Beyond Rank 1: Algebraic Semantics and Finite Models for Coalgebraic Logics

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Abstract. Coalgebras provide a uniform framework for the semantics of a large class of (mostly non-normal) modal logics, including e.g. monotone modal logic, probabilistic and graded modal logic, and coalition logic, as well as the usual Kripke semantics of modal logic. In earlier work, the finite model property for coalgebraic logics has been established w.r.t. the class of *all* structures appropriate for a given logic at hand; the corresponding modal logics are characterised by being axiomatised in rank 1, i.e. without nested modalities. Here, we extend the range of coalgebraic techniques to cover logics that impose global properties on their models, formulated as frame conditions with possibly nested modalities on the logical side (in generalisation of frame conditions such as symmetry or transitivity in the context of Kripke frames). We show that the finite model property for such logics follows from the finite algebra property of the associated class of complex algebras, and then investigate sufficient conditions for the finite algebra property to hold. Example applications include extensions of coalition logic and logics of uncertainty and knowledge.

1 Introduction

The coalgebraic semantics of modal logic has proved to be useful to establish results that apply uniformly to a large class of modal logics. For example, [17] provides a finite model construction and decidability results and [18] derives uniform PSPACE bounds for coalgebraic modal logics. The class of logics covered by the coalgebraic approach includes e.g. monotone modal logic and the standard logic K but also less well-studied specimen such as Pauly's coalition logic [16], probabilistic modal logic [13,9], and graded modal logic [8]. Moreover, the coalgebraic approach allows combining logics modularly [4] while preserving completeness [5] and complexity bounds [19].

However, the range of the coalgebraic techniques is hitherto limited to logics axiomatised in rank 1, i.e. with nesting depth of modal operator uniformly equal to 1, thus excluding standard logics such as $K4$ and $S5$. The reason for this limitation is that in previous work, only such modal logics have been considered that are interpreted over the class of *all* structures of an appropriate type. Indeed it is shown in [17] that the class of all structures of a given type is always axiomatisable in rank 1. By analogy, rank 1 axioms play the role of the K-axioms for Kripke frames: they ensure completeness w.r.t. the class of *all* frames.

However, it is often desirable to have completeness for a subclass of all structures that satisfy additional properties like transitivity or reflexivity in a relational context.

These additional properties are captured as frame conditions on the logical side; e.g. the (4) axiom $\Box a \to \Box\Box a$ ensures transitivity for Kripke frames. This is our starting point: we extend a given complete rank-1 axiomatisation of a class of structures (that we formalise as coalgebras for an endofunctor) by additional frame conditions and establish the finite model property (and hence completeness) with respect to the class of all structures that satisfy the additional axioms.

In view of our interest in finite model results, our main technical tool is finite Stone duality, i.e. the dual equivalence between finite sets and finite boolean algebras. Accordingly, the finite model property is established in two steps: the first step shows that the finite model property follows from the finite algebra property of an associated algebraic theory by transporting finite algebraic models to the coalgebraic side via Stone duality (the converse implication, i.e. that the finite model property implies the finite algebra property, is trivial). In the second step, we use algebraic filtrations in the style of Lemmon [14], adapted to a non-normal setting, to obtain the finite algebra property. In view of the duality between modal algebras and neighbourhood frames [6] this latter step is equivalent to establishing the finite model property with respect to neighbourhood semantics, albeit at the expense of losing the correspondence with the algebraic semantics.

The versatility of our approach is demonstrated by two extended examples. For logics combining uncertainty and knowledge as described in [7], we show that the finite model property can be derived purely synthetically. In particular, we derive the finite model property in the presence of axioms that stipulate interaction between belief and uncertainty, and our results are modular in the axiomatisation of agent belief. The second example uses our techniques to establish the finite model property for various extensions of Pauly's coalition logic [16].

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2 Preliminaries and Notation

The category of sets and functions is denoted by Set, and we write BA for the category of boolean algebras. We use Fin to denote the category of finite sets, and FBA is the category of finite boolean algebras. Stone duality [10] restricts to a dual equivalence between Fin and FBA given by the contravariant powerset functor $2 : \mathsf{Fin} \to \mathsf{FBA}$ and the functor $\mathsf{Uf} : \mathsf{FBA} \to \mathsf{Fin}$ that maps a finite boolean algebra to the set of its ultrafilters. If $A \in \mathsf{FBA}$, we write $\iota_A : 2 \circ \mathsf{Uf}(A) \to A$ for the canonical isomorphism.

