Linearizability of Non-expansive Semigroup Actions on Metric Spaces

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Abstract

We show that a non-expansive action of a topological semigroup S on a metric space X is linearizable iff its orbits are bounded. The crucial point here is to prove that X can be extended by adding a fixed point of S, thus allowing application of a semigroup version of the Arens-Eells linearization, iff the orbits of S in X are bounded.

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Introduction

By a well-known construction due to Arens and Eells [1], every metric space can be isometrically embedded as a closed metric subspace of a normed (linear) space. Using this construction (or other linear extensions like the free Banach space), one can show [5,3] that a non-expansive action π of a topological semigroup S on a metric space is linearizable, i.e. arises by restricting an action of S by linear contractions on a normed space V to a metric subspace of V, if π has a fixed point z (which then serves as the 0 of V). The question of when an action π is linearizable in general thus reduces to the question of when π can be extended by adding a fixed point.

It is trivial to observe that if X is bounded, then π may be extended by adding a fixed point: introduce a new point z, make z a fixed point of S, and put $d(z, x) = \mathrm{diam}(X)/2$ for all $x \in X$. It is then easy to check that the distance function d thus defined on $X \cup \{z\}$ is a metric, and that the action of S on

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the extended space is non-expansive. Here, we improve on this construction by giving a necessary and sufficient criterion: π may be extended by adding a fixed point iff its orbits are bounded sets. We thus obtain an exact linearizability criterion: π is linearizable iff its orbits are bounded.

1 Preliminaries

Throughout the exposition, fix a topological semigroup S (i.e. a semigroup S equipped with a topology such that the multiplication $S \times S \to S$ is jointly continuous). We shall generally be concerned with *non-expansive actions* $\pi : S \times X \to X$, with $\pi(s, x)$ denoted as $s \cdot x$, of S on metric spaces (X, d), i.e. $d(s \cdot x, s \cdot y) \leq d(x, y)$ for all $s \in S$ and all $x, y \in X$; if S is a monoid with identity element e, then S acts *monoidally* if $e \cdot x = x$ for all $x \in X$. In the special case that (X, d) is a real normed space V, we say that π is *linear* if the translation maps $\breve{s} : x \mapsto s \cdot x$ are linear maps on V. In this case, non-expansivity of π means that the $\breve{s}$ are *contracting*, i.e. $\|s \cdot x\| \leq \|x\|$ for all x. We say that a map $f : X \to Y$ is *equivariant* w.r.t. actions of S on X and Y if $f(s \cdot x) = s \cdot f(x)$ for all $s \in S$, $x \in X$.

We note an observation from [3], omitting the (straightforward) proof:

Lemma 1 *For a non-expansive action $\pi : S \times X \to X$ on a metric space (X, d), the following are equivalent.*

(1) The action π is jointly continuous.
(2) The action π is separately continuous.
(3) The restriction $\pi : S \times Y \to X$ to some dense subspace Y of X is separately continuous.

We shall henceforth implicitly include the requirement that $S \times X \to X$ is continuous in the term *non-expansive action* (thus avoiding the term 'non-expansive continuous action', which is a bit of a mouthful). Note that the monoid of all non-expansive self-maps of (X, d), equipped with the topology of pointwise convergence, is a topological semigroup which acts non-expansively and monoidally on (X, d) [3]. As an immediate consequence of the preceding lemma, we obtain the following extension result [3]:

Lemma 2 *A linear non-expansive action of S on a normed space V extends (uniquely) to a linear non-expansive action of S on the completion of V.*

We denote the orbit $\{s \cdot x \mid s \in S\}$ of $x \in X$ under S by $S \cdot x$. Note that orbits need not be disjoint, elements of an orbit need not have the same orbit, and x need not be contained in its orbit $S \cdot x$. In case S is a monoid and acts monoidally on X, however, $x \in S \cdot x$ for all $x \in X$.

