Cut Elimination in Coalgebraic Logics

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Abstract

We give two generic proofs for cut elimination in propositional modal logics, interpreted over coalgebras. We first investigate semantic coherence conditions between the axiomatisation of a particular logic and its coalgebraic semantics that guarantee that the cut-rule is admissible in the ensuing sequent calculus. We then independently isolate a purely syntactic property of the set of modal rules that guarantees cut elimination. Apart from the fact that cut elimination holds, our main result is that the syntactic and semantic assumptions are equivalent in case the logic is amenable to coalgebraic semantics. As applications we present a new proof of the (already known) interpolation property for coalition logic and newly establish the interpolation property for the conditional logics *CK* and *CK* + *ID*.

1 Introduction

Establishing the admissibility of the cut rule in a modal sequent calculus often allows proving many other properties of the particular logic under scrutiny. If the sequent calculus enjoys the subformula property, the conservativity property is immediate: each formula is provable using only those deductive rules that mention exclusively operators that occur in the formula. As a consequence, completeness of the calculus at large immediately entails completeness of every subsystem that is obtained by removing a set of modal operators and the deduction rules in which they occur. Moreover, cut-free sequent systems admit backward proof search, as the logical complexity of a formula usually decreases when passing from the conclusion to the premise

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of a deductive rule to the premise. Given that contraction is admissible in the proof calculus, this yields – in the presence of completeness – decidability and complexity bounds for the satisfiability problem associated with the logic under consideration [12, 3]. Finally, a cut-free system provides the necessary scaffolding to prove interpolation theorems by induction on cut-free proofs.

For normal modal logics, sequent calculi, often in the guise of tableau systems, have therefore – not surprisingly – received much attention in the literature [1, 7, 22]. In the context of non-normal logics, sequent calculi have been explored for regular and monotonic modal logics [9], for Pauly's coalition logic [10] and for a family of conditional logics [17]. All these logics are coalgebraic in nature: their standard semantics can be captured by interpreting them over coalgebras for an endofunctor on sets. This is the starting point of our investigation, and we set out to derive sequent systems for logics with coalgebraic semantics and study their properties. Given a (complete) axiomatisation of a logic w.r.t. its coalgebraic semantics, we systematically derive a (complete) sequent calculus. In general, this calculus will only be complete if we include the cut rule. We show that cut free completeness, and therefore eliminability of cut, follows if the axiomatisation is *one-step cut-free complete*: every valid clause containing operators applied to propositional variables can be derived using a *single* modal deduction rule. The existence of a cut-free sequent calculus for coalgebraic logics is then exploited to establish conservativity, complexity, and interpolation for modal logics in a coalgebraic framework. While conservativity and complexity of coalgebraic logics have already been established in [28] we believe that the results here offer additional conceptual insight. Regarding interpolation, we obtain a new proof of the (known) interpolation property for coalition logic [10] while interpolation for the conditional logics CK and $CK + ID$ [5] was left as future work for the (different) sequent systems considered in [17] and appears to be new.

On a technical level, we consider modal logics that are built from atomic propositions, propositional connectives and modal operators, in contrast to earlier work (e.g. [13, 18, 24, 28]) where atomic propositions were regarded as an optional feature, incorporated into the modal similarity type as nullary modalities. This does not only provide a better alignment with standard texts in modal logic [5, 4] but is moreover a prerequisite for formulating the interpolation property. As a consequence, we are lead to work with coalgebraic models, that is, coalgebras together with a valuation of the propositional variables, right from the start. Completeness and cut-free completeness is then proved via a terminal sequence argument, but over the extension of the signature functor to the slice category $\mathsf{Set}/\mathcal{P}(V)$ where V is the set of propositional variables. This provides an alternative route to the shallow proof property of [28].

In this setting, we observe that one-step cut-free completeness corresponds to eliminability of cut. We then isolate purely syntactic conditions under which cut elimination holds. In essence, the set of modal rules has to be rich enough so that cuts between conclusions of modal rules can be absorbed into a single rule. If the rules are moreover strong enough to propagate contraction, we show that cut can be eliminated. This essentially amounts to completing the rule set so that cuts involving rule conclusions are in fact absorbed in the rule set, in strong analogy with Mints' comparison [15] between resolution and sequent proofs. It is interesting to note that the respective strengths of the syntactic and the semantic approach are identical: we show that the semantic coherence condition that guarantees admissibility of cut is equivalent to the syntactic requirement which is needed for cut elimination.

We summarise the coalgebraic semantics of modal logics in Section 2 and introduce modal sequent calculi in 3. Section 4 then establishes cut-free completeness semantically, while a purely syntactic proof of cut elimination is given in Section 5. We discuss applications, in particular the interpolation property, in Section 6 before concluding with two open problems.

Related Work

Sequent systems, and dually tableau systems, for modal logics traditionally come in two flavours: *labelled* calculi employ extended formulas that speak about states and transitions explicitly, while *unlabelled* calculi work purely with formulas of the logic. Our generic approach employs unlabelled systems. A good overview of work on labelled systems for normal modal logics is found in [16], while unlabelled systems are surveyed in [30]. Tableau systems for normal modal logics are discussed in [8]. There is, as far as we are aware, only a limited amount of work on sequent systems for non-normal logics, with the exception of [17] where labelled sequent systems for conditional logics are studied. We use conditional logic as one of two running examples; it turns out that the treatment of unlabelled sequent systems for conditional logics is pleasantly simple, as illustrated also in our further work on modal sequent systems outside rank 1 [19]. We do not know of a systematic general study of sequent systems for non-normal modal logics.

Our principle of absorption of structural rules is broadly related to generic criteria for cut elimination in substructural logics [2, 6], where, however, rules are assumed to be of a format that does not fit typical modal rules — besides the structural rules, there can be only left and right introduction rules for the logical connectives which introduce only one occurrence of a connective. A general approach to cut elimination which does apply to modal logics is presented in [21]. The range of application of this method is very wide and encompasses e.g. first-order logic, the modal logic $S4$, linear logic, and intuitionistic propositional logic. This generality is reflected in the fact that the method as a whole is substantially more involved than ours; whether it applies also to non-normal logics in principle remains an open question.

2 Coalgebraic and Logical Preliminaries

Given a category $\mathbb{C}$ and an endofunctor $F : \mathbb{C} \to \mathbb{C}$, an *$F$-coalgebra* is a pair (C, γ) where $C \in \mathbb{C}$ is an object of $\mathbb{C}$ and $\gamma : C \to FC$ is a morphism of $\mathbb{C}$. A *morphism* between F-coalgebras (C, γ) and (D, δ) is a morphism $m : C \to D \in \mathbb{C}$ such that $\delta \circ m = Fm \circ \gamma$. The category of F-coalgebras will be denoted by $\mathsf{Coalg}(F)$.

In the sequel, we will be concerned with F-coalgebras both on the category Set of sets and (total) functions and on the slice category $\mathsf{Set}/\mathcal{P}(V)$, for V a denumerable set of propositional variables that we keep fixed throughout the paper. Working with the slice category $\mathsf{Set}/\mathcal{P}(V)$ allows a convenient treatment of propositional variables. In particular, coalgebras on $\mathsf{Set}/\mathcal{P}(V)$ play the role of Kripke models, i.e. they come equipped with a valuation of propositional variables. Recall that an object of $\mathsf{Set}/\mathcal{P}(V)$ is a function $f : X \to \mathcal{P}(V)$ and a morphism $m : (X \xrightarrow{f} \mathcal{P}(V)) \to (Y \xrightarrow{g} \mathcal{P}(V))$ is a commuting triangle, that is, a function $m : X \to Y$ such that $g \circ m = f$. The projection functor mapping $(X \to \mathcal{P}(V)) \mapsto X$ is denoted by $U : \mathsf{Set}/\mathcal{P}(V) \to \mathsf{Set}$. For the remainder of the paper, we fix an endofunctor $T : \mathsf{Set} \to \mathsf{Set}$ and denote its extension to $\mathsf{Set}/\mathcal{P}(V)$ by $T/\mathcal{P}(V) : (\mathsf{Set}/\mathcal{P}(V)) \to (\mathsf{Set}/\mathcal{P}(V))$; the functor $T/\mathcal{P}(V)$ maps objects $f : X \to \mathcal{P}(V)$ to the second projection mapping $TX \times \mathcal{P}(V) \to \mathcal{P}(V)$. *We assume w.l.o.g. that T is non-trivial, i.e. $TX \neq \emptyset$ for some set X; it follows that $TY = \emptyset$ only if $Y = \emptyset$.* Note that an object $M \in \mathsf{Coalg}(T/\mathcal{P}(V))$ is a commuting triangle necessarily of the form

$$\begin{array}{ccc} C & \xrightarrow{\langle\gamma,\vartheta\rangle} & TC \times \mathcal{P}(V) \\ & {}_{\vartheta}\searrow \qquad \swarrow_{\pi_2} & \\ & \mathcal{P}(V) & \end{array}$$

or equivalently a triple (C, γ, ϑ) where $(C, \gamma) \in \mathsf{Coalg}(T)$ and $\vartheta : C \to \mathcal{P}(V)$ is a co-valuation of the propositional variables. Passing from the co-valuation $\vartheta : C \to \mathcal{P}(V)$ to the valuation $\vartheta^\sharp : V \to \mathcal{P}(C)$ induced by the self-adjointness of the powerset functor, we can view $T/\mathcal{P}(V)$-coalgebras as T-coalgebras (C, γ) together with a valuation of propositional variables. $T/\mathcal{P}(V)$-coalgebras therefore play the role of T-*models* (T-coalgebras, which we see as frames, together with a valuation of propositional variables). In what follows, we will denote $T/\mathcal{P}(V)$-coalgebras as triples (C, γ, ϑ) as above and use $\mathsf{Mod}(T)$ to refer to the category $\mathsf{Coalg}(T/\mathcal{P}(V))$ of T-models. If $M = (C, \gamma, \vartheta)$ is a T-model, then we refer to $(C, \gamma) \in \mathsf{Coalg}(T)$ as the *underlying frame* of M.

On the syntactic side, we work with modal logics over an arbitrary modal similarity type (set of modal operators with associated arities) Λ. The set of Λ-*formulas* is given by the grammar

$$\mathcal{F}(\Lambda) \ni A, B ::= p \mid A \wedge B \mid \neg A \mid \heartsuit(A_1, \dots, A_n)$$

where $p \in V$ and $\heartsuit \in \Lambda$ is n-ary. We use the standard definitions of the other propositional connectives, i.e. we put $A \vee B = \neg(\neg A \wedge \neg B)$, $A \to B = \neg A \vee B$, $A \leftrightarrow B = (A \to B) \wedge (B \to A)$, $\bot = p \wedge \neg p$ for some $p \in V$, and $\top = \neg\bot$. If S is a set (of formulas or variables) then $\Lambda(S)$ denotes the set $\{\heartsuit(s_1, \dots, s_n) \mid \heartsuit \in \Lambda \text{ is } n\text{-ary}, s_1, \dots, s_n \in S\}$ of formulas comprising exactly one application of a modality to elements of S. We denote the set of propositional formulas over a set S by $\mathsf{Prop}(S)$. The *(modal) rank* of a formula A is the maximal nesting depth of modal operators in A (0 if A does not contain any modal operators). We denote the set of propositional variables occurring in a formula A by $\mathsf{FV}(A)$.

Remark 2.1. Above, we deviate slightly from the approach to propositional variables used in coalgebraic logics so far [13, 18, 24, 28]: Instead of emulating propositional variables as nullary modal operators, interpreted over $T \times \mathcal{P}(V)$-coalgebras in Set, we treat propositional variables as syntactic entities in their own right, and interpret them over $T/\mathcal{P}(V)$-coalgebras in $\mathsf{Set}/\mathcal{P}(V)$. As indicated in the introduction, this is motivated by the desire to stay as close as possible to the standard treatment of modal logic. In particular, the distinction between variables and modal operators has a bearing on the stratification by modal rank that pervades our exposition both semantically and syntactically: traditionally, and also according to the above definition, a propositional variable p is regarded as 'non-modal'; in particular, the formula p has rank 0. Contrastingly, if p is emulated as a modal operator, then the formula p has rank 1. Further technical implications of this distinction are discussed in Section 4.

To facilitate induction on the modal rank of a formula, we stratify the set $\mathcal{F}(\Lambda)$ by modal rank. That is, we put

$$\mathcal{F}_{-1}(\Lambda) = \emptyset \text{ and } \mathcal{F}_n(\Lambda) = \mathsf{Prop}(\Lambda(\mathcal{F}_{n-1}(\Lambda)) \cup V)$$

for $n \geq 0$. It is easy to see that $\mathcal{F}(\Lambda) = \bigcup_{n \in \omega} \mathcal{F}_n(\Lambda)$.

An *S-substitution* is a mapping $\sigma : V \to S$. We denote the result of simultaneously substituting $\sigma(p)$ for every $p \in V$ in a formula $A \in \mathcal{F}(\Lambda)$ by $A\sigma$. As usual, substitution associates to the right, i.e. $A\sigma\rho = (A\sigma)\rho$ for formulas $A \in \mathcal{F}(\Lambda)$ and substitutions $\sigma, \rho : V \to \mathcal{F}(\Lambda)$.

As in [18, 23], formulas of $\mathcal{F}(\Lambda)$ are interpreted over T-coalgebras provided that T extends to a Λ-*structure*, i.e. comes equipped with an assignment of predicate liftings (natural transformations)

$$[\![\heartsuit]\!] : 2^n \to 2 \circ T$$

to every n-ary modal operator $\heartsuit \in \Lambda$. Here $2 : \mathsf{Set}^{op} \to \mathsf{Set}$ is the contravariant powerset functor, and for any functor F, F^n denotes the n-fold product of F with itself, i.e. $F^n(X) = FX \times \cdots \times FX$. Explicitly, the naturality equation for $[\![\heartsuit]\!]$ translates into the requirement that $[\![\heartsuit]\!]$ commutes with inverse images, i.e.

$$[\![\heartsuit]\!]_X(f^{-1}[Z_1], \dots, f^{-1}[Z_n]) = (Tf)^{-1}[[\![\heartsuit]\!]_Y(Z_1, \dots, Z_n)]$$

for all maps $f : X \to Y$ and all subsets $Z_1, \dots, Z_n \subseteq Y$. We usually leave the assignment of predicate liftings to modal operators implicit and simply use T to refer to the entire Λ-structure.

Given a Λ-structure T and $M = (C, \gamma, \vartheta) \in \mathsf{Mod}(T)$, the semantics of $A \in \mathcal{F}(\Lambda)$ is inductively given by

$$[\![\heartsuit(A_1, \dots, A_n)]\!]_M = \gamma^{-1} \circ [\![\heartsuit]\!]_C([\![A_1]\!]_M, \dots, [\![A_n]\!]_M)$$

and

$$[\![p]\!]_M = \{c \in C \mid p \in \vartheta(c)\}$$

for $p \in V$, together with the standard clauses for the propositional connectives.

If $M = (C, \gamma, \vartheta)$ is a T-model, semantic validity $[\![A]\!]_M = C$ is denoted by $M \models A$. We write $\mathsf{Mod}(T) \models A$ if $M \models A$ for all $M \in \mathsf{Mod}(T)$.

The completeness results that we establish later rely heavily on exploiting the semantic relation between formulas of $\mathsf{Prop}(V)$ (describing properties of states) and formulas of $\mathsf{Prop}(\Lambda(V))$ that describe properties of successors, in close analogy to coalgebra structures mapping states (elements of C) to successors in TC. The following notation is convenient for this purpose:

If $A \in \mathsf{Prop}(V)$, then every valuation $\tau : V \to \mathcal{P}(X)$ inductively defines a subset $[\![A]\!]_X^\tau \subseteq X$ by evaluation in the boolean algebra $\mathcal{P}(X)$, and we write $X, \tau \models A$ if $[\![A]\!]_X^\tau = X$. For statements about successor states, i.e. formulas $A \in \mathsf{Prop}(\Lambda(V))$, we have that every valuation $\tau : V \to \mathcal{P}(X)$ induces a subset $[\![A]\!]_{TX}^\tau \subseteq TX$ given by inductively extending the assignment

$$[\![\heartsuit(p_1, \dots, p_n)]\!]_{TX}^\tau = [\![\heartsuit]\!]_V(\tau(p_1), \dots, \tau(p_n))$$

on atoms to the whole of $\mathsf{Prop}(\Lambda(V))$. We write $TX, \tau \models A$ if $[\![A]\!]_{TX}^\tau = TX$.

Our techniques will be illustrated by the following two running examples.