Given an endofunctor $T : \mathsf{Set} \to \mathsf{Set}$, a *$T$-coalgebra* is a pair (C, γ) consisting of a carrier $C \in \mathsf{Set}$ and a transition function $\gamma : C \to TC$. A *coalgebra morphism* $f : (C, \gamma) \to (D, \delta)$ is a function $f : C \to D$ for which $\delta \circ f = Tf \circ \gamma$. We denote the category of T-coalgebras by $\mathsf{Coalg}(T)$ and write $\mathsf{Coalg}(T)_f$ for the full subcategory of $\mathsf{Coalg}(T)$ consisting of all those $(C, \gamma) \in \mathsf{C}(T)$ for which the carrier C is finite. Dually, if $L : \mathsf{BA} \to \mathsf{BA}$ is a functor, we write $\mathsf{Alg}(L)$ for the category of L-algebras, that is pairs (A, α) where $A \in \mathsf{BA}$ and $\alpha : LA \to A$ is a morphism of boolean algebras. As for coalgebras, $\mathsf{Alg}(L)_f$ denotes the full subcategory of those $(A, \alpha) \in \mathsf{Alg}(L)$ whose carrier $A \in \mathsf{FBA}$ is finite. Throughout, we fix a denumerable set V of propositional variables. The set of propositional formulas over a set X is denoted by $\mathsf{Prop}(X)$ and the set of clauses over X by $\mathsf{Cl}(X)$.

3 Rank-1 Logics

We start by introducing rank-1 logics that we take as extensions of propositional logic with unary modal operators. The restriction to unary modalities is purely for convenience; all of our results generalise to polyadic modalities in a straightforward way. Rank-1 logics are the basic building blocks of our theory, as they provide a sound and complete axiomatisation of the class $\mathsf{Coalg}(T)$ of all T-coalgebras that we extend with frame conditions to effect specific properties later.

Definition 1 (Modal signatures, formulas). A *modal signature* or *modal similarity type* is a set Λ consisting of (unary) modal operators. For a set S, we write $\Lambda(S) = \{M(s) \mid M \in \Lambda, s \in S\}$ for the set of formulas that arise by prefixing elements of S by precisely one modality in Λ. The set $\mathcal{F}(\Lambda)$ of Λ-*formulas* is inductively given by

$$\mathcal{F}(\Lambda) \ni \phi, \psi ::= p \mid \bot \mid \phi \to \psi \mid M(\phi)$$

where $p \in V$ is a propositional variable and $M \in \Lambda$.

We interpret modal logics over T-coalgebras, where T is an endofunctor on Set. Modal operators are interpreted using predicate liftings [15].

Definition 2 (Structures, Coalgebraic Semantics). If Λ is a modal signature, a Λ-*structure* consists of an endofunctor $T : \mathsf{Set} \to \mathsf{Set}$ and a predicate lifting (a natural transformation)$[\![M]\!] : 2 \to 2 \circ T$ for every $M \in \Lambda$. A *morphism* between two Λ-structures S and T is a natural transformation $\mu : S \to T$ such that $[\![M]\!]_S = \mu^{-1} \circ [\![M]\!]_T$ for all $M \in \Lambda$. The *coalgebraic semantics* of $\phi \in \mathcal{F}(\Lambda)$ w.r.t. a T-coalgebra $\mathbb{C} = (C, \gamma)$ and a valuation $\pi : V \to \mathcal{P}(C)$ is inductively defined by

$$[\![p]\!]^\pi_\mathbb{C} = \pi(p) \qquad [\![M\phi]\!]^\pi_\mathbb{C} = \gamma^{-1} \circ [\![M]\!]_C([\![\phi]\!]^\pi_\mathbb{C})$$

and the standard clauses for propositional connectives. If $\Theta \subseteq \mathcal{F}(\Lambda)$ and $\mathbb{C} \in \mathsf{Coalg}(T)$ we write $\mathbb{C} \models \Theta$ if $[\![\phi]\!]^\pi_\mathbb{C} = \top$ for all $\pi : V \to \mathcal{P}(C)$ and all $\phi \in \Theta$ and denote the full subcategory of all $\mathbb{C} \in \mathsf{Coalg}(T)$ that satisfy every formula in Θ by $\mathsf{Coalg}(T, \Theta)$. Finally, a formula ϕ is *(finitely) satisfiable* in $\mathsf{Coalg}(T, \Theta)$ if there exists a (finite) coalgebra $\mathbb{C} \in \mathsf{Coalg}(T, \Theta)$ and a valuation $\pi : V \to \mathcal{P}(C)$ with $[\![\phi]\!]_\pi \neq \emptyset$.

