partial algebras, with applications to syntax*, Theoret. Comput. Sci. **286** (2002), 293–322.

[13] P.-L. Curien and A. Obtułowicz, *Partiality, cartesian closedness and toposes*, Inform. and Comput. **80** (1989), no. 1, 50–95.

[14] M. Fiore, *Axiomatic domain theory in categories of partial maps*, Cambridge University Press, 1996.

[15] M. Fiore and G. Plotkin, *An extension of models of axiomatic domain theory to models of synthetic domain theory*, Computer Science Logic, CSL 97 (D. van Dalen and M. Bezem, eds.), Lect. Notes Comput. Sci., vol. 1258, Springer, 1997, pp. 129–149.

[16] L. Henkin, *Completeness in the theory of types*, J. Symbolic Logic **15** (1950), 81–91.

[17] H. J. Hoehnke, *On partial recursive definitions and programs*, Fundamentals of Computation Theory, FCT 77 (M. Karpiński, ed.), Lect. Notes Comput. Sci., vol. 56, Springer, 1977, pp. 260–274.

[18] B. Jacobs, *Semantics of the second order lambda calculus*, Math. Struct. Comput. Sci. **1** (1991), 327–360.

[19] J. Lambek and P. J. Scott, *Introduction to higher order categorical logic*, Cambridge University Press, 1986.

[20] M. Makkai and A. M. Pitts, *Some results on locally finitely presentable categories*, Trans. Amer. Math. Soc. **299** (1987), 473–496.

[21] E. Moggi, *Categories of partial morphisms and the λ_p-calculus*, Category Theory and Computer Programming (D. Pitt, S. Abramsky, A. Poigné, and D. Rydeheard, eds.), Lect. Notes Comput. Sci., vol. 240, Springer, 1986, pp. 242–251.

[22] ———, *The partial lambda calculus*, Ph.D. thesis, University of Edinburgh, 1988.

[23] P. D. Mosses (ed.), CASL *reference manual*, Lect. Notes Comput. Sci., vol. 2960, Springer, 2004.

[24] J. Penon, *Quasi-topos*, C. R. Acad. Sci., Paris, Sér. A **276** (1973), 237–240.

[25] F. Regensburger, *HOLCF: Higher order logic of computable functions*, Theorem Proving in Higher Order Logics, TPHOLS 95 (E. Schubert, P. Windley, and J. Alves-Foss, eds.), Lect. Notes Comput. Sci., vol. 971, 1995, pp. 293–307.

[26] E. Robinson and G. Rosolini, *Categories of partial maps*, Inform. and Comput. **79** (1988), 95–130.

[27] L. Román, *On partial cartesian closed categories*, Categories in computer science and logic, Contemp. Math., vol. 92, Amer. Math. Soc., 1989, pp. 349–356.

[28] G. Rosolini, *Continuity and effectiveness in topoi*, Ph.D. thesis, Merton College, Oxford, 1986.

[29] L. Schröder, *Classifying categories for partial equational logic*, Category Theory and Computer Science, CTCS 02 (R. Blute and P. Selinger, eds.), Electron. Notes Theoret. Comput. Sci., vol. 69, Elsevier, 2002.

[30] ———, *Henkin models of the partial λ-calculus*, Computer Science Logic, CSL 03 (M. Baaz and J. Makowsky, eds.), Lect. Notes Comput. Sci., vol. 2803, Springer, 2003, pp. 498–512.

[31] ———, *The logic of the partial λ-calculus with equality*, Computer Science Logic, CSL 04 (J. Marcinkowski and A. Tarlecki, eds.), Lect. Notes Comput. Sci., vol. 3210, Springer, 2004, pp. 385–399.

[32] L. Schröder and T. Mossakowski, HASCASL – *Integrated functional specification and programming*, in preparation.

[33] ———, HASCASL*: Towards integrated specification and development of functional programs*, Algebraic Methodology And Software Technology, AMAST 02 (H. Kirchner and C. Ringeissen, eds.), Lect. Notes Comput. Sci., vol. 2422, Springer, 2002, pp. 99–116.

[34] ———, *Monad-independent Hoare logic in* HasCasl, Fundamental Aspects of Software Engineering, FASE 03 (M. Pezzè, ed.), Lect. Notes Comput. Sci., vol. 2621, 2003, pp. 261–277.

[35] ———, *Monad-independent dynamic logic in* HasCasl, J. Logic Comput. **14** (2004), 571–619.

[36] L. Schröder, T. Mossakowski, and C. Maeder, HasCasl – *Integrated functional specification and programming. Language summary*, available at `http://www.informatik.uni-bremen.de/agbkb/forschung/formal_methods/CoFI/HasCASL`, 2003.