Example 2.2 (Coalition logic and conditional logic)**.**

(i) Coalition logic [20] allows reasoning about coalitional power in games. We take $N = \{1, \dots, n\}$ to be a fixed set of *agents*, subsets of which are called *coalitions*. The similarity type Λ of coalition logic contains a unary modal operator $[C]$ for every coalition $C \subseteq N$. Informally, $[C]A$ expresses that coalition C has a collaborative strategy to force A. The coalgebraic semantics for coalition logic is based on the signature functor C defined by

$$\mathsf{C}X = \{(S_1, \dots, S_n, f) \mid \emptyset \neq S_i \subseteq \mathbb{N} \text{ finite for all } i; f : \textstyle\prod_{i \in N} S_i \to X\}.$$

(In order to enable arguments that use the terminal sequence of C, we restrict to finite rather than arbitrary sets of strategies to ensure that C is really set-valued, in contrast to earlier uses of this example where we did not need to worry about functors being class-valued [28]. We thus obtain a more restrictive semantics of coalition logic than considered in [20]; however, we retain weak completeness of the rule set to be introduced in Example 3.6 as discussed in Example 4.16). The elements of $\mathsf{C}X$ are understood as *strategic games* with set X of states, i.e. tuples consisting of nonempty finite sets $S_i \subseteq \mathbb{N}$ of *strategies* for all agents i, and an *outcome function* $(\prod S_i) \to X$. A C-coalgebra is a *game frame* [20] (with finite strategy sets.) We denote the set $\prod_{i \in C} S_i$ by S_C, and for $\sigma_C \in S_C, \sigma_{\bar{C}} \in S_{\bar{C}}$, where

$\bar{C} = N - C$, $(\sigma_C, \sigma_{\bar{C}})$ denotes the obvious element of $\prod_{i \in N} S_i$. A Λ-structure over C is defined by the predicate liftings

$$[\![[C]]\!]_X(B) = \{(S_1, \dots, S_n, f) \in \mathsf{C}X \mid \exists \sigma_C \in S_C . \forall \sigma_{\bar{C}} \in S_{\bar{C}} . f(\sigma_C, \sigma_{\bar{C}}) \in B\}.$$

(ii) The similarity type of the conditional logics CK and $CK + ID$ contains the single binary modal operator $\Rightarrow$ that represents a non-monotonic conditional. The selection function semantics of CK is captured coalgebraically via the functor $\mathsf{CK}X = (2(X) \to \mathcal{P}(X))$ with $\to$ representing function space, and CK-coalgebras are standard conditional models [5]. We extend CK to a Λ-structure by virtue of the predicate lifting

$$[\![\Rightarrow]\!]_X(A, B) = \{f : 2X \to \mathcal{P}X \mid f(A) \subseteq B\}$$

which induces the standard semantics of CK. The conditional logic $CK + ID$ additionally obeys the axiom $A \Rightarrow A$ and is interpreted over the functor $\mathsf{CK}_{\mathsf{Id}}X = \{f : 2(X) \to \mathcal{P}(X) \mid \forall A \subseteq X . f(A) \subseteq A\}$; note that $\mathsf{CK}_{\mathsf{Id}}$ is a subfunctor of CK. The functor $\mathsf{CK}_{\mathsf{Id}}$ extends to a Λ-structure by relativising the interpretation of $\Rightarrow$ given above, i.e.

$$[\![\Rightarrow]\!]_X(A, B) = \{f \in \mathsf{CK}_{\mathsf{Id}}X \mid f(A) \subseteq B\}$$

for subsets $A, B \subseteq X$. One possible way to understand a conditional model $\xi : X \to (2(X) \to \mathcal{P}(X))$ is to regard $\xi(x)(A)$ as the 'typical' worlds for property A from the perspective of world x; the restriction imposed by $\mathsf{CK}_{\mathsf{Id}}$ then states that typical A-worlds actually belong to A.

3 Sequent Systems for Coalgebraic Logics

We proceed to define a generic Gentzen-style sequent system for coalgebraic modal logics, thus complementing earlier work on Hilbert systems [18, 24, 28]. The system will be parametrised over a set of modal rules of the same format as in the Hilbert systems, so that the same data determine both a Hilbert and a Gentzen system.

If $S \subseteq \mathcal{F}(\Lambda)$ is a set of formulas, an *S-sequent*, or just a *sequent* in case $S = \mathcal{F}(\Lambda)$, is a finite multiset of elements of $S \cup \{\neg A \mid A \in S\}$ (following [29], we opt to treat sequents as multisets rather than sets in order to make the crucial issue of contraction more explicit). We write $\mathcal{S}(S)$ for the set of S-sequents, and $\mathcal{S}$ for the set of $\mathcal{F}(\Lambda)$-sequents. As the logics we consider here are extensions of classical propositional logic, we work with single-sided sequent calculi and read sequents disjunctively. That is, a sequent corresponds to the disjunction of its elements, and we write $\check{\Gamma} = \bigvee \Gamma$ for the associated formula. We use the standard set-theoretic notation of union and subset also for multisets, respecting multiplicity; i.e. for multisets Γ, Δ, we write $\Gamma \subseteq \Delta$ if every element that is contained in Γ with multiplicity n is contained in Δ with multiplicity at least n, and $\Gamma \cup \Delta$ denotes the multiset that contains x with multiplicity $n+m$ whenever Γ contains x with multiplicity n and Δ contains x with multiplicity m. We write $\mathrm{supp}(\Gamma)$ for the support of Γ, i.e. the set of elements of Γ, disregarding multiplicities; sets, in turn, are implicitly regarded as multisets with all multiplicities at most 1. E.g., if $\Delta \subseteq \mathrm{supp}(\Gamma)$ for multisets Γ, Δ, then Δ must be a set, i.e. cannot contain duplicates. We identify a formula A with the singleton sequent $\{A\}$ whenever convenient and denote the multiset union of sequents Γ and Δ by Γ, Δ. Combining both conventions, we write Γ, A for $\Gamma \cup \{A\}$.

Substitutions are applied pointwise to sequents: if σ is a substitution and Γ is a sequent, $\Gamma\sigma = \{A\sigma \mid A \in \Gamma\}$. In our terminology, a *sequent rule* is a tuple of sequents, usually written in the form

$$\frac{\Gamma_1 \quad \dots \quad \Gamma_n}{\Gamma_0} \quad \text{or} \quad \Gamma_1 \dots \Gamma_n / \Gamma_0$$

where we silently identify sequent rules modulo reordering of the sequents in the premise.

Given a set $\mathbf{S}$ of sequent rules and a set $H \subseteq \mathcal{S}$ of additional hypotheses, the notion of deduction is standard: *proofs* are finite trees with nodes labelled by sequents, constructed inductively from the rules in $\mathbf{S}$ (the rules themselves, not substitution instances thereof) and the hypotheses in H. We write $\mathbf{S} + H \vdash \Gamma$, and say that Γ is $\mathbf{S} + H$-derivable, if there exists such a proof of Γ; in case $H = \emptyset$, we write $\mathbf{S}$ instead of $\mathbf{S} + H$. The *depth* of a proof is its depth as a tree. A sequent rule $\Gamma_1 \dots \Gamma_n / \Gamma_0$ is *(depth-preserving)* $\mathbf{S}$-*admissible* if whenever $\mathbf{S} \vdash \Gamma_i$ for all $i = 1, \dots, n$ (with proofs of depth at most n for some n), then $\mathbf{S} \vdash \Gamma_0$ (with a proof of depth at most n).

Remark 3.1. Note that the above notion of derivability explicitly does not allow substituting into sequent rules. This facilitates restriction of the rule set in inductive proofs, e.g. to rules of bounded modal rank. The full rule set governing a given modal logic has closure under substitution built in; see Definition 3.5.

We use the following set $\mathbf{G}$ of sequent rules to account for the propositional part of our calculus

$$(Ax)\ \frac{}{\Gamma, p, \neg p} \qquad (\wedge)\ \frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \qquad (\neg\wedge)\ \frac{\Gamma, \neg A, \neg B}{\Gamma, \neg(A \wedge B)} \qquad (\neg\neg)\ \frac{\Gamma, A}{\Gamma, \neg\neg A}$$

where $p \in V$, $A, B \in \mathcal{F}(\Lambda)$ and $\Gamma \in \mathcal{S}$. We adopt the context-free version of the cut-rule, writing **C** for the set of rules of the form

$$(cut)\ \frac{\Gamma, A \quad \Delta, \neg A}{\Gamma, \Delta}$$

where $\Gamma, \Delta \in \mathcal{S}$ and $A \in \mathcal{F}(\Lambda)$. For the purpose of arguments by induction on the modal rank of a formula, we write

$$\mathbf{S}_n = \Big\{ \frac{\Gamma_1 \dots \Gamma_k}{\Gamma_0} \in \mathbf{S} \mid \Gamma_i \in \mathcal{S}(\mathcal{F}_n(\Lambda)) \text{ for all } i = 0, \dots, k \Big\}$$

for the set of rules in **S** whose premises and conclusions are restricted to sequents over $\mathcal{F}_n(\Lambda)$. In particular, this induces the sets $\mathbf{G}_n$ and $\mathbf{C}_n$ containing the propositional rules and instances of the cut rule, applied to formulas of modal rank at most n. We denote the union of sequent rule sets by juxtaposition; e.g. **GC** is propositional reasoning with cut, i.e. the system comprised of the rules of **G** and those of **C**. Juxtaposition binds more strongly than rank restriction; e.g. $\mathbf{GC}_n$ denotes $(\mathbf{GC})_n$ (not $\mathbf{G}(\mathbf{C}_n)$).

The system **G** appears under the name **G3c** in [29], where the meticulous reader may find proofs of both soundness and completeness. We note a few basic completeness properties of **G**:

Proposition 3.2.

1. *The system* **G** *is complete w.r.t. propositional validity, i.e.* $\mathbf{G} \vdash \Gamma$ *iff* $\check{\Gamma}$ *is a propositional tautology*
2. *The system* **GC** *is complete for propositional consequence, i.e. for a set* Φ *of sequents,* $\mathbf{GC} + \Phi \vdash \Gamma$ *iff* $\check{\Gamma}$ *is a propositional consequence of* $\{\check{\Delta} \mid \Delta \in \Phi\}$.

Proof. (i): Straightforward induction on the complexity of Γ.

(ii): If $\check{\Gamma}$ is a logical consequence of $\{\check{\Delta} \mid \Delta \in \Phi\}$, then there exist $\Delta_1, \dots, \Delta_n \in \Phi$ such that $\neg\check{\Delta}_1 \wedge \dots \neg\check{\Delta}_n \to \check{\Gamma}$ is a propositional tautology. By (i), we have $\mathbf{G} \vdash \neg\check{\Delta}_1, \dots, \neg\check{\Delta}_n, \Gamma$, and hence $\mathbf{GC} + \Phi \vdash \Gamma$. □

Our next task is to extend **G** with additional sequent rules to account for modal deduction. It has been shown in [23] that coalgebraic logics can always be completely axiomatised in *rank 1*, i.e. by a (possibly infinite) number of *one-step rules*, that is, rules whose premise is a purely propositional formula and which have a purely modalised conclusion.

Definition 3.3. A *one-step rule* over a modal similarity type Λ is an $n+1$-tuple of sequents $\Gamma_i \in \mathcal{S}(V)$, $i = 1, \dots, n$, and $\emptyset \neq \Gamma_0 \in \mathcal{S}(\Lambda(V))$, written as $\frac{\Gamma_1 \dots \Gamma_n}{\Gamma_0}$ or $\Gamma_1 \dots \Gamma_n / \Gamma_0$.

Remark 3.4. It is clear that one-step rules as defined above have the same expressive power as a more general type of rules where one allows $\Gamma_0 \in \mathcal{S}(\Lambda(\mathsf{Prop}(V)))$, as one can just introduce additional premises that abbreviate propositional formulas to single propositional variables. Typically, however, the natural formulation of the rules already has the format required above. For instance, the normal modal logic K can be axiomatised by the family of one-step rules

$$(RK_n) \quad \frac{\neg a_1, \dots, \neg a_n, b}{\neg \Box a_1, \dots, \neg \Box a_n, \Box b} \ (n \geq 0)$$

(where RK_0 is the necessitation rule $b/\Box b$, and the K-axiom $\Box(a \to b) \to (\Box a \to \Box b)$, dissolved into the sequent $\neg \Box(a \to b), \neg \Box a, \Box b$, may be derived by RK_2 from the tautologous sequent $\neg(a \to b), \neg a, b$). We illustrate this further in Example 3.6.

One-step rules describe the passage from statements about states (the premises) to a statement about successors (in the conclusion), analogously to the way in which the structure map $\gamma : C \to TC$ of a T-coalgebra (C, γ) provides us with a (structured) successor state for each world $c \in C$ of the model.

The definition above differs slightly from that given in [18, 23] in the sense that one-step rules in *op.cit.* are of the form ϕ/ψ where $\phi \in \mathsf{Prop}(V)$ is a purely propositional formula and ψ is a clause over atoms in $\Lambda(V)$. By passing from a propositional formula ϕ to its conjunctive normal form, every one-step rule in the sense of [18, 23] can be accommodated in the above definition in a straightforward way.

Every set of one-step rules gives rise to a set of sequent rules by passing from a one-step rule to all its substitution instances, augmented with an additional weakening context. The latter is standardly used in modal sequent rules in order to make the weakening rule admissible.

Definition 3.5. Let $\mathbf{R}$ be a set of one-step rules. The set $\mathcal{S}(\mathbf{R})$ of sequent rules associated with $\mathbf{R}$ consists of all *(substitution) instances* of $\mathbf{R}$, i.e. all rules

$$\frac{\Gamma_1 \sigma \quad \dots \Gamma_n \sigma}{\Gamma_0 \sigma, \Delta}$$

where $\Gamma_1 \dots \Gamma_n / \Gamma_0 \in \mathbf{R}$, $\sigma : V \to \mathcal{F}(\Lambda)$ is a substitution, and $\Delta \in \mathcal{S}$.

For our two running examples, the situation is as follows.

Example 3.6 (Coalition logic and conditional logic)**.**

(i) In [28], coalition logic has been axiomatised by the rules

$$\frac{\bigvee_{i=1}^{k} \neg a_i}{\bigvee_{i=1}^{k} \neg [C_i] a_i} \qquad \frac{\bigwedge_{i=1}^{k} a_i \to b \vee \bigvee_{j=1}^{l} c_j}{\bigwedge_{i=1}^{k} [C_i] a_i \to [D] b \vee \bigvee_{j=1}^{l} [N] c_j}$$

subject to the side condition that the C_i are pairwise disjoint; the second rule additionally requires that $C_i \subseteq D$ for all $i = 1, \dots, k$. These rules are one-step rules if we dissolve premise and conclusion into sequents, i.e. if we replace propositional clauses $\bigwedge_{i=1,\dots,n} A_i \to \bigvee_{j=1,\dots,m} B_j$ by sequents $\neg A_1, \dots, \neg A_n, B_1, \dots, B_m$. The arising set $\mathbf{R_C}$ of one-step rules is most economically presented if we abbreviate $\mathbf{A} = A_1, \dots, A_k$ for $A_1, \dots, A_k \in V$ and $\mathbf{C} = (C_1, \dots, C_k)$ for $C_1, \dots, C_k \subseteq N$; in this case $[\mathbf{C}]\mathbf{A}$ represents the sequent $[C_1]A_1, \dots, [C_k]A_k$. In this notation, $\mathbf{R_C}$ consists of the rules

$$(A)\frac{\neg \mathbf{A}}{\neg [\mathbf{C}]\mathbf{A}} \qquad (B)\frac{\neg \mathbf{A}, B, \mathbf{A}'}{\neg [\mathbf{C}]\mathbf{A},\ [D]B,\ [\mathbf{N}]\mathbf{A}'}$$

where $\mathbf{N} = N, \dots, N$ and $\neg\Delta = \{\neg A \mid A \in \Delta\}$ for $\Delta \in \mathcal{S}$. Both rule schemas are subject to the side condition that the coalitions appearing in $\mathbf{C}$ are disjoint; rule (B) moreover requires that their union is a subset of D.