The axiomatisation of the modal logics considered here consists of two parts: a set of rank-1 axioms that is (sound and) complete for the class of all T-coalgebras (and thus accounts for the structure of $\mathsf{Coalg}(T)$) and a set of frame conditions that specify additional properties. The logic of $\mathsf{Coalg}(T)$ can always be axiomatised by rank-1 axioms [17].

Definition 3. Suppose Λ is a modal similarity type. A *rank-1 axiom* over Λ is a propositional formula over $\Lambda(\mathsf{Prop}(V))$; propositional formulas over $\Lambda(\mathsf{Prop}(V)) \cup V$ are called *rank-0/1*. A *rank-1 logic* is a pair $\mathcal{L} = (\Lambda, \mathcal{A})$ where $\mathcal{A}$ is a set of rank-1 axioms over Λ. An *extended rank-1 logic* is a triple $\mathcal{L} = (\Lambda, \mathcal{A}, \Theta)$ where $(\Lambda, \mathcal{A})$ is a rank-1 logic and $\Theta \subseteq \mathcal{F}(\Lambda)$ is a set of additional axioms. The logic $\mathcal{L}$ is *rank-0/1* if every $\phi \in \Theta$ is rank-0/1.

Deduction over (extended) rank-1 logics is standard:

Definition 4 (Derivability). Suppose $\mathcal{L} = (\Lambda, \mathcal{A}, \Theta)$ is an extended rank-1 logic. We write $\mathcal{L} \vdash \phi$ if ϕ is contained in the least set closed under the rules

$$\frac{\phi \in \mathcal{A} \cup \Theta}{\mathcal{L} \vdash \phi\sigma} \quad \frac{\mathcal{L} \vdash \phi \to \psi \quad \mathcal{L} \vdash \phi}{\mathcal{L} \vdash \psi} \quad \frac{\phi \in \mathsf{Taut}(V)}{\mathcal{L} \vdash \phi\sigma} \quad \frac{\mathcal{L} \vdash \phi \leftrightarrow \psi}{\mathcal{L} \vdash M\phi \leftrightarrow M\psi}$$

where $\mathsf{Taut}(V)$ is the set of propositional tautologies over the set V of (propositional) variables and $\sigma : V \to \mathcal{F}(\Lambda)$ ranges over $\mathcal{F}(\Lambda)$-substitutions. This is extended to sets of formulas, and we write $\Psi \vdash_{\mathcal{L}} \phi$ if there exist $\psi_1, \dots, \psi_n \in \Psi$ such that $\mathcal{L} \vdash \psi_1 \wedge \dots \wedge \psi_n \to \phi$.

To prove completeness in the presence of frame conditions, we require that the rank-1 axioms are one-step sound and one-step complete; both notions are as in [15,5]. For the sake of brevity, we (ab)use subsets of a set X as propositional variables with the obvious interpretation in the boolean algebra $\mathcal{P}(X)$.

Definition 5 (One-step Soundness and Completeness). Let $\mathcal{L} = (\Lambda, \mathcal{A})$ be a rank-1 logic, and let T be a Λ-structure. The *one-step semantics* $[\![\phi]\!]_{TX} \subseteq TX$ of $\phi \in \mathsf{Prop}(\Lambda(\mathcal{P}(X)))$ at a set X is defined inductively by the standard clauses for propositional connectives and $[\![MA]\!]_{TX} = [\![M]\!]_X(A)$ for $M \in \Lambda$, $A \subseteq X$. We write $TX \models \phi$ if $[\![\phi]\!]_{TX} = TX$. The relation $\mathcal{A}, X \vdash \phi$ of *one-step derivability* at a set X is generated by

$$\frac{\phi \in \mathsf{Taut}(\Lambda(\mathcal{P}(X)))}{\mathcal{A}, X \vdash \phi} \qquad \frac{\mathcal{A}, X \vdash \phi \to \psi \quad \mathcal{A}, X \vdash \phi}{\mathcal{A}, X \vdash \psi} \qquad \frac{\phi \in \mathcal{A}}{\mathcal{A}, X \vdash \phi\sigma},$$

where $\sigma : V \to \mathcal{P}(X)$ is a $\mathcal{P}(X)$-valuation in the last rule.