[37] L. Schröder, T. Mossakowski, A. Tarlecki, P. Hoffman, and B. Klin, *Amalgamation in the semantics of* Casl, Theoret. Comput. Sci. **331** (2005), 215–247.

[38] D. Scott, *Continuous lattices*, Toposes, Algebraic Geometry and Logic, Lect. Notes Math., vol. 274, Springer, 1972, pp. 97–136.

[39] ———, *Identity and existence in intuitionistic logic*, Applications of Sheaves, Lect. Notes Math., vol. 753, Springer, 1979, pp. 660–696.

[40] ———, *Relating theories of the λ-calculus*, To H. B. Curry: Essays in Combinatory Logic, Lambda Calculus and Formalisms (J. Seldin and J. Hindley, eds.), Academic Press, 1980, pp. 403–450.

[41] O. Wyler, *Lecture notes on topoi and quasitopoi*, World Scientific, 1991.

A Existence of β-normal Forms

In the partial λ-calculus, β-reduction takes the form

$$(\lambda\bar{x}.\,\alpha)\bar{\beta} \to^p_\beta \alpha[\bar{\beta}/\bar{x}] \restriction \bar{\beta},$$

where the two sides are strongly equal (strong equalities can be applied within the scope of a λ-abstraction). Here, we have to treat conditioned terms *syntactically* as a primitive feature, since otherwise (i.e. with the coding $\alpha \restriction \beta \equiv (\lambda x, y.\,x)(\alpha, \beta)$), e.g. the term $(\lambda x, y.\,x)(\alpha, \beta)$ would β-reduce infinitely. We conjecture that strong normalization holds w.r.t. this reduction; here, we need only that each term has a β-normal form, which is easily proved by quoting normalization for the simply typed total λ-calculus as follows.

The set $TFV(\bar{\alpha})$ of *top level free variables* of a (multi-)term $\bar{\alpha}$ is defined by

$$\begin{aligned} TFV(x) &= \{x\} \\ TFV(\bar{\alpha}) &= \textstyle\bigcup_i TFV(\alpha_i) \\ TFV(f(\bar{\alpha})) &= TFV(\bar{\alpha}) \\ TFV(\lambda\bar{x}.\,\alpha) &= \emptyset; \end{aligned}$$

in other words, the top level free variables of α are those that appear outside the scope of λ-abstractions. If, for any subterm of α of the form $\lambda\bar{x}.\,\beta$, the variables $\bar{x}$ are in $TFV(\beta)$, then α is called *saturated*. For saturated terms, we can soundly apply the usual form of β-reduction

$$(\lambda\bar{x}.\,\alpha)\bar{\beta} \to_\beta \alpha[\bar{\beta}/\bar{x}].$$

Lemma 67 *If α is saturated and $\alpha \to_\beta \alpha'$, then α' is saturated.*

PROOF. Let α' arise from α by replacing a redex $r = (\lambda\bar{x}.\,\beta)\bar{\gamma}$ with its contractum $c = \beta[\bar{\gamma}/\bar{x}]$. We have to check the top level variable condition for λ-abstractions in α':

- If a λ-abstraction appears in $\bar{\gamma}$, then it appears also in α and hence satisfies the top level variable condition.
- Other λ-abstractions contained in c are of the form $(\lambda\bar{y}.\,\delta)[\bar{\gamma}/\bar{x}]$, with $\lambda\bar{y}.\,\delta$ appearing in β, so that the $\bar{y}$ are in $TFV(\delta)$. Since substitution does not affect the $\bar{y}$, the top level variable condition remains intact.
- If a λ-abstraction in α' contains c, then it arises from a λ-abstraction $\lambda\bar{y}.\,\delta$ in α by replacing r with c. Now if $y_i \in TFV(r)$, then necessarily $y_i \in TFV(\bar{\gamma})$ and thus, since r is saturated, $y_i \in TFV(c)$; thus, the top level variable condition remains intact.
- λ-abstractions in α' that are neither contained in nor contain c appear in identical form in α. □

Thus, we can apply β-reduction to saturated terms in the same way as in the simply typed total λ-calculus (regarding conditioned terms syntactically as applications of distinguished constants); in particular, every saturated term has a β-normal form. Observing finally that every term in the partial λ-calculus can be transformed into an equivalent saturated term by recursively replacing subterms of the form $\lambda\bar{x}.\,\alpha$ with $\lambda\bar{x}.\,(\alpha \restriction \bar{x})$, we arrive at

Theorem 68 *Every term in the partial λ-calculus has a β-normal form.*