(ii) The axiomatisation of the conditional logic CK in [5] consists of the rules

$$(RCK)\ \frac{\bigwedge_{i=1,\dots,n} b_i \to b_0}{\bigwedge_{i=1,\dots,n} (a \Rightarrow b_i) \to (a \Rightarrow b_0)} \qquad (RE)\ \frac{a \leftrightarrow a'}{(a \Rightarrow b) \to (a' \Rightarrow b)}$$

from which we obtain a set $\mathbf{R_{CK_0}}$ of one-step rules by replacing $a \leftrightarrow a'$ with the sequents $\neg a, a'$ and $\neg a', a$ in (RE). Merging these rules yields the rule set $\mathbf{R_{CK}}$ consisting of the one-step rules

$$(C)\ \frac{\neg b_1, \dots, \neg b_n, b_0 \quad \neg a_0, a_1 \quad \dots \quad \neg a_0, a_n \quad \neg a_1, a_0 \quad \dots \quad \neg a_n, a_0}{\neg(a_1 \Rightarrow b_1), \dots, \neg(a_n \Rightarrow b_n), (a_0 \Rightarrow b_0)}$$

for every $n \in \omega$. (It is clear that these rules are derivable from (RCK) and (RE). Conversely, $\mathbf{R_{CK}}$ contains (RE) as the case for $n = 1$, and (RCK) can be derived using the axiom rule of $\mathbf{G}$.) As above, we abbreviate $\mathbf{B} = B_1, \dots, B_n$, $\mathbf{A} = A_1, \dots, A_n$ and $\mathbf{A} \Rightarrow \mathbf{B} = A_1 \Rightarrow B_1, \dots, A_n \Rightarrow B_n$. The rules (C) can then be written in the form

$$(C)\frac{\neg \mathbf{B}, B_0 \quad \neg A_0, A_1 \quad \dots \quad \neg A_0, A_n \quad \neg A_1, A_0 \quad \dots \quad \neg A_n, A_0}{\neg(\mathbf{A} \Rightarrow \mathbf{B}),\ A_0 \Rightarrow B_0}.$$

The rules (C) express that the second argument of $\Rightarrow$ obeys normality whereas the first behaves like the modal $\Box$ of neighbourhood frames. The set of one-step rules needed to axiomatise $CK + ID$ contains the additional rule

$$(ID)\frac{\neg A_0, A_1 \qquad \neg A_1, A_0}{A_0 \Rightarrow A_1}$$

which is equivalent to the identity axiom $A \Rightarrow A$ used in Hilbert-style formulations of $CK + ID$. Integrating (C) and (ID) into a single rule schema, we let the set $\mathbf{R_{CKId}}$ consist of the rules

$$(CI)\frac{\neg A_0, \neg \mathbf{B}, B_0 \quad \neg A_0, A_1 \quad \dots \quad \neg A_0, A_n \quad \neg A_1, A_0 \quad \dots \quad \neg A_n, A_0}{\neg(\mathbf{A} \Rightarrow \mathbf{B}),\ A_0 \Rightarrow B_0}.$$

It is the special format of one-step rules that facilitates inductive arguments over the modal rank of formulas. For the case of one-step rules, we have the following characterisation:

Lemma 3.7. *Let* $\mathbf{R}$ *be a set of one-step rules. Then*

$$\mathcal{S}(\mathbf{R})_n = \Big\{ \frac{\Gamma_1\sigma \dots \Gamma_k\sigma}{\Gamma_0\sigma, \Delta} \mid \Gamma_1 \dots \Gamma_k/\Gamma_0 \in \mathbf{R}, \sigma : V \to \mathcal{F}_{n-1}(\Lambda), \Delta \in \mathcal{S}(\mathcal{F}_n(\Lambda)) \Big\}$$

(using the notation $\mathbf{S}_n$ *for sets* $\mathbf{S}$ *of sequent rules introduced earlier).*

Proof. Immediate from the definitions. □

In the remainder of the paper, we will use sequent calculi that are induced by several different rule sets. In particular, we will consider sequent calculi with and without cut, and also calculi whose rules are restricted to formulas of fixed modal rank. This is reflected by the following convention:

Convention 3.8. If $\mathbf{S}_1, \dots, \mathbf{S}_n$ are sets of sequent rules and $H_1, \dots, H_k \subseteq \mathcal{S}$ is a set of additional hypotheses, we use the short form and write

$$\mathbf{S}_1 \dots \mathbf{S}_n + H_1 + \dots + H_m \vdash \Gamma$$

in case $(\mathbf{S}_1 \cup \dots \cup \mathbf{S}_n) + (H_1 \cup \dots \cup H_m) \vdash \Gamma$ for $\Gamma \in \mathcal{S}$. Moreover, if $\mathbf{R}$ is a set of one-step rules, we write $\mathbf{GR}$ for the the rule set $\mathbf{G} \cup \mathcal{S}(\mathbf{R})$. As a consequence, note that $\mathbf{GR}_n = \mathbf{G}_n \cup (\mathcal{S}(\mathbf{R}))_n$ for $n \in \omega$.

We start our analysis of the provability predicate $\mathbf{GR} \vdash$ by establishing that weakening and inversion are admissible in the relativised calculi $\mathbf{GR}_n$. This is most easily established using the following characterisation of $\mathbf{GR}_n$-provability: a sequent is $\mathbf{GR}_n$-provable iff it is $\mathbf{G}_n$-provable from the set of conclusions of $\mathcal{S}(\mathbf{R})_n$-rules whose premises are $\mathbf{GR}_{n-1}$-provable. That is, we have the following:

Lemma 3.9. *Let* $\mathbf{R}$ *be a set of one-step rules and* $n \in \omega$*. Then* $\mathbf{GR}_n \vdash \Gamma$ *iff*

$$\mathbf{G}_n + \{\Gamma_0\sigma, \Delta \mid \Gamma_1 \dots \Gamma_k/\Gamma_0 \in \mathbf{R}, \Delta \in \mathcal{S}(\mathcal{F}_n(\Lambda)), \\ \sigma : V \to \mathcal{F}_{n-1}(\Lambda), \forall_{1 \le i \le k}(\mathbf{GR}_{n-1} \vdash \Gamma_i\sigma)\} \vdash \Gamma$$

whenever $\Gamma \in \mathcal{S}(\mathcal{F}_n(\Lambda))$*.*

The proof relies on the following fact:

Lemma 3.10. *For* $\Gamma \in \mathcal{S}(\mathcal{F}_n(\Lambda))$*,* $\mathbf{GR} \vdash \Gamma$ *iff* $\mathbf{GR}_n \vdash \Gamma$*.*

Proof. 'If' is trivial. 'Only if' is a standard induction on the **GR**-proof of Γ where we use that backwards application of the rules in **GR** does not increase modal rank. □

Proof of Lemma 3.9. 'If' is trivial. To prove 'only if', we proceed by induction on the $\mathbf{GR}_n$-proof of Γ. The cases for the propositional rules, i.e. the rules of $\mathbf{G}_n$, are trivial. So assume that $\mathbf{GR}_n \vdash \Gamma$ has been established using a modal rule in $\mathcal{S}(\mathbf{R})_n$. By Lemma 3.7 we find a one-step rule $\Gamma_1 \dots \Gamma_k/\Gamma_0$, a substitution $\sigma : V \to \mathcal{F}_{n-1}(\Lambda)$ and $\Delta \in \mathcal{S}(\mathcal{F}_n(\Lambda))$ such that $\Gamma = \Gamma_0\sigma, \Delta$ and $\mathbf{GR}_n \vdash \Gamma_i\sigma$ for all $i = 1, \dots, k$. Then $\Gamma_i\sigma \in \mathcal{S}(\mathcal{F}_{n-1}(\Lambda))$, whence $\mathbf{GR}_{n-1} \vdash \Gamma_i\sigma$ for $i = 1, \dots, k$ by Lemma 3.10. This proves the claim. □

One ingredient in the construction of sequent rules from one-step rules was the addition of a weakening context (Δ, in the notation of Definition 3.5) to the conclusion of every substituted one-step rule. As a consequence, weakening is admissible:

Lemma 3.11 (Weakening lemma)**.** *Let* $\mathbf{R}$ *be a set of one-step rules. Then* $\mathbf{GR}_n \vdash \Gamma, A$ *whenever* $\mathbf{GR}_n \vdash \Gamma$ *and* $A \in \mathcal{F}_n(\Lambda)$*.*

Proof. Immediate from Lemma 3.9 and the fact (proved by a straightforward induction over proofs) that weakening is admissible in $\mathbf{G}_n + H$ for a set H of hypotheses, provided that H is closed under weakening. □

The same argument allows us to prove that *inversion*, i.e. converse application of the rules $(\wedge)$, $(\neg\wedge)$, and $(\neg\neg)$ for the propositional connectives, is admissible. For future reference, we formulate admissibility of inversion for propositional reasoning with hypotheses explicitly:

Lemma 3.12. *Let* $H \subseteq \mathcal{S}(\mathcal{F}_n(\Lambda))$ *be closed under inversion (i.e. if* $A \wedge B \in H$ *then* $A \in H$ *and* $B \in H$*, etc.). Then*

1. $\mathbf{G}_n + H \vdash \Gamma, A$ *and* $\mathbf{G}_n + H \vdash \Gamma, B$ *whenever* $\mathbf{G}_n + H \vdash \Gamma, A \wedge B$
2. $\mathbf{G}_n + H \vdash \Gamma, A$ *whenever* $\mathbf{G}_n + H \vdash \Gamma, \neg\neg A$
3. $\mathbf{G}_n + H \vdash \Gamma, \neg A, \neg B$ *whenever* $\mathbf{G}_n + H \vdash \Gamma, \neg(A \wedge B)$

with proofs of at most the same depth.

Proof. This is as in [29], where the hypotheses in H play the role of axioms. □

The modal inversion lemma can now be formulated as follows:

Lemma 3.13 (Inversion lemma)**.** *Let $n \in \omega$, and let $\mathbf{R}$ be a set of one-step rules. Then all instances of the inversion rules*

$$\frac{\Gamma, \neg\neg A}{\Gamma, A} \qquad \frac{\Gamma, \neg(A_1 \wedge A_2)}{\Gamma, \neg A_1, \neg A_2} \qquad \frac{\Gamma, A_1 \wedge A_2}{\Gamma, A_1} \qquad \frac{\Gamma, A_1 \wedge A_2}{\Gamma, A_2},$$

where $A_1, A_2 \in \mathcal{F}_n(\Lambda)$ and $\Gamma \in \mathcal{S}(\mathcal{F}_n(\Lambda))$, are depth-preserving $\mathbf{GR}_n$-admissible.

Proof. Immediate from Lemmas 3.9 and 3.12, noting that the set of hypotheses in the statement of Lemma 3.9 is trivially closed under inversion, as conclusions of one-step rules never contain top-level propositional connectives except isolated occurrences of $\neg$. □

Finally, we show that **GRC**-derivability is closed under uniform substitution. Again, this is carried out relative to the modal rank of formulas. From now on, we denote the set of non-atomic axioms of rank at most k by

$$\mathbf{Ax}_k = \{\neg A, A, \Gamma \mid A \in \mathcal{F}_k(\Lambda), \Gamma \in \mathcal{S}(\mathcal{F}_k(\Lambda))\}.$$

Lemma 3.14 (Substitution lemma)**.** *Let $\mathbf{R}$ be a set of one-step rules, let $H \subseteq \mathcal{S}(\mathcal{F}_n(\Lambda))$, and suppose that $\mathbf{GRC}_n + H \vdash \Gamma$. If $\sigma : V \to \mathcal{F}_k(\Lambda)$, then $\mathbf{GRC}_{n+k} + \mathbf{Ax}_k + \{A\sigma \mid A \in H\} \vdash \Gamma\sigma$.*

Proof. Induction on the proof of $\mathbf{GRC}_n + H \vdash \Gamma$, where the hypotheses $\mathbf{Ax}_k$ take care of the case for the rule (Ax). □

By Lemma 3.10, Lemma 3.11 and Lemma 3.13 entail the admissibility of weakening and inversion also in the full calculus **GR**; we refrain from stating this formally as we shall need only the bounded-rank versions in the sequel.

Finally, we note a weaker form of Lemma 3.10 that applies to the system with cut:

Proposition 3.15. *Let* $\mathbf{R}$ *be a set of one-step rules, and let* $\Gamma \in \mathcal{S}$*. Then* $\mathbf{GRC} \vdash \Gamma$ *iff* $\mathbf{GRC}_n \vdash \Gamma$ *for some* $n \in \omega$*.*

Proof. As **GRC** is finitary, any proof in **GRC** can be simulated in $\mathbf{GRC}_n$ where n is large enough, i.e. such that all formulas occurring in the proof are elements of $\mathcal{F}_n(\Lambda)$. (Formally, this argument amounts to a simple proof by induction over the structure of proofs in **GRC**.) □

This concludes our discussion of the basic properties of sequent systems induced by one-step rules. The next two sections are devoted to establish admissibility of cut and contraction, first semantically in the next section and then by a purely syntactic argument.

4 Soundness and Cut-Free Completeness

We now study the relationship between **GR**-derivability and semantic validity. As in previous work, soundness and completeness will be implied by one-step soundness and one-step completeness, respectively of the rule set **R**. The proof of this known fact that we present here will, however, shed additional light on the structure of proofs. In particular, we will see that a one-step complete rule set in general necessitates the use of cut to obtain completeness, while eliminability of cut amounts to one-step cut-free completeness.

We recall the definition of one-step soundness and one-step completeness, adapted from [18, 23] to a sequent calculus setting . Here, we liberally extend notation introduced for formulas in Section 2 to sequents in the obvious way by regarding sequents as disjunctions of formulas; e.g. $\mathsf{FV}(\Gamma) = \mathsf{FV}(\hat{\Gamma})$, and $[\![\Gamma]\!]_M = [\![\hat{\Gamma}]\!]_M$ for a T-model M.

Definition 4.1. A set **R** of one-step rules is *one-step sound* (w.r.t. the Λ-structure T) if, whenever $\Gamma_1 \dots \Gamma_n/\Gamma_0 \in \mathbf{R}$, we have $TX, \tau \models \Gamma_0$ for each set X and each valuation $\tau : V \to \mathcal{P}(X)$ such that $X, \tau \models \Gamma_i$ for all $i = 1, \dots, n$. The set **R** is *one-step complete* if

$$\mathbf{GC}_1 + \{\Gamma_0\sigma, \Delta \mid \Delta \in \mathcal{S}(\Lambda(V)), \Gamma_1 \dots \Gamma_n/\Gamma_0 \in \mathbf{R},\\ \sigma : V \to \mathsf{Prop}(V), \forall_{1 \le i \le n}(X, \tau \models \Gamma_i\sigma)\} \vdash \Gamma \quad (1)$$

whenever $TX, \tau \models \Gamma$ for a set X, $\Gamma \in \mathcal{S}(\Lambda(V))$, and a $\mathcal{P}(X)$-valuation τ. Finally, **R** is *one-step cut-free complete* if it satisfies the same condition, but with $\mathbf{GC}_1$ replaced by $\mathbf{G}_1$.

Remark 4.2. In the definition of one-step completeness, we may equivalently omit the weakening context Δ appearing in the entailment (1), since weakening is derivable under the cut rule. For one-step *cut-free* completeness, however, the weakening context is essential.

In the sequel, we will work with the following reformulation of one-step cut-free completeness.

Lemma 4.3. *A set* $\mathbf{R}$ *of one-step rules is one-step cut-free complete iff whenever* $TX, \tau \models \Gamma$ *for* $\Gamma \in \mathcal{S}(\Lambda(V))$*, we have*

$$\Gamma_0\sigma \subseteq \Gamma$$

for some $\Gamma_1 \dots \Gamma_n/\Gamma_0 \in \mathbf{R}$ *and some renaming* $\sigma : V \to V$ *such that* $X, \tau \models \Gamma_i\sigma$ *for all* $i = 1, \dots, n$.

Proof. 'If' is trivial. 'Only if': By the definition of one-step cut-free completeness, we have $\mathbf{G}+\Psi \vdash \Gamma$ for $\Psi = \{\Gamma_0\sigma, \Delta \mid \Delta \in \mathcal{S}(\Lambda(V)), \Gamma_1 \dots \Gamma_n/\Gamma_0 \in \mathbf{R}, \sigma : V \to \mathsf{Prop}(V), \forall_{1 \leq i \leq n}(X, \tau \models \Gamma_i\sigma)\}$. As $\Gamma \in \mathcal{S}(\Lambda(V))$, the 'last' step in the corresponding proof cannot involve any of the rules of $\mathbf{G}_1$ (including the axiom rule, because none of the formulas in Γ is atomic). Therefore, Γ must be one of the hypotheses in $\Gamma_0\sigma, \Delta \in \Psi$, where necessarily $\sigma(a) \in V$ for all $a \in \mathsf{FV}(\Gamma_0)$; this proves the claim. $\square$

Remark 4.4. Note that the above definition of one-step (cut-free) completeness deviates slightly from definitions of *(strict) one-step completeness* we have used previously [24, 28] in that the axiom rule is excluded in propositional reasoning over $\Lambda(V)$. As a consequence, one-step complete rule sets in the sense of the present work always implicitly contain the congruence rule

$$\frac{\neg a_1, b_1 \quad \dots \quad \neg a_k, b_k \quad \neg b_1, a_1 \quad \dots \quad \neg b_k, a_k}{\neg\heartsuit(a_1, \dots, a_n), \heartsuit(b_1, \dots, b_n)}$$

for every k-ary modal operator $\heartsuit \in \Lambda$. This is formulated and proved explicitly for the case of one-step cut-free completeness in Section 5; the situation is similar with one-step completeness.

The above notions of one-step soundness, completeness and cut-free completeness may equivalently be restricted to finite sets, i.e. the relevant conditions are required to hold for all *finite* sets X rather than for all sets X. We indicate the corresponding restricted notions by the qualification *'on finite sets'*. For later use, we note this explicitly for one-step completeness:

Proposition 4.5. *A set* $\mathbf{R}$ *of one-step rules is one-step complete iff* $\mathbf{R}$ *is one-step complete on finite sets.*

(Up to adaptation of terminology and notation, the above statement has appeared, without proof, as Proposition 3.10 in [25].)