The logic $\mathcal{L}$ is *one-step sound* w.r.t. a structure T if, for all sets X and all $\phi \in \mathcal{F}(\Lambda)$, $\mathcal{A}, X \vdash \phi$ implies $TX \models \phi$; $\mathcal{L}$ is *one-step complete* if the converse implication holds.

One-step soundness guarantees soundness in the standard sense:

Proposition 6. *Let $\mathcal{L} = (\Lambda, \mathcal{A}, \Theta)$ be an extended rank-1 logic which is one-step sound w.r.t. a structure T. Then $\mathcal{L}$ is sound w.r.t. $\mathsf{Coalg}(T, \Theta)$, i.e. if $\mathcal{L} \vdash \phi$ for $\phi \in \mathcal{F}(\Lambda)$, then $\mathsf{Coalg}(T, \Theta) \models \phi$.*

We prove the converse, and simultaneously establish the finite model property, by constructing finite algebraic models that we then transport to the coalgebraic side.

4 Algebraic Semantics

We recall the concept of the functorial presentation of a rank-1 logic, due to Kurz and collaborators [12,1]. Again, this concept is most conveniently introduced by using elements of a boolean algebra as variables with the obvious interpretation.

Definition 7. Let $\mathcal{L} = (\Lambda, \mathcal{A})$ be a rank-1 logic, let A be a boolean algebra, and let $\phi, \psi \in \mathsf{Prop}(\Lambda(A))$. We write $\phi =_A \psi$ if $\phi = \psi$ can be derived equationally from the axioms of boolean algebra augmented with the set $\{\psi\sigma = \top \mid \psi \in \mathcal{A}, \sigma : V \to A\}$. The functor $L : \mathsf{BA} \to \mathsf{BA}$ defined by $L(A) = \mathsf{Prop}(\Lambda(A))/{=_A}$ is called the *functorial presentation* of $\mathcal{L}$.

The functorial presentation of a logic allows us to view formulas as terms that are interpreted over L-algebras.

Definition 8. Let L be the functorial presentation of a rank-1 logic $\mathcal{L} = (\Lambda, \mathcal{A})$, and let $\mathbb{A} = (A, \alpha) \in \mathsf{Alg}(L)$. The *algebraic semantics* of $\phi \in \mathcal{F}(\Lambda)$ w.r.t. $\mathbb{A}$ and a valuation $\pi : V \to A$ is defined inductively by the clauses

$$[\![p]\!]^\pi_\mathbb{A} = \pi(p) \qquad [\![M(\phi)]\!]^\pi_\mathbb{A} = \alpha \circ q(M[\![\phi]\!]^\pi_\mathbb{A})$$

where $p \in V$ and $q : \mathsf{Prop}(\Lambda(A)) \to \mathsf{Prop}(\Lambda(A))/{=_A}$ is the quotient mapping, with propositional connectives interpreted via the boolean algebra structure of $\mathbb{A}$. If $\mathbb{A} = (A, \alpha) \in \mathsf{Alg}(L)$ and $\Theta \subseteq \mathcal{F}(\Lambda)$, we write $\mathbb{A} \models \Theta$ if $[\![\phi]\!]^\pi_\mathbb{A} = \top$ for all $\phi \in \Theta$ and all $\pi : V \to A$. We denote the full subcategory of all $\mathbb{A} \in \mathsf{Alg}(L)$ with $\mathbb{A} \models \Theta$ by $\mathsf{Alg}(L, \Theta)$. Finally, a formula is (finitely) satisfiable in $\mathsf{Alg}(L, \Theta)$ if there exists a (finite) $\mathbb{A} = (A, \alpha) \in \mathsf{Alg}(L, \Theta)$ such that $[\![\phi]\!]^\pi_\mathbb{A} \neq \bot$ for some valuation $\pi : V \to A$.

To capitalise on the duality between Fin and FBA we need to insist that L restricts to an endofunctor on FBA which is the case if the set of modalities is finite. Despite its simplicity, we state this fact as a lemma as this will be a recurrent theme later.