2 Fixed Points and Linearizations

We now give the announced criterion for extendability by a fixed point:

Theorem 3 *Let (X, d) be a metric space equipped with a non-expansive action of S. Then the following are equivalent:*

(1) The space X can be extended by adding a fixed point of S, i.e. there exists a metric space (Y, d) equipped with a non-expansive action of S that has a fixed point, and an isometric and equivariant embedding of X into Y.
(2) The orbits $S \cdot x$ of S in X are bounded sets.

The following definition will be useful in the proof:

Definition 4 *Let (X, d) be a metric space. For $A \subseteq X$ and $x \in X$, we put*

$$\operatorname{supdist}(x, A) = \sup_{y \in A} d(x, y) \in [0, \infty].$$

PROOF. (Theorem 3) *(1) ⇒ (2):* We can assume that X is a subspace of Y. Let $z \in Y$ be a fixed point of S. Then we have, for $x \in X$ and $s, t \in S$,

$$\begin{aligned} d(s \cdot x, t \cdot x) &\leq d(s \cdot x, z) + d(z, t \cdot x) \\ &= d(s \cdot x, s \cdot z) + d(t \cdot z, t \cdot x) \\ &\leq 2d(x, z), \end{aligned}$$

i.e. $\operatorname{diam}(S \cdot x) \leq 2d(x, z)$.

(2) ⇒ (1): To begin, we reduce to the case that S is a monoid, as follows. For a semigroup S, we have the free monoid S_e over S, constructed by taking $S_e = S \cup \{e\}$, where $e \notin S$ is a new isolated point, and putting $es = se = s$ for all $s \in S_e$. The action of S on X is extended to a non-expansive (continuous) action of S_e by putting $e \cdot x = x$ for all $x \in X$. The orbits $S_e \cdot x = \{x\} \cup S \cdot x$ are bounded (by $d(x, s \cdot x) + \operatorname{diam}(S \cdot x)$ for any $s \in S$). By the monoid case of the theorem, we obtain an extended space (Y, d) in which S_e has a fixed point z; the action of S_e restricts to an action of S on Y, and z trivially remains a fixed point of S.

When S is a monoid, then $x \in S \cdot x$ for all $x \in X$. We can assume w.l.o.g. that there exists a point $x_0 \in X$ which is not fixed under S. We put $Y = X \cup \{z\}$, where $z \notin X$, and define the distance function d on Y by

$$d(z,x) = d(x,z) = \operatorname{supdist}(x_0, S \cdot x)$$

for $x \in X$, and $d(z,z) = 0$. We have to check that this makes (Y,d) a metric space. To begin, $d(x,z) > 0$ for $x \in X$: we have $\operatorname{supdist}(x_0, S \cdot x_0) > 0$ because x_0 is not fixed under S, and for $x \neq x_0$, $\operatorname{supdist}(x_0, S \cdot x) \geq d(x_0, x) > 0$ (using $x \in S \cdot x$). Symmetry holds by construction. Moreover, for $x \in X$, $d(x_0, s \cdot x) \leq d(x_0, x) + d(x, s \cdot x) \leq d(x_0, x) + \operatorname{diam}(S \cdot x)$ for all $s \in S$ (again using $x \in S \cdot x$) and hence $d(x,z) \leq d(x_0,x) + \operatorname{diam}(S \cdot x) < \infty$ by (2). It remains to prove the triangle inequality. There are only two non-trivial cases to prove:

(a) $d(x,z) \leq d(x,y) + d(y,z)$ for $x, y \in X$, and
(b) $d(x,y) \leq d(x,z) + d(y,z)$ for $x, y \in X$.

Ad (a): Let $s \in S$. Then $d(x_0, s \cdot x) \leq d(x_0, s \cdot y) + d(s \cdot y, s \cdot x) \leq d(x_0, s \cdot y) + d(y,x)$. Thus, $\operatorname{supdist}(x_0, S \cdot x) \leq d(x,y) + \operatorname{supdist}(x_0, S \cdot y)$.

Ad (b): We have

$$\begin{aligned} d(x,y) &\leq d(x,x_0) + d(y,x_0) \\ &\leq \operatorname{supdist}(S \cdot x, x_0) + \operatorname{supdist}(S \cdot y, x_0) \\ &= d(x,z) + d(y,z), \end{aligned}$$

where the second inequality uses $x \in S \cdot x$.