Proof. 'Only if' is trivial. To prove 'if', let $TX, \tau \models \Gamma$ for a (possibly infinite) set X, $\Gamma \in \mathcal{S}(\Lambda(V))$, and a $\mathcal{P}(X)$-valuation τ. Let 2 denote the set $\{\bot, \top\}$. The valuation τ induces a map $X \to 2^{\mathsf{FV}(\Gamma)}$, whose image we denote by X_τ, so that we have a surjective map $\bar{\tau} : X \to X_\tau$. As $\mathsf{FV}(\Gamma)$ is finite, X_τ is a finite set. We define the $\mathcal{P}(X_\tau)$-valuation τ^* by $\tau^*(a) = \{\kappa \in X_\tau \subseteq 2^{\mathsf{FV}(\Gamma)} \mid \kappa(a) = \top\}$. Then $\bar{\tau}^{-1}[\tau^*(a)] = \tau(a)$: we have $x \in \tau(a)$ iff $\bar{\tau}(x)(a) = \top$ iff $\bar{\tau}(x) \in \tau^*(a)$ iff $x \in \bar{\tau}^{-1}[\tau^*(a)]$. Using naturality of predicate liftings and commutation of preimage with Boolean operations, we thus obtain that $TX_\tau, \tau^* \models \Gamma$. By assumption, this implies

$$\mathbf{GC}_1 + \left\{\Gamma_0\sigma \mid \frac{\Gamma_1 \dots \Gamma_n}{\Gamma_0} \in \mathbf{R}, \sigma : V \to \mathsf{Prop}(V), \forall_{1 \le i \le n}(X_\tau, \tau^* \models \Gamma_i\sigma)\right\} \vdash \Gamma.$$

By commutation of preimage with Boolean operations, $X_\tau, \tau^* \models \Gamma_i\sigma$ iff $X, \tau \models \Gamma_i\sigma$, which implies the claim. □

It is best to understand the notions introduced above as *coherence conditions* between the axiomatisation of a particular logic using one-step rules and its semantics, given in terms of predicate liftings. In particular, they can be checked without reference to (coalgebraic) models, by comparing the interpretation of propositional formulas (premises of one-step rules) over a set X with the interpretation of purely modalised formulas (the conclusions) over the set TX. In a nutshell, one-step soundness asserts that a rule conclusion is valid over the set TX of structured successors whenever all its premises are valid over the set X. Dually, one-step completeness requires that whenever a purely modalised sequent is valid over TX, it can be derived – with the help of cut – from the conclusions of one-step rules whose premises are valid over X. Finally, one-step cut-free completeness asserts that valid sequents can be obtained from the conclusion of a one-step rule purely in terms of weakening. The following example highlights the difference between one-step completeness and one-step cut-free completeness.

Example 4.6. We consider the modal logic K (see e.g. [4]) interpreted over coalgebras for the powerset functor $TX = \mathcal{P}(X)$. The syntax of K is given by the similarity type $\Lambda = \{\Box\}$ that contains a single, unary operator. The functor T extends to a Λ-structure by putting

$$[\![\Box]\!]_X(A) = \{B \in TX \mid B \subseteq A\}$$

which gives rise to the standard semantics of K. We show that the rule set

$$(K)\quad \frac{\neg p_1, \dots, \neg p_n, p_0}{\neg\Box p_1, \dots, \neg\Box p_n, \Box p_0}(n \geq 0)$$

is one-step cut-free complete, leaving the routine verification of one-step soundness to the reader. Thus let X be a set, let $\tau : V \to \mathcal{P}(X)$, and let $\Gamma = \neg\Box p_1, \dots, \neg\Box p_n, \Box q_1, \dots, \Box q_k \in \mathcal{S}(\Lambda(V))$ such that $TX, \tau \models \Gamma$, i.e.

$$\bigcap_{i=1,\dots,n} [\![\Box]\!]_X(\tau(p_i)) \subseteq \bigcup_{j=1,\dots,k} [\![\Box]\!]_X(\tau(q_j)).$$

By the definition of $[\![\Box]\!]$, we have $\bigcap_{i=1,\dots,n} \tau(p_i) \in \bigcap_{i=1,\dots,n}[\![\Box]\!]_X(\tau(p_i))$, so that by the above set inclusion there exists $j \in \{1, \dots, k\}$ such that

$$(\dagger) \qquad \bigcap_{i=1,\dots,n} \tau(p_i) \subseteq \tau(q_j).$$

This means that $X, \tau \models \neg p_1, \dots, \neg p_n, q_j$, and applying (K) we derive the sequent $\neg\Box p_1, \dots, \neg\Box p_n, \Box q_j$ which is contained in Γ as required.

Contrastingly, consider the rule set consisting of necessitation (N) and distribution (D)

$$(N)\quad \frac{p}{\Box p} \qquad (D)\quad \frac{\neg p, \neg q_j, r}{\neg\Box p, \neg\Box q_j, \Box r}.$$

This set is one-step sound (being a subset of the previous one) and one-step complete, but not one-step cut-free complete. One-step completeness follows from the fact that, as shown by a simple induction, all rules of (K) are derivable using (N) and (D); however, this requires cut. We refrain from proving the failure of strict one-step completeness formally, and instead illustrate how cut comes up in the proof of one-step completeness. Consider, e.g., the case $n = 3$ in the above proof. Then we have that $X, \tau \models \neg(p_1 \wedge p_2), \neg p_3, q_j$ and $X, \tau \models \neg p_1, \neg p_2, p_1 \wedge p_2$. By two applications of (D), we derive $\neg\Box(p_1 \wedge p_2), \neg\Box p_3, \Box q_j$ and $\neg\Box p_1, \neg\Box p_2, \Box(p_1 \wedge p_2)$, and we need to use (cut) to derive $\neg\Box p_1, \neg\Box p_2, \neg\Box p_3, \Box q_j$.

It is an easy exercise to show that both **GR** and **GRC** are sound provided the rule set **R** is one-step sound. Recall that the *interpretation* of a sequent Γ w.r.t. $M \in \mathsf{Mod}(V)$ is the semantics of the associated propositional formula, i.e. $[\![\Gamma]\!]_M = [\![\check{\Gamma}]\!]_M$, and accordingly $M \models \Gamma$ iff $M \models \check{\Gamma}$, $\mathsf{Mod}(T) \models \Gamma$ if $\mathsf{Mod}(T) \models \check{\Gamma}$.

Theorem 4.7 (Soundness)**.** *Let* **R** *be one-step sound for* T*. Then* $\mathsf{Mod}(T) \models \Gamma$ *if* **GRC** $\vdash \Gamma$ *and, a fortiori,* $\mathsf{Mod}(T) \models \Gamma$ *if* **GR** $\vdash \Gamma$.

Proof. We proceed by induction over the length of the derivation, where the only interesting cases are applications of rules $\Gamma_1 \ldots \Gamma_n/\Gamma_0 \in \mathcal{S}(\mathbf{R})$. So suppose that $M = (C, \gamma, \vartheta) \in \mathsf{Mod}(T)$ and that Γ has been derived via an application of $\Gamma_1 \ldots \Gamma_n/\Gamma_0$. That is, we have $\Gamma'_1 \ldots \Gamma'_n/\Gamma'_0 \in \mathbf{R}$ and a substitution $\sigma : V \to \mathcal{F}(\Lambda)$ such that $\Gamma_i = \Gamma'_i\sigma$ for $i = 1, \ldots, n$ and $\Gamma_0 = \Gamma'_0\sigma, \Delta$ for some $\Delta \in \mathcal{S}$. By the induction hypothesis, $[\![\Gamma_i\sigma]\!]_M = \top$ for all $i = 1, \ldots, n$. Let τ be the $\mathcal{P}(C)$-valuation defined by $\tau(p) = [\![\sigma(p)]\!]_M$; note that $[\![\Gamma'_0\sigma]\!]_M = \gamma^{-1}[[\![\Gamma'_0]\!]^{\tau}_{TC}]$. We obtain $C, \tau \models \Gamma'_i$ for all $i = 1, \ldots, n$, and one-step soundness implies $TC, \tau \models \Gamma'_0$. Consequently, $[\![\Gamma_0]\!]_M = [\![\Gamma'_0\sigma, \Delta]\!]_M \supseteq [\![\Gamma'_0\sigma]\!]_M = \gamma^{-1}[[\![\Gamma'_0]\!]^{\tau}_{TC}] = \gamma^{-1}[TC] = C$. □

We now proceed to establish completeness and cut-free completeness directly by means of a semantic argument, and present a purely syntactic reconstruction in the following section. For the semantic approach, we prove completeness using a terminal sequence argument in the style of [18], which ties in well with the proof of cut elimination in the next section. As we are dealing with models, i.e. coalgebras equipped with a valuation, we consider the terminal sequence of the endofunctor $T/\mathcal{P}(V)$ in the category $\mathsf{Set}/\mathcal{P}(V)$. We briefly recapitulate the terminal sequence construction, as used in [18] but phrased in a general categorical setting.

If $F : \mathbb{C} \to \mathbb{C}$ is an endofunctor on a category $\mathbb{C}$ with terminal object 1 and unique morphisms $A \to 1$ denoted $!_A$ (or just '!'), the finitary part of the terminal sequence of F is the diagram consisting of

- the objects $F^n 1$ for $n \in \omega$, where F^n denotes n-fold application of F, and
- the morphisms $p_n : F^{n+1}1 \to F^n 1$ defined for $n \in \omega$ by $p_n = F^n(!_{F1})$.

Every F-coalgebra (C, γ) gives rise to a *canonical cone* $(C, (\gamma_n : C \to F^n 1)_{n \in \omega})$ over the finitary part of the terminal sequence defined by $\gamma_0 = !_C : C \to 1 = F^0 1$ and $\gamma_{n+1} = F\gamma_n \circ \gamma$. The terminal sequence of the functor $F = T/\mathcal{P}(V)$ is visualised in the following diagram (observing that the terminal object of $\mathsf{Set}/\mathcal{P}(V)$ is $1 \times \mathcal{P}(V)$).

$$
\begin{array}{ccccccc}
\overbrace{1 \times \mathcal{P}(V)}^{S_0} & \xleftarrow{p_0 = ! \times \mathsf{id}} & \overbrace{TS_0 \times \mathcal{P}(V)}^{S_1} & \xleftarrow{p_1 = Tp_0 \times \mathsf{id}} & \overbrace{TS_1 \times \mathcal{P}(V)}^{S_2} & \xleftarrow{p_2 = Tp_1 \times \mathsf{id}} & \cdots \\
\downarrow \pi_2 & & \downarrow \pi_2 & & \downarrow \pi_2 & & \\
\mathcal{P}(V) & = & \mathcal{P}(V) & = & \mathcal{P}(V) & = & \cdots
\end{array}
$$

Remark 4.8. At first sight, the upper row of the above diagram looks identical to the terminal sequence of the functor $T \times \mathcal{P}(V)$ on Set. Note however that the above sequence begins with $1 \times \mathcal{P}(V)$, thus already capturing the interpretation of propositional variables at S_0, while the terminal sequence of $T \times \mathcal{P}(V)$ begins with 1.

The key technique in the proof of completeness via a terminal sequence argument is to associate to every formula A of modal rank $\leq n$ an n-step semantics $[\![A]\!]_n$ over the n-th approximant $(T/\mathcal{P}(V))^n 1$ of the terminal sequence. In our case, we take a predicate over $(T/\mathcal{P}(V))^n 1$ to be a subset of $S_n = U((T/\mathcal{P}(V))^n 1)$. The formal definition is as follows:

Definition 4.9. The *n-step semantics* of $A \in \mathcal{F}_n(\Lambda) \subseteq S_n$ is inductively defined by $S_0 = \mathcal{P}(V)$ and

$$[\![p]\!]_0 = \{S \in \mathcal{P}(V) \mid p \in S\},$$

and $S_n = TS_{n-1} \times \mathcal{P}(V)$ together with

$$[\![p]\!]_n = \pi_2^{-1}[\{S \in \mathcal{P}(V) \mid p \in S\}]$$

and

$$[\![\heartsuit(A_1, \dots, A_k)]\!]_n = \pi_1^{-1}[[\![\heartsuit]\!]_{S_{n-1}}([\![A_1]\!]_{n-1}, \dots, [\![A_k]\!]_{n-1})]$$

for $n > 0$, $A_1, \dots, A_k \in \mathcal{F}_{n-1}(\Lambda)$ and $\heartsuit \in \Lambda$ a k-ary modality.

Note that $S_n = U((T/\mathcal{P}(V))^n 1)$. We can mediate between the n-step semantics and the semantics w.r.t $\mathsf{Mod}(T)$ as follows:

Lemma 4.10. *Let $A \in \mathcal{F}_n(\Lambda)$, let $M = (C, \gamma, \vartheta) \in \mathsf{Mod}(T)$, and let $(M, (\gamma_n)_{n \in \omega})$ be the canonical cone of M over the terminal sequence of $T/\mathcal{P}(V)$. Then $[\![A]\!]_M = (U\gamma_n)^{-1}[[\![A]\!]_n]$ for all $A \in \mathcal{F}_n(\Lambda)$.*

Proof. By induction on n. For $n = 0$ we have $U\gamma_0 = \vartheta$ and $\vartheta^{-1}[[\![p]\!]_0] = \vartheta^{-1}[\{S \subseteq V \mid p \in S\}] = \{c \in C \mid p \in \vartheta(c)\} = [\![p]\!]_M$. For $n > 0$, we obtain inductively $U\gamma_n = \langle TU\gamma_{n-1} \circ \gamma, \vartheta \rangle : C \to TS_{n-1} \times \mathcal{P}(V)$. This gives $(U\gamma_n)^{-1}[[\![p]\!]_n] = (\pi_2 \circ \langle TU\gamma_n \circ \gamma, \vartheta \rangle)^{-1}[\{S \subseteq V \mid p \in S\}] = \vartheta^{-1}[\{S \subseteq V \mid p \in S\}] = \{c \in C \mid p \in \vartheta(c)\} = [\![p]\!]_M$ as above. The cases for boolean operators are easily discharged using commutation of preimage with boolean set operations. For modal formulas $\heartsuit(A_1, \dots, A_k)$ with $A_1, \dots, A_k \in \mathcal{F}_{n-1}(\Lambda)$

we obtain

$$
\begin{aligned}
&(U\gamma_n)^{-1}[[\![\heartsuit(A_1,\dots,A_k]\!]_n)] \\
&= \langle TU\gamma_{n-1}\circ\gamma,\vartheta\rangle^{-1}[\pi_1^{-1}[[\![\heartsuit]\!]_{S_{n-1}}([\![A_1]\!]_{n-1},\dots,[\![A_k]\!]_{n-1})]] \\
&= \gamma^{-1}[(TU\gamma_{n-1})^{-1}[[\![\heartsuit]\!]_{S_{n-1}}([\![A_1]\!]_{n-1},\dots,[\![A_k]\!]_{n-1})]] \\
&= \gamma^{-1}[[\![\heartsuit]\!]_C((U\gamma_{n-1})^{-1}[[\![A_1]\!]_{n-1}]\times\cdots\times(U\gamma_{n-1})^{-1}[[\![A_k]\!]_{n-1}])] \\
&= \gamma^{-1}[[\![\heartsuit]\!]([\![A_1]\!]_M,\dots,[\![A_k]\!]_M)] \\
&= [\![\heartsuit(A_1,\dots,A_k)]\!]_M
\end{aligned}
$$

using the induction hypothesis and naturality of $[\![\heartsuit]\!]$. □

We recall the following lemma, whose proof directly translates to a general categorical setting, from [18]:

Lemma 4.11. *Let $f^0 : 1 \to F1$ be a morphism of $\mathbb{C}$ and let $f^n = F^n(f^0) : F^n 1 \to F^{n+1}1$, an F-coalgebra on $F^n 1$. Then the component f^n_n of the canonical cone $(F^n 1, (f^n_k : F^n 1 \to F^k 1)_{k\in\omega})$ is $\mathsf{id}_{F^n 1}$ for all $n \in \omega$.*

This immediately implies that validity of a sequent $\Gamma \in \mathcal{S}(\mathcal{F}_n(\Lambda))$ is equivalent to validity w.r.t. the n-step semantics:

Corollary 4.12. *Let $\Gamma \in \mathcal{S}(\mathcal{F}_n(\Lambda))$. Then $\mathsf{Mod}(T) \models \Gamma$ iff $[\![\Gamma]\!]_n = \top$.*

Proof. The 'if'-part is a consequence of Lemma 4.10 above. For the 'only if'-part assume that $\mathsf{Mod}(T) \models \Gamma$ and pick $f^0 : 1 \to (T/\mathcal{P}(V))(1)$ in $\mathsf{Set}/\mathcal{P}(V)$ where 1 is the terminal object of $\mathsf{Set}/\mathcal{P}(V)$ (f^0 exists by our global assumption that T is non-trivial). Consider $M = (C,\gamma) \in \mathsf{Coalg}(T/\mathcal{P}(V))$ where $C = (T/\mathcal{P}(V))^n(1)$ and $\gamma = (T/\mathcal{P}(V))^n(f^0)$. As $\mathsf{Mod}(T) \models \Gamma$ we have that $M \models \Gamma$, and by Lemmas 4.10 and 4.11, it follows that $[\![\Gamma]\!]_n = \top$. □

The proof of (cut-free) completeness relies on the stratification of the provability predicate $\mathbf{GR}_n \vdash$ of $\mathbf{GR}$, indexed by modal rank. The following proposition is the key stepping stone in the completeness proof, relating validity in the n-step semantics to derivability in rank n.