Lemma 9. *Let $\mathcal{L} = (\Lambda, \mathcal{A})$ be a rank-1 logic with Λ finite, and let L be the functorial presentation of $\mathcal{L}$. Then LA is finite for all $A \in \mathsf{FBA}$.*

The functorial presentation of a logic with a finite number of modalities gives rise to a (dual) functor that we denote by $L^\star = \mathsf{Uf} \circ L \circ 2 : \mathsf{Set} \to \mathsf{Set}$. The equivalence between FBA and Fin now extends to a dual equivalence between *finite* $L^\star$-coalgebras and *finite* L-algebras:

Lemma 10. *Let $\mathcal{L} = (\Lambda, \mathcal{A})$ be a rank-1 logic over a finite similarity type Λ, and let L be the functorial presentation of $\mathcal{L}$. Then the functors $\mathsf{A} : \mathsf{Coalg}(L^\star)_f \to \mathsf{Alg}(L)_f$ and $\mathsf{C} : \mathsf{Alg}(L)_f \to \mathsf{Coalg}(L^\star)_f$ defined by $\mathsf{A}(C, \gamma) = (\mathcal{P}(A), \gamma^{-1} \circ \iota_{L2C})$ and $\mathsf{C}(A, \alpha) = (\mathsf{Uf}(A), \mathsf{Uf} \circ L\iota_A \circ \mathsf{Uf}\alpha)$ define a dual equivalence between finite $L^\star$-coalgebras and finite L-algebras.*

Under this equivalence, the carrier of the L-algebra associated with $(C, \gamma) \in \mathsf{Coalg}(L^\star)_f$ is the powerset $\mathcal{P}(C)$ of C, so that the interpretation of a formula ϕ in the algebra $\mathsf{A}(C, \gamma)$ is in fact a subset of C. This allows us to relate the algebraic and the coalgebraic semantics conveniently as follows:

Lemma 11. *Let $(C, \gamma) \in \mathsf{Coalg}(L^\star)$, and let $(A, \alpha) = \mathsf{A}(C, \gamma)$. Then $[\![\phi]\!]^\pi_\mathbb{A} = [\![\phi]\!]^\pi_\mathbb{C}$ for all $\phi \in \mathcal{F}(\Lambda)$ and all $\pi : V \to \mathcal{P}(C)$.*

In particular, we can reduce satisfiability in $\mathsf{Coalg}(L^\star)$ to satisfiability on $\mathsf{Alg}(L)$ also in the presence of extralogical axioms.

Corollary 12. *Let $\mathcal{L} = (\Lambda, \mathcal{A}, \Theta)$ be a rank-1 logic over a finite similarity type Λ. Then the following are equivalent for $\phi \in F(\Lambda)$:*

1. *ϕ is finitely satisfiable in $\mathsf{Alg}(L, \Theta)$*
2. *ϕ is finitely satisfiable in $\mathsf{Coalg}(L^\star, \Theta)$.*

We now describe the functor $L^\star$ to the extent that that is needed for purposes of this work. A more complete description can be found in [20].

Definition 13. Let $\mathcal{L} = (\Lambda, \mathcal{A})$ be a rank-1 logic, and let X be a set. A subset $\Phi \subseteq \mathsf{Prop}(\Lambda(\mathcal{P}(X))$ is *one-step $\mathcal{L}$-inconsistent at X* iff there are finitely many $\phi_1, \ldots, \phi_n \in \Phi$ such that $\mathcal{A}, X \vdash \phi_1 \wedge \cdots \wedge \phi_n \to \bot$, and *one-step $\mathcal{L}$-consistent at X* otherwise. Φ is *one-step $\mathcal{L}$-maximally consistent at X* if it is maximal (w.r.t. $\subseteq$) among the one-step $\mathcal{L}$-consistent sets at X.

Lemma 14. *Let $\mathcal{L} = (\Lambda, \mathcal{A})$ be a rank-1 logic with Λ finite. Then $L^\star X \cong \{\Phi \subseteq \mathsf{Prop}(\Lambda(\mathcal{P}(X))) \mid \Phi$ one-step $\mathcal{L}$-maximally consistent$\}$.*

Thus, one can go back and forth between $L^\star$ and its equivalent description by introducing and dissolving equivalence classes. In the sequel, we will silently use the description of $L^\star$ that is most convenient for our purposes. We note that $L^\star$ gives rise to a canonical structure for a rank-1 logic $\mathcal{L}$.