We then extend the action of S to Y by letting z be fixed under S. It is clear that this really defines an action of S; we have to check that this action is non-expansive. For $x \in X$ and $s \in S$, we have

$$\begin{aligned} d(s \cdot x, s \cdot z) &= d(s \cdot x, z) \\ &= \operatorname{supdist}(x_0, S \cdot (s \cdot x)) \\ &\leq \operatorname{supdist}(x_0, S \cdot x) \\ &= d(x,z), \end{aligned}$$

where the inequality uses $S \cdot (s \cdot x) \subseteq S \cdot x$.

It remains to prove that $S \times Y \to Y$ is continuous, i.e. by Lemma 1 that the orbit maps $S \to Y, s \mapsto s \cdot y$, are continuous. For $y \in X$, this follows from continuity of the action on X, and for $y = z$, the orbit map is constant. □

Remark 5 In case S is a group, one can identify the space Y constructed in the above proof with the subspace $\{\{x\} \mid x \in X\} \cup \{S \cdot x_0\}$ of the space of bounded subsets of X, equipped with the Hausdorff pesudometric

$$d(A,B) := \max\{\sup\{d(a,B) : a \in A\}, \sup\{d(A,b) : b \in B\}\}$$

and the natural action taking A to $s \cdot A$ for $s \in S$. For arbitrary semigroups S, however, orbits will in general fail to be fixed points under the natural action.

Remark 6 For $x, y \in X$, we generally have $\operatorname{diam}(S \cdot y) \leq 2d(x, y) + \operatorname{diam}(S \cdot x)$, since for $s, t \in S$,

$$\begin{aligned} d(s \cdot x, t \cdot x) &\leq d(s \cdot x, s \cdot y) + d(s \cdot y, t \cdot y) + d(t \cdot y, t \cdot x) \\ &\leq 2d(x, y) + d(s \cdot y, t \cdot y). \end{aligned}$$

Thus, for boundedness of all orbits it suffices to require that there exists a bounded orbit.

We now briefly recall the Arens-Eells extension of a pointed metric space (X, d, z) (i.e. $z \in X$). One constructs a real normed space $(A(X), \|\_\|)$ by taking as the elements of $A(X)$ the formal linear combinations

$$\sum_{i=1}^{n} c_i(x_i - y_i),$$

with $x_i, y_i \in X$ and $c_i \in \mathbb{R}$ and putting for $u \in A(X)$

$$\|u\| = \inf \left\{ \sum_{i=1}^{n} |c_i| d(x_i, y_i) \mid u = \sum_{i=1}^{n} c_i(x_i - y_i) \right\}.$$

The space (X, d) is isometrically embedded into $A(X)$ (as a closed subspace) by taking $x \in X$ to $x - z$. It is shown in [3] (Proposition 2.10) that a non-expansive action of S on X can be extended to a linear non-expansive action of S on $A(X)$ by putting

$$s \cdot \sum_{i=1}^{n} c_i(x_i - y_i) = \sum_{i=1}^{n} c_i(s \cdot x_i - s \cdot y_i).$$

(A similar construction can be found already in [5]; moreover, the Arens-Eells extension may be replaced by other linear extensions [2], e.g. the free Banach space over X as in [5].)

We then immediately obtain the announced exact linearizability criterion.

Theorem 7 *For a non-expansive action of S on a metric space (X, d), the following are equivalent:*

(1) There exists a Banach space V, equipped with a linear non-expansive action of S, and an equivariant isometric embedding of (X, d) into V.
(2) The orbits $S \cdot x$ of S in X are bounded sets.

PROOF. *(1) ⇒ (2):* By the corresponding direction of Theorem 3, as 0 is a fixed point of S in V.

(2) ⇒ (1): By Theorem 3, we may assume that S has a fixed point z in X. By Lemma 2, it suffices to construct V as a normed space. We thus may take V as the Arens-Eells extension of (X, d, z), equipped with the S-action described above. □

Remark 8 Recent results by Pestov [4] indicate that *every* non-expansive action can be extended to an action by *affine* maps on a Banach space.

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