Proposition 4.13. *Let $\Gamma \in \mathcal{S}(\mathcal{F}_n(\Lambda)$ be a sequent over $\mathcal{F}_n(\Lambda)$ such that $[\![\Gamma]\!]_n = \top$. If $\mathbf{R}$ is one-step complete, then $\mathbf{GRC}_n \vdash \Gamma$, and if $\mathbf{R}$ is one-step cut-free complete, then $\mathbf{GR}_n \vdash \Gamma$.*

Proof. By induction on n. If $n = 0$ the statement follows from completeness of $\mathbf{G}$. By the inversion lemma (Lemma 3.13), it suffices to consider, for $n > 0$, the case

$$\Gamma = \neg\heartsuit_1 A_1,\dots,\neg\heartsuit_k A_k,\neg q_1,\dots,\neg q_m,\heartsuit'_1 A'_1,\dots,\heartsuit'_{k'} A'_{k'},q'_1,\dots,q'_{m'}$$

where the $A_i, A'_{i'}$ are tuples of formulas in $\mathcal{F}_{n-1}(\Lambda)$ according to the arity of $\heartsuit_i$ and $\heartsuit'_{i'}$ and $q_j, q'_{j'} \in V$. By the definition of $[\![\cdot]\!]_n$ and elementary boolean algebra, we deduce that either

$$[\![\neg\heartsuit_1 A_1, \dots, \neg\heartsuit_k A_k, \heartsuit'_1 A'_1, \dots, \heartsuit'_{k'} A'_{k'}]\!]_n = \top$$

or, alternatively,

$$[\![\neg q_1, \dots, \neg q_m, q'_1, \dots, q'_{m'}]\!]_n = \top$$

holds. In the latter case, it follows from the definition of $[\![\cdot]\!]_n$ that $\neg q_1, \dots, \neg q_m, q'_1, \dots, q'_{m'}$ is a propositionally valid sequent, hence necessarily an axiom and thus provable as required. So assume that the upper identity holds. This allows us to write $\Gamma = \Delta\tau$ where

$$\Delta = \neg\heartsuit_1 p_1, \dots, \neg\heartsuit_k p_k, \heartsuit'_1 p'_1, \dots, \heartsuit'_{k'} p'_{k'}$$

where $p_i, p'_{i'}$ are tuples of propositional variables according to the arity of $\heartsuit_i$ and $\heartsuit'_{i'}$, respectively, and $\tau : V \to \mathcal{F}_{n-1}(\Lambda)$ is a substitution mapping every component of p_i to the corresponding component of A_i, and similarly for p'_i. Write τ_{n-1} for the $\mathcal{P}(S_{n-1})$-valuation $p \mapsto [\![\tau(p)]\!]_{n-1}$. Then $TS_{n-1}, \tau_{n-1} \models \Delta$.

We first prove the second part of the statement. Thus assume that $\mathbf{R}$ is one-step cut-free complete. Then from $TS_{n-1}, \tau_{n-1} \models \Delta$ we conclude by Lemma 4.3 that there exist a one-step rule $\Gamma_1 \dots \Gamma_m/\Gamma_0 \in \mathbf{R}$ and a renaming $\sigma : V \to V$ such that $S_{n-1}, \tau_{n-1} \models \Gamma_i\sigma$ for $i = 1, \dots, m$, and $\Gamma_0\sigma \subseteq \Delta$. This means that $[\![\Gamma_i\sigma\tau]\!]_{n-1} = \top$ for $i = 1, \dots, m$, so that $\mathbf{GR}_{n-1} \vdash \Gamma_i\sigma\tau$ for $i = 1, \dots, m$ by induction. Since $\Gamma_0\sigma \subseteq \Delta$ we can find $\Sigma \in \mathcal{S}(\mathcal{F}_n(\Lambda))$ such that $\Gamma_0\sigma\tau, \Sigma = \Delta\tau = \Gamma$, which implies that there is a sequent rule $\Gamma_1\sigma\tau \dots \Gamma_m\sigma\tau/\Delta\tau \in \mathcal{S}(\mathbf{R})$. We thus obtain $\mathbf{GR}_n \vdash \Gamma$ as required.

This finishes the proof of the second claim. To prove the first claim, assume that $\mathbf{R}$ is one-step complete. Then there exist $k \geq 0$ and one-step rules $\Gamma^l_1 \dots \Gamma^l_{m_l}/\Gamma^l_0 \in \mathbf{R}$ together with substitutions $\sigma_l : V \to \mathsf{Prop}(V)$ for each $l = 1, \dots, k$ such that

(i) $\mathbf{GC}_1 + \{\Gamma^l_0\sigma_l \mid l = 1, \dots, k\} \vdash \Delta$, and hence $\mathbf{GC}_n + \mathbf{Ax}_{n-1} + \{\Gamma^l_0\sigma_l\tau \mid l = 1, \dots, k\} \vdash \Delta\tau$ by the substitution lemma 3.14

(ii) $S, \tau_{n-1} \models \Gamma^l_m\sigma_l$ for all $l = 1, \dots, r$ and all $m = 1, \dots, m_l$.

As above, we obtain from (ii) by induction that $\mathbf{GRC}_{n-1} \vdash \Gamma^l_m\sigma_l\tau$ for all $l = 1, \dots r$ and $m = 1, \dots, m_l$, whence $\mathbf{GRC}_n \vdash \Gamma^l_0\sigma_l\tau$ for $1 \leq l \leq r$, and thus, by (i) and the fact that all elements of $\mathbf{Ax}_{n-1}$ are derivable by induction hypothesis, that $\mathbf{GRC}_n \vdash \Delta\tau = \Gamma$. □

Completeness is now an easy corollary.

Corollary 4.14 (Completeness and cut free completeness)**.** *Let* $\mathbf{R}$ *be one-step complete for* T *and* $\mathsf{Mod}(T) \models \Gamma$ *for a sequent* $\Gamma \in \mathcal{S}(\mathcal{F}(\Lambda))$*. Then* $\mathbf{GRC} \vdash \Gamma$*. If moreover* $\mathbf{R}$ *is one-step cut-free complete, then* $\mathbf{GR} \vdash \Gamma$*.*

In particular, this gives us a semantic proof of cut elimination and admissibility of contraction.

Theorem 4.15. *Let* $\mathbf{R}$ *be one-step cut-free complete. Then all instances of the cut and contraction rules*

$$\frac{\Gamma, A \qquad \Delta, \neg A}{\Gamma, \Delta} \quad \text{and} \quad \frac{\Gamma, A, A}{\Gamma, A}$$

where $\Gamma, \Delta \in \mathcal{S}$ *and* $A \in \mathcal{F}(\Lambda)$*, are admissible in* $\mathbf{GR}$*.*

Proof. This follows directly from soundness and completeness. Clearly, both the contraction rule and the cut rule are sound. To see that they are admissible, suppose that X contains all instances of cut and contraction. If $\mathbf{GR} + X \vdash \Gamma$, we have that $\mathsf{Mod}(T) \models \Gamma$ hence $\mathbf{GR} \vdash \Gamma$. □

One may argue that the above semantic proof yields a slightly weaker result than the syntactic proof of Section 5, as we pre-suppose soundness and completeness w.r.t. a given Λ-structure. However, for every rule set $\mathbf{R}$ satisfying the absorption conditions used in Section 5, we can construct a Λ-structure for which $\mathbf{R}$ is one-step sound and one-step cut-free complete using results of Section 5 and of [26]. We conclude the section by re-visiting our two running examples.

Example 4.16. (i) The set $\mathbf{R_C}$ of one-step rules axiomatising coalition logic is one-step cut-free complete; we defer the proof to Example 5.13. As a consequence, cut is admissible in $\mathbf{GR_C}$.

(ii) We leave it to the reader to show that $\mathbf{R_{CK_0}}$ is one-step complete either directly or as a corollary to one-step cut-free completeness of $\mathbf{R_{CK}}$, which we now set out to prove. Let $\Gamma = \{\neg(p_i \Rightarrow q_i) \mid i \in I\} \cup \{p'_j \Rightarrow q'_j \mid j \in J\}$, and let τ be a $\mathcal{P}(X)$-valuation such that $\mathsf{CK}(X), \tau \models \Gamma$. We claim that there exists $j \in J$ such that

$$\bigcap_{i \in I_j} \tau(q_i) \subseteq \tau(q'_j), \qquad (*)$$

where $I_j = \{i \in I \mid \tau(p_i) = \tau(p'_j)\}$. Assume, for a contradiction, that this is not the case. Then, for every $j \in J$, $\bigcap_{i \in I_j} \tau(q_i) \not\subseteq \tau(q'_j)$. Define the function $f : 2(X) \to \mathcal{P}(X)$ by

$$f(S) = \begin{cases} \bigcap_{i \in I_j} \tau(q_i) & S = \tau(p'_j) \\ \emptyset & \text{otherwise.} \end{cases}$$

(This is well-defined since $I_j = I_k$ whenever $\tau(p'_j) = \tau(p'_k)$.) Then $f(\tau(p_i)) \subseteq \tau(q_i)$ for all $i \in I$ and $f(\tau(p'_j)) \not\subseteq q'_j$ for all $j \in J$, contradicting $\mathsf{CK}(X), \tau \models \Gamma$. Having thus proved the claim, we pick $j \in J$ satisfying $(*)$. Writing $I_j = \{i_1, \dots, i_k\}$, we now derive Γ using a single instance of the $\mathbf{R_{CK}}$-rule

$$\frac{\{\neg q_i \mid i \in I_j\}, q'_j \quad \neg p'_j, p_{i_1} \quad \dots \quad \neg p'_j, p_{i_k} \quad \neg p_{i_1}, p'_j \quad \dots \quad \neg p_{i_k}, p'_j}{\{\neg(p_i \Rightarrow q_i) \mid i \in I_j\}, p_j \Rightarrow q_j},$$

whose premises hold in X, τ by $(*)$ and the definition of I_j.
This proof is easily modified to establish that also the rule set $\mathbf{R_{CKId}}$ is one-step cut-free complete for $\mathsf{CK_{Id}}$: if Γ is as above, one proves that there exists $j \in J$ satisfying the weaker condition

$$\tau(p'_j) \cap \bigcap_{i \in I_j} \tau(q_i) \subseteq \tau(q'_j). \qquad (+)$$

This is proved by constructing f as above, but with

$$f(\tau(p'_j)) = \tau(p'_j) \cap \bigcap_{i \in I_j} \tau(q_i),$$

which defines an element of $\mathsf{CK_{Id}}(X)$. From j satisfying $(+)$, one obtains an instance of (CI) that proves Γ. As a consequence, cut is admissible in $\mathbf{GR_{CK}}$ and $\mathbf{GR_{CKId}}$.

5 Cut Elimination, Syntactically

In the previous section, we have seen that one-step cut-free completeness is a sufficient criterion to ensure that an ensuing sequent calculus enjoys cut-free completeness, and we have deduced admissibility of contraction on the way. We now complement these results with a purely syntactic criterion for admissibility of congruence, cut and contraction. As we will see, syntactic conditions imposed on the set of modal rules under scrutiny will be equivalent to one-step cut-free completeness.

We start with admissibility of congruence and contraction, which is – unlike weakening and inversion – not automatic, and only holds if the underlying rule set satisfies an additional property. Recall that $\mathbf{GC}_0$ consists of all propositional sequent rules and the cut rule, but restricted to purely propositional formulas.

Definition 5.1. A set $\mathbf{R}$ of one-step rules *absorbs congruence* if for every n-ary $\heartsuit \in \Lambda$ and all $p_1, \ldots, p_n$ and $q_1, \ldots, q_n \in V$, there exists a rule $\Gamma_1 \ldots \Gamma_k/\Gamma_0$ and a substitution $\sigma : V \to \mathsf{Prop}(V)$ such that $\Gamma_0\sigma \subseteq \{\neg\heartsuit(p_1, \ldots, p_n), \heartsuit(q_1, \ldots, q_n)\}$ as multisets, and

$$\mathbf{GC}_0 + \neg p_1, q_1 + \cdots + \neg p_n, q_n + p_1, \neg q_1 + \cdots + p_n, \neg q_n \vdash \Gamma_i\sigma$$

for all $i = 1, \ldots, k$.

The requirement of absorption of congruence essentially amounts to the fact that the congruence rule (Remark 4.4) is contained in the given set of one-step rules, up to possible weakening of the premises. It is easy to see that this indeed holds in our examples. As we will see later, absorption of congruence implies that the congruence rule is admissible in the ensuing sequent system.

Example 5.2. We consider the rules presented in Example 4.6. First, suppose that $\mathbf{R}$ consists of (N) and (D) only. We claim that $\mathbf{R}$ does *not* absorb congruence. If this were the case, there would be a rule $R = \Gamma_1 \ldots \Gamma_n/\Gamma_0 \in \mathbf{R}$ and a substitution $\sigma : V \to \mathsf{Prop}(V)$ such that $\Gamma_0\sigma \subseteq \neg\Box p, \Box q$ as multisets and $\mathbf{GC}_0 + \neg p, q + p, \neg q \vdash \Gamma_i$ for all $i = 1, \ldots, n$. Owing to the fact that $\mathbf{R}$ consists of (N) and (D) only, and the inclusion $\Gamma_0\sigma \subseteq \neg\Box p, \Box q$ is required to hold in the sense of multisets, the only choice for R and σ is to take r as necessitation $p/\Box p$ and σ with $\sigma(p) = q$. But clearly the substituted premise $p\sigma \equiv q$ of the necessitation rule is not provable from the assumptions $\neg p, q$ and $p, \neg q$.

The situation is different for the rules (K), as the substitution instance $\neg p, q/\neg\Box p, \Box q$ of (K) fulfils the requirements of the definition.

Definition 5.3. A set $\mathbf{R}$ of one-step rules *absorbs contraction* if for every rule $\Gamma_1 \ldots \Gamma_n/\Gamma_0 \in \mathbf{R}$ and every renaming $\sigma : V \to V$, there exists a rule $\Delta_1 \ldots \Delta_m/\Delta_0 \in \mathbf{R}$ and a renaming $\rho : V \to V$ such that $\Delta_0\rho \subseteq \mathrm{supp}(\Gamma_0\sigma)$ (in particular, the multiset $\Delta_0\rho$ is a set, i.e. Δ_0 is a set and ρ does not identify any literals occurring in Δ_0) and

$$\mathbf{GC}_0 + \{\Gamma_i\sigma \mid 1 \leq i \leq n\} \vdash \Delta_j\rho$$

for all $j = 1, \ldots, m$.

Given any rule $r = \Gamma_1 \dots \Gamma_n/\Gamma_0$, a renaming $\sigma : V \to V$ may identify literals in the conclusion $\Gamma_0\sigma$ of r. In this case, $\Gamma_0\sigma$ is a proper super-multiset of its support, and the contraction rule would be needed to conclude $\mathrm{supp}(\Gamma_0\sigma)$ from $\Gamma_0\sigma$. If the rule set $\mathbf{R}$ absorbs contraction, these two deduction steps can be absorbed into the application of a single modal rule: we can find a (generally different) one-step rule $s = \Delta_1 \dots \Delta_m/\Delta_0$ together with a renaming that proves the contracted conclusion $\mathrm{supp}(\Gamma_0\sigma)$ of the original rule with the help of weakening (this amounts to the condition $\Delta_0\rho \subseteq \mathrm{supp}(\Gamma_0\sigma)$). In order to replace an instance of r by an instance of s, we furthermore need to require that all premises $\Delta_j\rho$ of s are provable, given the premises $\Gamma_i\sigma$ of r. Since the premises of a one-step rule have a strictly smaller modal rank than the conclusion, we are inductively permitted to use the cut rule in order to prove the conclusions $\Delta_j\rho$ from the set of premises $\{\Gamma_1\sigma, \dots, \Gamma_n\sigma\}$ of s. Formally, this is captured by the condition $\mathbf{GC}_0 + \{\Gamma_i\sigma \mid 1 \le i \le n\} \vdash \Delta_j\rho$ for $j = 1, \dots, m$. We make this argument precise when we we prove admissibility of cut and contraction by induction on the modal rank of the endsequent (Proposition 5.6). For the basic modal logic K, the situation is as follows:

Example 5.4. Consider the rule set $\mathbf{R}$ consisting of (N) and (D) introduced in Example 4.6. It is easy to see that $\mathbf{R}$ does *not* absorb contraction: consider a substitution σ with $\sigma(p) = \sigma(q) = p$ and $\sigma(r) = r$. This substitution identifies the literals $\neg\Box p$ and $\neg\Box q$ in the conclusion of (D). If $\mathbf{R}$ were closed under contraction, there would be a rule $R = \Delta_1 \dots \Delta_m/\Delta_0$ and a substitution ρ such that $\Delta_0\rho \subseteq \{\neg\Box p, \Box r\}$ as multisets and $\mathbf{GC}_0 + \{\neg p, \neg p, r\} \vdash \Delta_j\rho\}$ for all $j = 1, \dots, m$. Then R must be the necessitation rule $p/\Box p$, and $\rho(p) = r$, but of course $\neg p, \neg p, r$ does not propositionally entail the substituted premise r of the necessitation rule.

The situation is different if we adopt the rule set (K), which does absorb contraction as we now show. For the sake of readability, we treat only the case of a renaming σ that identifies precisely two literals in the conclusion of (K) (it is readily seen that this is in fact generally sufficient); w.l.o.g. we may thus assume that σ identifies the variables p_n and p_{n-1} and acts as the identity on all other propositional variables, so that we obtain a substitution instance of (K) of the form

$$\frac{\neg p_1, \dots, \neg p_{n-1}, \neg p_{n-1}, p_0}{\neg\Box p_1, \dots, \neg\Box p_{n-1}, \neg\Box p_{n-1}, \Box p_0}. \qquad (*)$$

Then we take $\Delta_1 \dots \Delta_m/\Delta_0$ to be $\neg p_1, \dots, \neg p_{n-1}, p_0/\neg\Box p_1, \dots, \neg\Box p_{n-1}, \Box p_0$, and ρ to be the identity substitution. Then $\Delta_0\rho \equiv \neg\Box p_1, \dots, \neg\Box p_{n-1}, \Box p_0 \subseteq$

supp($\neg\Box p_1, \dots, \neg\Box p_{n-1}, \neg\Box p_{n-1}, \Box p_0$) as multisets, and the premise $\neg p_1, \dots, \neg p_{n-1}, p_0$ of the new rule instance is propositionally entailed by (in fact equivalent to) the premise $\neg p_1, \dots, \neg p_{n-1}, \neg p_{n-1}, p_0$ of the original rule instance $(*)$, as required.

The definition of absorption of cut is modelled on the same idea: an application of cut to the conclusions of two one-step rules r_1, r_2 can be replaced by a different one-step rule r_0 such that all the premises of r_0 are propositional consequences of (i.e. can be derived with the help of cut from) the premises of r_1, r_2.

Definition 5.5. A set $\mathbf{R}$ of one-step rules *absorbs cut* if for all $\Gamma_1 \dots \Gamma_n / \Gamma_0$ and all $\Delta_1 \dots \Delta_m / \Delta_0 \in \mathbf{R}$ and all renamings $\sigma, \rho : V \to V$ such that $\Gamma_0 \sigma = \Gamma, A$ and $\Delta_0 \rho = \Delta, \neg A$, there exists a rule $\Sigma_1 \dots \Sigma_l / \Sigma_0$ and a renaming $\kappa : V \to V$ such that $\mathrm{supp}(\Sigma_0 \kappa) \subseteq \Gamma, \Delta$ and

$$\mathbf{GC}_0 + \{\Gamma_i \sigma \mid 1 \leq i \leq n\} + \{\Delta_i \rho \mid 1 \leq i \leq m\} \vdash \Sigma_j \kappa$$

for all $j = 1, \dots, l$.

We note that all absorption properties defined above are local in the sense that they can be checked by considering just the set of modal (one-step) rules, without having to take into account cuts involving propositional rules.

We formulate admissibility of cut, contraction, and the non-atomic axiom rule in relativised form as follows.

Proposition 5.6. *If* $\mathbf{R}$ *absorbs cut, contraction, and congruence, then*

- $\mathbf{GR}_n \vdash \Gamma, A, \neg A$
- $\mathbf{GR}_n \vdash \Gamma, A$ *whenever* $\mathbf{GR}_n \vdash \Gamma, A, A$
- $\mathbf{GR}_n \vdash \Gamma$ *whenever* $\mathbf{GRC}_n \vdash \Gamma$

for all $\Gamma \in \mathcal{S}(\mathcal{F}_n(\Lambda))$ *and all* $A \in \mathcal{F}_n(\Lambda)$.

Proof. We proceed by induction on n, where the base case $n = 0$ is just a collection of known statements about $\mathbf{G}$. For $n > 0$, we note that, as a consequence of Lemma 3.9, $\mathbf{GR}_n \vdash \Gamma$ iff $\mathbf{G}_n + H \vdash \Gamma$ where

$$H = \{\Gamma_0 \sigma, \Delta \mid \Gamma_1 \dots \Gamma_k / \Gamma_0 \in \mathbf{R}, \sigma : v \to \mathcal{F}_{n-1}(\Lambda), \forall_{1 \leq i \leq k} (\mathbf{GR}_{n-1} \vdash \Gamma_i \sigma)\}$$

for all $\Gamma \in \mathcal{S}(\mathcal{F}_n(\Lambda))$.

We first show that $\mathbf{GR}_n \vdash \Gamma, A, \neg A$. By definition of $\mathcal{F}_n(\Lambda)$, we have that $\Gamma, A, \neg A \in \mathcal{S}(\mathsf{Prop}(\Lambda(\mathcal{F}_{n-1}(\Lambda) \cup V)))$ and therefore $\Gamma, A, \neg A = \Gamma'\sigma, B\sigma, \neg B\sigma$ for a formula $B \in \mathsf{Prop}(V)$, a sequent $\Gamma' \in \mathcal{S}(\mathsf{Prop}(V))$, and a substitution $\sigma : V \to \Lambda(\mathcal{F}_{n-1}(\Lambda)) \cup V$. By admissibility of the non-atomic axiom rule in $\mathbf{G}$, we have $\mathbf{G}_0 \vdash \Gamma', \neg B, B$. The claim will thus follow from the more general statement that $\mathbf{G}_0 \vdash \Delta$ implies that $\mathbf{GR}_n \vdash \Delta\sigma$ for all $\Delta \in \mathcal{S}(\mathsf{Prop}(V))$ and all $\sigma : V \to \Lambda(\mathcal{F}_{n-1}(\Lambda)) \cup V$, which we prove by induction on the proof of $\mathbf{G}_0 \vdash \Delta$. The interesting case, in which absorption of congruence is needed, is $\Delta = \neg p, p, \Delta'$, i.e. Δ is an axiom. We show that $\mathbf{GR}_n \vdash p\sigma, \neg p\sigma, \Delta'\sigma$, for which it suffices to establish that that $\mathbf{GR}_n \vdash \neg p\sigma, p\sigma$ by admissibility of weakening (Lemma 3.11). As σ takes values in $\Lambda(\mathcal{F}_{n-1}(\Lambda)) \cup V$ and the case $\sigma(p) \in V$ is readily discharged by the (atomic) axiom rule, we may assume that $\sigma(p) = \heartsuit(A_1, \dots, A_k)$ where $A_1, \dots, A_k \in \mathcal{F}_{n-1}(\Lambda)$. By absorption of congruence, we may find a rule $\Gamma_1 \dots \Gamma_l / \Gamma_0$ and a substitution $\rho : V \to \mathsf{Prop}(V)$ such that $\mathbf{GC}_0 + \neg p_1, q_1 + \dots + \neg p_k, q_k + p_1, \neg q_1 + \dots + p_k, \neg q_k \vdash \Gamma_i\rho$ for $i = 1, \dots, l$, and $\Gamma_0\rho \subseteq \heartsuit(p_1, \dots, p_k), \neg\heartsuit(p_1, \dots, p_k)$. By substitutivity (Lemma 3.14) this entails that $\mathbf{GRC}_{n-1} + \mathbf{Ax}_{n-1} \vdash \Gamma_i\rho\tau$ for $i = 1, \dots, l$ where $\tau(p_i) = \tau(q_i) = A_i$. By the induction hypothesis, cut may be eliminated and all elements of $\mathbf{Ax}_n$ are $\mathbf{GR}_{n-1}$-derivable, hence $\mathbf{GR}_{n-1} \vdash \Gamma_i\rho\tau$. Therefore, $\mathbf{GR}_n \vdash \Gamma_0\rho\tau$, and hence $\mathbf{GR}_n \vdash \heartsuit(A_1, \dots, A_k), \neg\heartsuit(A_1, \dots, A_k)$ by weakening (Lemma 3.11). This finishes the inductive case for the axiom rule. The cases for the other rules are standard; as an example, we deal with the case that $(\wedge)$ is the last rule applied in the proof of $\mathbf{G}_0 \vdash \Delta$. Then $\Delta = (B_0 \wedge B_1), \Gamma$ and both $\mathbf{G}_0 \vdash B_0, \Gamma$, $\mathbf{G}_0 \vdash B_1, \Gamma$. As these proofs are shorter, we may apply the induction hypothesis to obtain that $\mathbf{GR}_n \vdash B_0\sigma, \Gamma\sigma$ and $\mathbf{GR}_n \vdash B_1\sigma, \Gamma\sigma$, whence $\mathbf{GR}_n \vdash (B_0 \wedge B_1)\sigma, \Gamma\sigma$ as required.

We now deal with contraction. So suppose that $\mathbf{GR}_n \vdash \Gamma$, or equivalently, $\mathbf{G}_n + H \vdash \Gamma$ with H as above. Strengthening the claim, we show that $\mathbf{GR}_n \vdash \mathrm{supp}(\Gamma)$ by induction on the $\mathbf{G}_n$-proof of Γ from the additional assumptions in H. First suppose that $\Gamma \in H$, that is, $\Gamma = \Gamma_0\sigma, \Delta$ for $\Gamma_1 \dots \Gamma_k / \Gamma_0 \in \mathbf{R}$, $H \in \mathcal{S}(\mathcal{F}_n(\Lambda))$, and $\sigma : V \to \mathcal{F}_{n-1}(\Lambda)$. We may factorise $\sigma = \sigma_m \circ \sigma_e$ where $\sigma_e : V \to V$ is a renaming and $\sigma_m : V \to \mathcal{F}_{n-1}(\Lambda)$ is an injective substitution. As $\mathbf{R}$ absorbs contraction, we can find a rule $\Delta_1 \dots \Delta_l / \Delta_0$ and a renaming $\rho : V \to V$ such that $\Delta_0\rho \subseteq \mathrm{supp}(\Gamma_0\sigma_e)$ and $\mathbf{GC}_0 + \{\Gamma_i\sigma_e \mid i = 1, \dots, k\} \vdash \Delta_j\rho$ for $j = 1, \dots, l$. By substitutivity (Lemma 3.14) this entails

$$\mathbf{GRC}_{n-1} + \mathbf{Ax}_{n-1} + \{\Gamma_i\sigma_e\sigma_m \mid i = 1, \dots, k\} \vdash \Delta_j\rho\sigma_m$$

for $j = 1, \dots, l$.

As all elements of $\mathbf{Ax}_{n-1}$ as well as the additional assumptions $\Gamma_i \sigma_e \sigma_m$ are $\mathbf{GR}_{n-1}$-derivable, we conclude that $\mathbf{GRC}_{n-1} \vdash \Delta_j \rho \sigma_m$ for all $j = 1, \ldots, l$, and hence $\mathbf{GR}_{n-1} \vdash \Delta_j \rho \sigma_m$ by induction. Thus, $\mathbf{GR}_n \vdash \Delta_0 \rho \sigma_m$. As σ_m is injective and $\Delta_0 \rho \subseteq \mathrm{supp}(\Gamma_0 \sigma_e)$, we have $\Delta_0 \rho \sigma_m \subseteq \mathrm{supp}(\Gamma_0 \sigma_e \sigma_m) = \mathrm{supp}(\Gamma_0 \sigma)$. By weakening (Lemma 3.11) we finally obtain $\mathbf{GR}_n \vdash \mathrm{supp}(\Gamma)$.

The remaining cases, where Γ has been proved using rules of $\mathbf{G}_n$, are standard, and we exemplify the argument for the case of the $(\neg\wedge)$-rule. Suppose that $\mathbf{G}_n + H \vdash \Gamma$ has a proof of depth h and $(\neg\wedge)$ was the last rule applied in this proof. Then $\Gamma = \Delta, \neg(B \wedge C)$ and $\mathbf{G}_n + H \vdash \Delta, \neg B, \neg C$ with a proof of depth $< h$. We distinguish two cases. First, if $\neg(B \wedge C) \notin \Delta$, we have, by induction hypothesis (and possibly an application of weakening), that $\mathbf{G}_n + H \vdash \mathrm{supp}(\Delta), \neg B, \neg C$ whence $\mathbf{G}_n + H \vdash \mathrm{supp}(\Delta), \neg(B \wedge C) = \mathrm{supp}(\Delta, \neg(B \wedge C)) = \mathrm{supp}(\Gamma)$. Now consider the case $\neg(B \wedge C) \in \Delta$, i.e. the multiplicity of $\neg(B \wedge C)$ in Δ is $m > 0$. By repeated application of inversion (Lemma 3.12) we have that $\mathbf{G}_n + H \vdash \Delta_0, \Delta_1$ where Δ_1 consists of $m + 1$ copies each of $\neg B$ and $\neg C$ and Δ_0 arises by removing all m occurrences of $\neg(B \wedge C)$ from Δ, with a proof of depth $< h$. By induction, it follows that $\mathbf{G}_n + H \vdash \mathrm{supp}(\Delta_0), \neg B, \neg C$ whence $\mathbf{G}_n + H \vdash \mathrm{supp}(\Delta_0), \neg(B \wedge C) = \mathrm{supp}(\Delta_0, \neg(B \wedge C)) = \mathrm{supp}(\Delta, \neg(B \wedge C))$ as required.

We turn to admissibility of cut, where it suffices to show that $\mathbf{GR}_n \vdash \Gamma, \Delta$ whenever $\mathbf{GR}_n \vdash \Gamma, A$ and $\mathbf{GR}_n \vdash \Delta, \neg A$. If this is the case, we find that $\mathbf{G}_n + H \vdash \Gamma, A$ and $\mathbf{G}_n + H \vdash \Delta, \neg A$ with H as above. We show that $\mathbf{G}_n + H \vdash \Gamma, \Delta$ using the classical double induction method, with outer induction on the rank of the cut formula A and inner induction on the sum of the size of the proof trees of $\mathbf{G}_n + H \vdash \Gamma, A$ and $\mathbf{G}_n + H \vdash \Delta, \neg A$. We distinguish three different types of cut: (a) cuts between elements of H, (b) cuts between elements of H and conclusions of $\mathbf{G}_n$-rules and (c) cuts between conclusions of $\mathbf{G}_n$-rules. As regards (a), we have that $\Gamma, A = \Gamma_0 \sigma, \Gamma'$ and $\Delta, \neg A = \Delta_0 \rho, \Delta'$ for two substitutions $\sigma, \rho : V \to \mathcal{F}_{n-1}(\Lambda)$ and two rules $\Gamma_1 \ldots \Gamma_k / \Gamma_0$ and $\Delta_1 \ldots \Delta_l / \Delta_0 \in \mathbf{R}$. In case $A \in \Gamma'$ or $\neg A \in \Delta'$, we are done immediately as $\Gamma, \Delta \in H$. To see this in the case $A \in \Gamma'$ suppose that $\Gamma' = A, \Gamma''$. Then $\Gamma, \Delta = \Gamma_0 \sigma, \Gamma'', \Delta \in H$. The case $\neg A \in \Delta'$ is entirely analogous.

The remaining case is that $A \in \Gamma_0 \sigma$ and $\neg A \in \Delta_0 \rho$. As $\mathbf{R}$ absorbs cut, we may use the substitution lemma 3.14 to find a rule $\Sigma_1 \ldots \Sigma_m / \Sigma_0$ and a substitution $\kappa : V \to \mathcal{F}_{n-1}(\Lambda)$ such that $\mathrm{supp}(\Sigma_0 \kappa) \subseteq \Gamma, \Delta$ and

$$\mathbf{GC}_{n-1} + \mathbf{Ax}_{n-1} + \{\Gamma_i \sigma \mid i = 1, \ldots, k\} + \{\Delta_j \rho \mid j = 1, \ldots, l\} \vdash \Sigma_j \kappa$$

for all $j = 1, \ldots, m$. As all elements of $\mathbf{Ax}_{n-1}$ and all assumptions $\Gamma_i \sigma$ ($i =$

$1, \dots k$) and $\Delta_j \rho$ ($j = 1, \dots, l$) are $\mathbf{GR}_{n-1}$-derivable and cut is admissible in $\mathbf{GR}_{n-1}$ by induction hypothesis, we have that $\mathbf{GR}_{n-1} \vdash \Sigma_j \kappa$ for all $j = 1, \dots, m$. As contraction is admissible in $\mathbf{GR}_n$, we finally obtain $\mathbf{GR}_n \vdash \mathrm{supp}(\Sigma_0 \kappa_0) \subseteq \Gamma, \Delta$, and $\mathbf{GR}_n \vdash \Gamma, \Delta$ follows from the relativised weakening lemma 3.11.

We now look at cuts of type (b), that is, cuts between propositional rules and hypotheses in H. We treat the case that $\Gamma, A = \Gamma_0 \sigma, \Gamma' \in H$ and $\Delta, \neg A$ has been derived using a propositional rule (the case $\Gamma, \neg A \in H$ is almost identical but slightly simpler, as it leaves fewer cases for A). If $A \in \Gamma'$, that is, $\Gamma' = \Gamma'', A$, the cut happens on a formula in the weakening context Γ' and we have $\Gamma, \Delta = \Gamma_0 \sigma, \Gamma'', \Delta \in H$ whence $\mathbf{G}_n + H \vdash \Gamma, \Delta$. Now suppose that $A \in \Gamma_0 \sigma$, i.e. $\Gamma_0 \sigma = A, \Gamma''$. We proceed by a case distinction over the last rule applied in the proof of $\mathbf{G}_n + H \vdash \Delta, \neg A$.

Rule ($\wedge$)*:* Since A occurs in the conclusion of a modal rule, A is not of the form $B \wedge C$, so that necessarily $\Delta = \Delta', B \wedge C$, and we have $\mathbf{G}_n + H \vdash \Delta', B, \neg A$ and $\mathbf{G}_n + H \vdash \Delta', C, \neg A$ with shorter proofs. By the inner induction hypothesis, cutting the latter two endsequents with Γ, A is admissible, that is, $\mathbf{G}_n + H \vdash \Gamma, \Delta', B$ and $\mathbf{G}_n + H \vdash \Gamma, \Delta', C$. Applying ($\wedge$) yields $\mathbf{G}_n + H \vdash \Gamma, \Delta', B \wedge C = \Gamma, \Delta$.

Rule ($\neg\wedge$)*:* As in the previous case, necessarily $\Delta = \Delta', \neg(B \wedge C)$ and $\mathbf{GR} \vdash \Delta', \neg B, \neg C, \neg A$ with a shorter proof. Again by the inner induction hypothesis, cuts on A with the latter endsequent are admissible, that is, $\mathbf{G}_n + H \vdash \Gamma, \Delta', \neg B, \neg C$ and consequently $\mathbf{G}_n + H \vdash \Gamma, \Delta', \neg(B \wedge C) = \Gamma, \Delta$.

Rule ($\neg\neg$)*:* First suppose that $\neg A$ is principal whence $A = \neg A'$ and $\mathbf{G}_n + H \vdash \Delta, A'$ with a shorter proof. We may now use the inner induction hypothesis to obtain $\mathbf{G}_n + H \vdash \Gamma, \Delta$ by an admissible cut on A'. Now suppose that $\neg A$ is not principal in the application of ($\neg\neg$). Then $\Delta = \Delta', \neg\neg B$, and $\mathbf{G}_n + H \vdash \Delta', B, \neg A$ with a shorter proof. Again using the inner induction hypothesis, an admissible cut on A yields $\mathbf{G}_n + H \vdash \Delta', B, \Gamma$, and applying ($\neg\neg$) yields $\mathbf{G}_n + H \vdash \Delta', \neg\neg B, \Gamma = \Delta, \Gamma$.

Rule (Ax)*:* As in the case for ($\wedge$), necessarily $\Delta = p, \neg p, \Delta''$. Then $\Gamma, \Delta = \Gamma, p, \neg p, \Delta''$ is again an axiom and $\mathbf{G}_n + H \vdash \Gamma, \Delta$.

This finishes the case of cuts of type (b). The elimination of cuts of type (c) between conclusions of propositional rules is standard, and follows from the $\mathbf{GR}_n$-admissibility of contraction (that we have already established) and the inversion lemma 3.13. □

The following theorem, which readily follows from Proposition 5.6 and Proposition 3.15, therefore provides a purely syntactic counterpart of Theorem 4.15.

Theorem 5.7. *If* **R** *absorbs cut, contraction and congruence, then all instances of the cut and contraction rules*

$$\frac{\Gamma, A \qquad \Delta, \neg A}{\Gamma, \Delta} \qquad \frac{\Gamma, A, A}{\Gamma, A},$$

where $\Gamma, \Delta \in \mathcal{S}$ *and* $A \in \mathcal{F}(\Lambda)$, *are admissible in* **GR**.

Our last main result in this section is that in the presence of one-step completeness, the absorption properties are actually equivalent with one-step cut-free completeness, the condition used in the semantic proof of cut eliminability. We split the equivalence into two separate lemmas.

Proposition 5.8. *Let* **R** *be one-step complete. Then* **R** *is one-step cut-free complete if* **R** *absorbs cut and contraction.*

Proof. Let X be a set, and let τ be a $\mathcal{P}(X)$-valuation. Consider the set

$$\Psi = \{\Gamma_0\sigma, \Delta \mid \Delta \in \mathcal{S}(\Lambda(V)), \Gamma_1 \dots \Gamma_n/\Gamma_0 \in \mathbf{R},$$
$$\sigma : V \to \mathsf{Prop}(V), \forall_{1 \le i \le n}(X, \tau \models \Gamma_i\sigma)\}.$$

As rule conclusions do not contain top-level propositional connectives, Ψ is trivially closed under inversion. Moreover, Ψ is closed under weakening, i.e. $\Gamma \in \Psi$ implies that $\Gamma, \Delta \in \Psi$ for $\Delta \in \mathcal{S}(\Lambda(V))$.

We now establish that Ψ is closed under contraction, i.e. $\Gamma \in \Psi$ implies that $\mathrm{supp}(\Gamma) \in \Psi$. If $\Gamma \in \Psi$, we can find a rule $\Gamma_1 \dots \Gamma_n/\Gamma_0 \in \mathbf{R}$ and a substitution $\sigma : V \to \mathsf{Prop}(V)$ such that $X, \tau \models \Gamma_i\sigma$ for all $i = 1, \dots, n$ and $\Gamma = \Gamma_0\sigma, \Gamma'$ for some $\Gamma' \in \mathcal{S}(\Lambda(V))$. It suffices to show that $\mathrm{supp}(\Gamma_0\sigma) \in \Psi$, as Ψ is closed under weakening. The idea is to replace every (propositional) formula A occurring in $\Gamma_0\sigma$ by a propositional variable p_A and then use absorption of contraction. We therefore choose pairwise distinct propositional variables p_A for all $A \in \{\sigma(p) \mid p \in V\}$ and consider the renaming σ_0 defined by $\sigma_0(p) = p_{\sigma(p)}$. We may moreover choose an injective substitution θ that satisfies $\theta(p_A) = A$. This allows us to factor $\sigma = \theta \circ \sigma_0$. Using the fact that **R** absorbs contraction, we now find a rule $\Delta_1 \dots \Delta_m/\Delta_0$ and a renaming $\rho : V \to V$ such that $\Delta_0\rho \subseteq \mathrm{supp}(\Gamma_0\sigma_0)$ and

$$\mathbf{GC}_0 + \{\Gamma_i\sigma_0 \mid i = 1, \dots, n\} \vdash \Delta_j\rho$$

for all $j = 1, \dots, m$. As $X, \tau \models \Gamma_i\sigma$ for all $i = 1, \dots, n$ we have that $X, \tau \models \Delta_j\rho\theta$ for $j = 1, \dots, m$ by soundness of propositional reasoning. Hence $\Delta_0\rho\theta \in \Psi$. As $\Delta_0\rho \subseteq \mathrm{supp}(\Gamma_0\sigma_0)$ and θ is injective, we have $\Delta_0\rho\theta \subseteq \mathrm{supp}(\Gamma_0\sigma_0\theta) = \mathrm{supp}(\Gamma_0\sigma)$. This shows that $\mathrm{supp}(\Gamma_0\sigma) \in \Psi$ as required.

Next, we show that Ψ is closed under cut, i.e. assuming that Γ, A and $\Delta, \neg A$ are in Ψ we show that $\Gamma, \Delta \in \Psi$. By definition, we have rules $\Gamma_1 \dots \Gamma_n / \Gamma_0$ and $\Delta_1 \dots \Delta_m / \Delta_0 \in \mathbf{R}$ and substitutions $\sigma, \rho : V \to \mathsf{Prop}(V)$ such that $X, \tau \models \Gamma_i \sigma$ and $X, \tau \models \Delta_j \tau$ for $i = 1, \dots n$ and $j = 1, \dots, m$, as well as

$$\Gamma, A = \Gamma_0 \sigma, \Gamma' \text{ and } \Delta, \neg A = \Delta_0 \tau, \Delta'$$

for some $\Gamma', \Delta' \in \mathcal{S}(\Lambda(V))$. In case $A \in \Gamma'$ we have $\Gamma, \Delta = \Gamma_0 \sigma, \Xi$ for some $\Xi \in \mathcal{S}(\Lambda(V))$ so there is nothing to show. By the same argument, we are done if $\neg A \in \Delta'$, so we can assume that $A \in \Gamma_0 \sigma$ and $\neg A \in \Delta_0 \tau$. Thus, we have

$$\Gamma_0 = B, \Gamma'' \text{ and } \Delta_0 = \neg C, \Delta''$$

with $B\sigma = A = C\tau$. As above, we choose pairwise distinct propositional variables p_A for all $A \in \{\sigma(p) \mid p \in V\} \cup \{\tau(p) \mid p \in V\}$ and pick an injective substitution $\theta : V \to \mathsf{Prop}(V)$ such that $\theta(p_A) = A$. If $\sigma_0, \tau_0 : V \to V$ are defined by $\sigma_0(p) = p_{\sigma(p)}$ and $\tau_0(p) = p_{\tau(p)}$, we can factor $\sigma = \theta \circ \sigma_0$ and $\tau = \theta \circ \tau_0$. Moreover, $B\sigma_0 = C\tau_0$ by injectivity of θ. We may therefore invoke absorption of cut to find a rule $\Sigma_1 \dots \Sigma_l / \Sigma_0$ and a renaming $\kappa : V \to V$ such that $\mathrm{supp}(\Sigma_0 \kappa) \subseteq \Gamma'' \sigma_0, \Delta'' \tau_0$ and

$$\mathbf{GC}_0 + \{\Gamma_i \sigma_0 \mid i = 1, \dots, n\} + \{\Delta_i \tau_0 \mid i = 1, \dots, m\} \vdash \Sigma_k \kappa$$

for all $k = 1, \dots, l$. Soundness of $\mathbf{GC}_0$ now entails that $X, \tau \models \Sigma_k \kappa \theta$ for all $k = 1, \dots, l$ as $X, \tau \models \Gamma_i \sigma_0 \theta$ and $X, \tau \models \Delta_j \tau_0 \theta$ for $i = 1, \dots, n$ and $j = 1, \dots, m$; therefore $\Sigma_0 \kappa \theta \in \Psi$. Applying θ to the inclusion $\mathrm{supp}(\Sigma_0 \kappa) \subseteq \Gamma'' \sigma_0, \Delta'' \tau_0$ we obtain $\mathrm{supp}(\Sigma_0 \kappa \theta) \subseteq \Gamma, \Delta$ (recall that $\Gamma, A = \Gamma_0 \sigma, \Gamma' = A, \Gamma'' \sigma, \Gamma'$ and $\Delta, \neg A = \neg A, \Delta'' \tau, \Delta'$). As Ψ is closed under weakening and contraction, we obtain that $\Gamma, \Delta \in \Psi$ as claimed.

Finally, we establish that $\mathbf{R}$ is one-step cut-free complete, using the criterion of Lemma 4.3. So let $\Gamma \in \mathcal{S}(\Lambda(V))$, and let $\tau : V \to \mathcal{P}(X)$ such that $TX, \tau \models \Gamma$. We need to show that there exist $\Gamma_1 \dots \Gamma_n / \Gamma_0 \in \mathbf{R}$ and a renaming $\sigma : V \to V$ such that $\Gamma_0 \sigma \subseteq \Gamma$ and $X, \tau \models \Gamma_i, \sigma$, $i = 1, \dots, n$. As $\mathbf{R}$ is one-step complete, $\mathbf{GC}_1 + \Psi \vdash \Gamma$. By Lemma 5.9 below, we conclude $\Gamma \in \Psi$, which finishes the proof. □

To complete the proof of Proposition 5.8 we need to supply the following lemma.

Lemma 5.9. *Let $\Psi \subseteq \mathcal{S}(\Lambda(V))$ be closed under cut, contraction, weakening, and inversion. Then $\mathbf{GC}_1 + \Psi \vdash \Gamma$ iff $\mathbf{G}_1 + \Psi \vdash \Gamma$. In particular, for $\Gamma \in \mathcal{S}(\Lambda(V))$, we have $\mathbf{GC}_1 + \Psi \vdash \Gamma$ iff $\Gamma \in \Psi$.*

Proof. This is a standard cut-elimination proof for $\mathbf{G}$ where the fact that Ψ is closed under cut, contraction, weakening, and inversion allows propagating instances of the respective rules to the leaves; see [29, Section 4.4] for details. □

The converse of Proposition 5.8 requires more semantic considerations. We start with a simple property of the semantics of propositional logic.

Lemma and Definition 5.10. *Let $H \subseteq \mathcal{S}(\mathsf{Prop}(V))$. Let $X_0 = 2^V$, where 2 is the set $\{\bot, \top\}$ truth values, and let $\tau_0 : V \to \mathcal{P}(X_0)$ be given by $\tau(a) = \{\kappa \mid \kappa(a) = \top\}$. The* canonical model *of H is the pair X, τ defined by $X = \bigcap_{\Delta \in H} [\![\Delta]\!]_{X_0}^{\tau_0}$ and $\tau(p) = \tau_0(p) \cap X$. Then $X, \tau \models \Gamma$ iff $\mathbf{GC}_0 + H \vdash \Gamma$ for every $\Gamma \in \mathcal{S}(\mathsf{Prop}(V))$; in particular, $X, \tau \models \Delta$ for all $\Delta \in H$.*

Proof. The statement reduces immediately to the case that Γ is a single formula A and H contains a single sequent consisting of a single formula B. By soundness and completeness of $\mathbf{GC}_0$, it suffices to show that $X, \tau \models A$ iff $B \to A$ is a propositional tautology. As by construction, $X, \tau \models A$ iff $X_0, \tau_0 \models B \to A$, this reduces to showing that for all $A \in \mathsf{Prop}(V)$, $X_0, \tau_0 \models A$ iff A is a propositional tautology. This follows from the more general claim, proved by an easy induction over the structure of A, that for $\kappa \in 2^V$, $\kappa \in [\![A]\!]_{X_0}^{\tau_0}$ iff A evaluates to true under the valuation κ. □

We can now show that one-step cut-free completeness entails the absorption properties.

Proposition 5.11. *Let $\mathbf{R}$ be one-step sound and one-step cut-free complete. Then $\mathbf{R}$ absorbs cut and contraction.*

Proof. We first establish that $\mathbf{R}$ absorbs contraction. So let $\Gamma_1 \dots \Gamma_n / \Gamma_0 \in \mathbf{R}$, and let $\sigma : V \to V$ be a renaming. We have to show that there exists a rule $\Delta_1 \dots \Delta_m / \Delta_0$ and a renaming $\rho : V \to V$ such that $\Delta_0 \rho \subseteq \mathrm{supp}(\Gamma_0 \sigma)$ and

$$\mathbf{GC}_0 + \{\Gamma_i \sigma \mid i = 1, \dots, n\} \vdash \Delta_j \rho$$

for all $j = 1, \dots, m$. Let X, τ be the canonical model of $\{\Gamma_1 \sigma, \dots, \Gamma_n \sigma\}$ according to Lemma 5.10. By one-step soundness, $TX, \tau \models \Gamma_0 \sigma$, and hence $TX, \tau \models \mathrm{supp}(\Gamma_0 \sigma)$. Since $\mathbf{R}$ is one-step cut-free complete, we can find a rule $\Delta_1 \dots \Delta_m / \Delta_0$ and a renaming $\rho : V \to V$ such that $X, \tau \models \Delta_i \rho$ for $i = 1, \dots, m$ and $\Delta_0 \rho \subseteq \mathrm{supp}(\Gamma_0 \sigma)$. By Lemma 5.10,

$$\mathbf{GC}_0 + \{\Gamma_i \sigma \mid i = 1, \dots, n\} \vdash \Delta_j \tau$$

for $j = 1, \ldots, m$ as required.

We use a very similar argument to show that $\mathbf{R}$ absorbs cut. If $\Gamma_1 \ldots \Gamma_n/\Gamma_0$ and $\Delta_1 \ldots \Delta_m/\Delta_0 \in \mathbf{R}$ and $\sigma, \rho : V \to V$ are renamings with $\Gamma_0\sigma = \Gamma, A$ and $\Delta_0\rho = \Delta, \neg A$, let X, τ be the canonical model of $\{\Gamma_1\sigma, \ldots, \Gamma_n\sigma, \Delta_1\rho, \ldots, \Delta_m\rho\}$. By one-step soundness, $TX, \tau \models \Gamma, \Delta$, and by one-step cut-free completeness we find a rule $\Sigma_1 \ldots \Sigma_l/\Sigma_0$ and a renaming $\kappa : V \to V$ such that $X, \tau \models \Sigma_i\kappa$ for $i = 1, \ldots, l$ and $\Sigma_0\kappa \subseteq \Gamma, \Delta$. By Lemma 5.10

$$\mathbf{GC}_0 + \{\Gamma_i\sigma \mid i = 1, \ldots, n\} + \{\Delta_i\rho \mid i = 1, \ldots, m\} \vdash \Sigma_j\kappa$$

for $j = 1, \ldots, l$ as required. □

Taken together, Propositions 5.11 and 5.8 establish that *under one-step completeness*, one-step cut-free completeness is equivalent to absorption of cut and contraction. The syntactic proof of cut elimination (Proposition 5.6 and Theorem 5.7) however requires a third condition: absorption of congruence. We now show that this condition is also implied by one-step cut-free completeness. (Note that this does not imply that absorption of congruence follows from absorption of cut and contraction.)

Proposition 5.12. *Suppose that* $\mathbf{R}$ *is one-step cut-free complete. Then* $\mathbf{R}$ *absorbs congruence.*

Proof. Let $\heartsuit \in \Lambda$ be an n-ary modal operator, pick propositional variables $p_1, \ldots, p_n, q_1, \ldots, q_n \in V$, and let Γ denote the sequent $\neg\heartsuit(p_1, \ldots, p_n), \heartsuit(q_1, \ldots, q_n)$. Let X, τ be the canonical model of $H = \{\neg p_i, q_i \mid i = 1, \ldots, n\} \cup \{\neg q_i, p_i \mid i = 1, \ldots, n\}$ according to Lemma 5.10. Then $TX, \tau \models \Gamma$. By one-step cut-free completeness, we have a rule $\Gamma_1 \ldots \Gamma_m/\Gamma_0 \in \mathbf{R}$ and a renaming $\sigma : V \to V$ such that $X, \tau \models \Gamma_j\sigma$ for all $j = 1, \ldots, m$ and $\Gamma_0\sigma \subseteq \Gamma$. By Lemma 5.10,

$$\mathbf{GC}_0 + H \vdash \Gamma_j\sigma$$

as required. □

Example 5.13. In order to discharge the pending proof of one-step cut-free completeness for the set $\mathbf{R_C}$ of one-step rules axiomatising coalition logic, we can proceed as in [28]. *Mutatis mutandis*, it has been shown in [28] that $\mathbf{R_C}$ absorbs cut and contraction. By Proposition 5.8, it remains to prove one-step completeness. As noted in [28], one-step completeness follows from Proposition 3.2 in [20]; however, we now have to pay attention to the fact that we modified the semantics of coalition logic by restricting to finite sets of strategies. For finite sets of outcomes, the proof of Proposition 3.2 in [20] constructs finite sets of strategies (and then, of course, strategies can be assumed to be natural numbers). In other words, the said proof shows that $\mathbf{R_C}$ is one-step complete on finite sets w.r.t. C. By Proposition 4.5, this is sufficient to establish one-step completeness.

We conclude the section with a short methodological digression on the construction of cut-free complete rule sets.

Remark 5.14. The syntactic approach to cut elimination provides us with a methodology to construct cut-free complete rule sets. Any one-step complete system of rules can be turned into a one-step cut-free complete system by adding instances of cut and contraction until both cut and contraction are absorbed (the question is only whether there is a tractable description of the resulting rule set). It is evident that this preserves one-step soundness.

6 Applications

This section presents, from a syntactic viewpoint, some applications of cut-free completeness of $\mathbf{GR}$ for a one-step cut-free complete set $\mathbf{R}$ of one-step rules. The first application, the subformula property, is immediate:

Theorem 6.1. *Let* $\mathbf{R}$ *be a set of one-step rules. Then* $\mathbf{GR}$ *has the subformula property, i.e. every deduction* $\mathbf{GR} \vdash \Gamma$ *only mentions subformulas, or negations thereof, of formulas occurring in* Γ*.*

Proof. By induction on the derivation of $\mathbf{GR} \vdash \Gamma$, where both the case of propositional connectives and the application of an instance of a one-step rule are immediate by the rule format. □

As a consequence, we obtain alternative proofs of two results of [25] regarding conservativity and complexity of coalgebraic logics.

Corollary 6.2 (Conservativity)**.** *Let* $\Lambda_0 \subseteq \Lambda$ *be a sub-similarity type, and let* $\mathbf{R}$ *be one-step sound and one-step cut-free complete for a* Λ*-structure* T*. If* $\mathbf{R}_0$ *consists of those* $\Gamma_1 \dots \Gamma_n / \Gamma_0 \in \mathbf{R}$ *for which* $\Gamma_0 \in \mathcal{S}(\Lambda_0(V))$*, then* $\mathbf{GR}_0$ *is complete for* T *as a* Λ_0*-structure.*

Proof. Let Γ be a valid sequent over $\mathcal{F}(\Lambda_0)$. Then $\mathbf{GR} \vdash \Gamma$. By the subformula property, all rules used in this derivation belong to $\mathbf{R}_0$. □

As the design of the system **GR** is such that the logical complexity of the formula strictly decreases when passing from conclusion to premise, these systems can be used to establish both decidability and complexity of the satisfiability problem. Simply put, proof search in **GR** terminates if for every sequent Γ there are only finitely many substitution instances of rule conclusions equal to Γ with properly different premises. If moreover rules can be represented by codes in such a way that one only needs to consider rules of polynomial-sized codes in Γ, and the rule set satisfies some additional sanity conditions, we say that the rule set is *PSPACE-tractable*; we refer to [25, Definition 6.12] for the exact definition. Under *PSPACE*-tractability, proof search in **GR** can be performed in polynomial space using a depth-first strategy. We thus re-prove the main result (Theorem 6.13) of [25], which is reformulated as follows in the sequent calculus setting used in the present work:

Theorem 6.3. *Let* **R** *be one-step sound and one-step cut-free complete. If moreover* **R** *is PSPACE-tractable, then the satisfiability problem for* $\mathcal{F}(\Lambda)$ *w.r.t.* $\mathsf{Mod}(T)$ *is decidable in polynomial space.*

Cut-free proof calculi also provide all the necessary scaffolding to prove Craig interpolation by induction on cut-free proofs. We recall that $\mathsf{FV}(A)$ denotes the set of propositional variables occurring in $A \in \mathcal{F}(\Lambda)$, and similarly for sequents. Interpolation then takes the following form:

Definition 6.4. $\mathcal{F}(\Lambda)$ has the *Craig Interpolation Property* (CIP) with respect to $\mathsf{Mod}(T)$ if whenever $\mathsf{Mod}(T) \models A \to B$ for $A, B \in \mathcal{F}(\Lambda)$, then there exists an *interpolant* $F \in \mathcal{F}(\Lambda)$ such that $\mathsf{Mod}(T) \models A \to F$, $\mathsf{Mod}(T) \models F \to B$ and $\mathsf{FV}(F) \subseteq \mathsf{FV}(A) \cap \mathsf{FV}(B)$.

Syntactic proofs of the CIP proceed by induction on cut-free proofs. The following definition introduces the necessary terminology.

Definition 6.5. A *split sequent* is a pair (Γ_0, Γ_1) of sequents, written $\Gamma_0 \mid \Gamma_1$. We say that $\Gamma_0 \mid \Gamma_1$ is a *splitting* of Γ if $\Gamma = \Gamma_0, \Gamma_1$. A formula F is an *interpolant* of a split sequent $\Gamma_0 \mid \Gamma_1$ if $\mathsf{FV}(F) \subseteq \mathsf{FV}(\Gamma_0) \cap \mathsf{FV}(\Gamma_1)$, $\mathbf{GR} \vdash \Gamma_0, F$, and $\mathbf{GR} \vdash \neg F, \Gamma_1$. We say that a sequent Γ *admits interpolation* if every splitting of Γ has an interpolant. The system **GR** has the *Craig interpolation property* (CIP) if every derivable sequent admits interpolation.

The idea of the syntactic proof of Craig interpolation [29, Chapter 4], in contrast to the semantic proofs via amalgamation (see [14] for the case of normal modal logics and [11] for monotone modal logic) is to construct interpolants inductively – clearly this fails in the presence of the cut-rule. Completeness provides the link between the syntactic and the semantic versions of the CIP.

Proposition 6.6. *Let* $\mathbf{R}$ *be one-step sound and one-step cut-free complete w.r.t the* Λ*-structure* T*. Then* $\mathbf{GR}$ *has the CIP iff* $\mathcal{F}(\Lambda)$ *has the CIP with respect to* $\mathsf{Mod}(T)$*.*

Proof. To prove 'if', let $\mathbf{GR} \vdash \Gamma$, and let $\Gamma_0 \mid \Gamma_1$ be a splitting of Γ. By soundness, $\mathsf{Mod}(T) \models (\neg\check{\Gamma}_0) \to \check{\Gamma}_1$. Therefore, we have $F \in \mathcal{F}(\Lambda)$ such that $\mathsf{FV}(F) \subseteq \mathsf{FV}(\check{\Gamma}_0) \cap \mathsf{FV}(\check{\Gamma}_1) = \mathsf{FV}(\Gamma_0) \cap \mathsf{FV}(\Gamma_1)$, $\mathsf{Mod}(T) \models (\neg\check{\Gamma}_0) \to F$, and $\mathsf{Mod}(T) \models F \to \check{\Gamma}_1$. By cut-free completeness (Section 4), $\mathbf{GR} \vdash \Gamma_0, F$ and $\mathbf{GR} \vdash \neg F, \Gamma_1$, i.e. F is the required interpolant of Γ_0, Γ_1. 'Only if' is proved similarly. □

Inductive proofs of the CIP for $\mathbf{GR}$ are often straightforward. Below, we show that the systems used in our running examples, coalition logic and conditional logic have the CIP. For coalition logic, this is not a new result [10] but our proof is shorter due to the smaller number of modal proof rules. For the conditional logics *CK* and *CK + ID* the CIP is – to the best of our knowledge – a new result which was explicitly left as future work in [17], where a substantially different proof calculus is used.

The proof of the CIP in both examples benefits from the following notions.

Definition 6.7. A sequent rule $\Gamma_1 \dots \Gamma_n/\Gamma_0$ *supports interpolation* if Γ_0 admits interpolation provided all of $\Gamma_1, \dots, \Gamma_n$ admit interpolation. A set $\mathbf{S}$ of sequent rules *supports interpolation* if all rules in $\mathbf{S}$ support interpolation.

As it is well known (and shown e.g. in [29]) that all (instances of) rules of $\mathbf{G}$ support interpolation, the following is evident.

Lemma 6.8. *If* $\mathcal{S}(\mathbf{R})$ *supports interpolation, then* $\mathbf{GR}$ *has the CIP.*

Moreover, we may restrict ourselves to rule instances without context formulas:

Lemma 6.9. *The set* $\mathcal{S}(\mathbf{R})$ *supports interpolation iff for every rule* $\Gamma_1 \dots \Gamma_n/\Gamma_0$ *in* $\mathbf{R}$ *and every substitution* $\sigma : V \to \mathcal{F}(\Lambda)$*, the sequent rule* $\Gamma_1\sigma \dots \Gamma_n\sigma/\Gamma_0\sigma$ *supports interpolation.*

Proof. Let $\Gamma_1 \dots \Gamma_n/\Gamma_0$ be a one-step rule in $\mathbf{R}$, let $\sigma : V \to \mathcal{F}(\Lambda)$ be a substitution, and let Δ be a sequent. Moreover, let $\Gamma_i\sigma$ admit interpolation for all $i = 1, \dots, n$; we have to show that the arising rule conclusion $\Gamma_0\sigma, \Delta$ admits interpolation. Every splitting of $\Gamma_0\sigma, \Delta$ is of the form $\Gamma_0^0\sigma, \Delta_0 \mid \Gamma_0^1\sigma, \Delta_1$, where $\Gamma_0^0\sigma \mid \Gamma_0^1\sigma$ is a splitting of $\Gamma_0\sigma$ and $\Delta_0 \mid \Delta_1$ is a splitting of Δ. By assumption, $\Gamma_0\sigma$ admits interpolation, so that there exists an interpolant F for the splitting $\Gamma_0^0\sigma \mid \Gamma_0^1\sigma$. By admissibility of weakening, F is also an interpolant for the given splitting of $\Gamma_0\sigma, \Delta$. □

We turn to our running examples:

Theorem 6.10. *Coalition logic, i.e. the system* **GC***, has the CIP.*

Proof. By the above lemmas, we only have to check that the given one-step rules support interpolation.

Rule (A). If $S = \neg[\mathbf{C}_0]\mathbf{A}_0 \mid \neg[\mathbf{C}_1]\mathbf{A}_1$ is a splitting of the (substituted) rule conclusion (recall the notation of Example 3.6) and F is an interpolant of $\neg\mathbf{A}_0 \mid \neg\mathbf{A}_1$, then $G = [\cup\mathbf{C}_0]F$ is an interpolant of S: From $\neg F, \neg\mathbf{A}_1$, we deduce $\neg G, \neg[\mathbf{C}_1]\mathbf{A}_1$ by rule (A), whose side condition is met as $\cup\mathbf{C}_0$ and the elements of $\mathbf{C}_1$ are pairwise disjoint. Moreover, from $\neg\mathbf{A}_0, F$ we deduce $\neg[\mathbf{C}_0]\mathbf{A}_0, G$ by rule (B), where, in the notation of the rule, we match G to the literal $[D]B$, and the side condition is met by construction of G.

Rule (B). There are two cases to distinguish, depending on which part of the splitting the literal $[D]B$ belongs to. First consider splittings of the rule conclusion of the form

$$S = \neg[\mathbf{C}_0]\mathbf{A}_0, [D]B, [\mathbf{N}]\mathbf{B}_0 \mid \neg[\mathbf{C}_1]\mathbf{A}_1, [\mathbf{N}]\mathbf{B}_1.$$

If F is an interpolant of $\neg\mathbf{A}_0, B, \mathbf{B}_0 \mid \neg\mathbf{A}_1, \mathbf{B}_1$, then $\neg[\cup\mathbf{C}_1]\neg F$ is an interpolant of S: from $\neg\mathbf{A}_0, F, B, \mathbf{B}_0$ we first derive $\neg\mathbf{A}_0, \neg\neg F, B, \mathbf{B}_0$ and then $\neg[\mathbf{C}_0]\mathbf{A}_0, \neg[\cup\mathbf{C}_1]\neg F, [D]B, [\mathbf{N}]\mathbf{B}_0$ using rule (B). Moreover, from $\neg\mathbf{A}_1, \neg F, \mathbf{B}_1$ we derive $\neg[\mathbf{C}_1]\mathbf{A}_1, [\cup\mathbf{C}_1]\neg F, [\mathbf{N}]\mathbf{B}_1$ using rule (B), and further $\neg[\mathbf{C}_1]\mathbf{A}_1, \neg\neg[\cup\mathbf{C}_1]\neg F, [\mathbf{N}]\mathbf{B}_1$.

Now consider a splitting of the rule conclusion of the form

$$S = \neg[\mathbf{C}_0]\mathbf{A}_0, [\mathbf{N}]\mathbf{B}_0 \mid \neg[\mathbf{C}_1]\mathbf{A}_1, [D]B, [\mathbf{N}]\mathbf{B}_1.$$

In this case, if F is an interpolant of $\neg\mathbf{A}_0, \mathbf{B}_0 \mid \neg\mathbf{A}_1, B, \mathbf{B}_1$, then $[\cup\mathbf{C}_0]F$ is an interpolant of S: from $\neg\mathbf{A}_0, F, \mathbf{B}_0$, we derive $R\neg[\mathbf{C}_0]\mathbf{A}_0, [\cup\mathbf{C}_0]F, [\mathbf{N}]\mathbf{B}_0$ by rule (B), and from $\neg F, \neg\mathbf{A}_1, B, \mathbf{B}_1$ we derive $\neg[\cup\mathbf{C}_0]F, \neg[\mathbf{C}_1]\mathbf{A}_1, [D]B, [\mathbf{N}]\mathbf{B}_1$ by rule (B). □

By a similar argument we establish the CIP for the conditional logics *CK* and *CK* + *ID*.

Theorem 6.11. *The conditional logics CK and CK + ID have the CIP.*

Proof. First consider **GCK**; we have to show that rule (C) supports interpolation. First consider splittings of the rule conclusion of the form $S = \neg(\mathbf{A}_0 \Rightarrow \mathbf{B}_0), A \Rightarrow B \mid \neg(\mathbf{A}_1 \Rightarrow \mathbf{B}_1)$. If F is an interpolant of $\neg\mathbf{B}_0, B \mid \neg\mathbf{B}_1$, then $\neg(A \Rightarrow \neg F)$ is an interpolant of S. Now consider splittings of the form $S = \neg(\mathbf{A}_0 \Rightarrow \mathbf{B}_0) \mid \neg(\mathbf{A}_1 \Rightarrow \mathbf{B}_1), A \Rightarrow B$. If F interpolates $\neg\mathbf{B}_0 \mid \neg\mathbf{B}_1, B$ then $A \Rightarrow F$ interpolates S.

We now consider interpolation for **GCKId**, which follows the same pattern. To show that the rule (CI) supports interpolation, first consider a splitting of the conclusion of (CI) of the form $S = \neg(\mathbf{A}_0 \Rightarrow \mathbf{B}_0), A \Rightarrow B \mid \neg(\mathbf{A}_1 \Rightarrow \mathbf{B}_1)$. If F is an interpolant of $\neg A_0, \neg\mathbf{B}_0, B \mid \neg\mathbf{B}_1$, then $\neg(A \Rightarrow \neg F)$ is an interpolant of S. Similarly, if $S = \neg(\mathbf{A}_0 \Rightarrow \mathbf{B}_0) \mid \neg(\mathbf{A}_1 \Rightarrow \mathbf{B}_1), A \Rightarrow B$ and F interpolates $\neg\mathbf{B}_0 \mid \neg\mathbf{B}_1, B, \neg A$ then $A \Rightarrow F$ interpolates S. □

7 Conclusions

We have shown that local absorption of congruence, contraction, and cut by a system of modal one-step rules automatically results in a sequent system that admits cut, and that under a localised completeness assumption the sequent system is (cut-free) complete w.r.t. coalgebraic semantics, a result which applies to many and widely differing examples of modal logics found in the literature. Cut free sequent systems are the key to a number of typical applications, including in particular proofs of the Craig interpolation property (CIP) which plays an important role in the modularisation of proofs. We have established the CIP for our two running examples; here, the CIP for the conditional logics *CK* and *CK + ID* is apparently a new result. It remains an open problem to find a quickly verifiable general criterion for a set of rules, or, semantically, a coalgebraic modal logic, to have the CIP. It is worthwhile to point out that for coalition logic, the inductive step in the proof of the CIP is not entirely straightforward as the newly constructed interpolant uses a modality that does not necessarily appear in the rule at hand. We phrase this problem explicitly as

Open Problem 7.1. *Find easily verifiable and general semantic or syntactic criteria for a coalgebraic modal logic to have the CIP.*

Secondly, we have observed that the crucial notion of absorption of cut by a set of rules is reflected semantically by what we have termed one-step cut-free completeness. The purely syntactic approach to cut elimination via local absorption of cut carries over to logics outside rank 1 [19] (see [27] for an exact definition of coalgebraic modal logics with general, i.e. not necessarily rank-1, frame conditions). Contrastingly, it is unclear which semantic criteria would apply in the general case. We formulate this explicitly as

Open Problem 7.2. *Find semantic criteria for a coalgebraic modal logic with general frame conditions to admit cut elimination.*

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