Lemma and Definition 15. *Let L be the functorial presentation of a rank-1 logic $\mathcal{L} = (\Lambda, \mathcal{A})$. Then the natural transformations defined by $[\![M]\!]_X(A) = \{\Phi \in L^\star(X) \mid MA \in \Phi\}$ for $M \in \Lambda$, $A \subseteq X$ define a Λ-structure for $L^\star$; we call this structure the* canonical *Λ-structure.*

It can be shown that the canonical Λ-structure is final in the category of all Λ-structures [20]. For our purposes, the following suffices:

Lemma 16. *Let T be a $\mathcal{L}$-structure, and let L be the functorial presentation of $\mathcal{L}$. Then the family of maps*

$$\mu(X) : TX \to L^\star X, t \mapsto \{\phi \in \mathsf{Prop}(\Lambda(\mathcal{P}(X))) \mid t \in [\![\phi]\!]_{TX}\}$$

defines a morphism between T and the canonical $L^\star$-structure.

As a consequence, the semantics of modal formulas is preserved if we move between an arbitrary Λ-structure and the canonical such.

Lemma 17. *Let $\mathbb{C} = (C, \gamma) \in \mathsf{Coalg}(T)$, and let $\mathbb{D} = (C, \mu(C) \circ \gamma)$. Then $[\![\phi]\!]^\pi_{\mathbb{C}} = [\![\phi]\!]^\pi_{\mathbb{D}}$ for all $\phi \in \mathcal{F}(\Lambda)$.*

In order to exploit the duality between finite sets and finite boolean algebras also in the presence of infinitely many modalities, we restrict attention to those modalities that occur either in the frame conditions or in the particular formula that we seek to satisfy. This cutting out of modalities is effected formally as follows:

Definition 18. Let $\mathcal{L} = (\Lambda, \mathcal{A})$ be a rank-1 logic, and let $\Gamma \subseteq \Lambda$. The *Γ-reduct* of $\mathcal{L}$ is the rank-1 logic $\mathcal{L}_\Gamma = (\Gamma, \mathcal{A}_\Gamma)$ where $\mathcal{A}_\Gamma = \{\phi \in \mathsf{Prop}(\Gamma(\mathsf{Prop}(V))) \mid \mathcal{L} \vdash \phi\}$.

Γ-reducts of complete rank-1 logics remain complete:

Lemma 19. *Let $\mathcal{L} = (\Lambda, \mathcal{A})$ be a rank-1 logic which is one-step sound and one-step sound complete w.r.t. a structure T. Then $\mathcal{L}_\Gamma$ is one-step sound and one-step complete for the structure given by T together with the predicate liftings $[\![M]\!]$ for $M \in \Gamma$.*

We have already seen in Lemma 17 that semantics is preserved when moving from an arbitrary Λ-structure to the canonical structure. The next lemma is the key result as it provides for a passage in the other direction and thus allows us to reduce satisfiability over $\mathsf{Coalg}(T)$ to satisfiability over $\mathsf{Coalg}(L^\star)$.

Lemma 20. *Let $\mathcal{L} = (\Lambda, \mathcal{A})$ be a rank-1 logic, let $\Gamma \subseteq \Lambda$ be finite, and let L be the functorial presentation of $\mathcal{L}_\Gamma$. If $X \in \mathsf{Fin}$ is a finite set, then $\mu_X : TX \to L^\star X$, with μ as in Lemma 16, is surjective.*

The reduction of satisfiability over $\mathsf{Coalg}(T)$ to satisfiability over $\mathsf{Coalg}(L^\star)$ can now be achieved by picking a one-sided inverse of μ.

Lemma 21. *Let $\mathcal{L} = (\Lambda, \mathcal{A})$ be a rank-1 logic, let $\Gamma \subseteq \Lambda$ be finite, and let L be the functorial presentation of $\mathcal{L}_\Gamma$. Then for every $\mathbb{C} = (C, \gamma) \in \mathsf{Coalg}(L^\star)$ with C finite, there exists $\mathbb{D} = (C, \delta) \in \mathsf{Coalg}(T)$ with the same carrier such that $[\![\phi]\!]^\pi_\mathbb{C} = [\![\phi]\!]^\pi_\mathbb{D}$ for all $\phi \in \mathcal{F}(\Gamma)$, $\pi : V \to \mathcal{P}(C)$ and $c \in C$.*

As the passage from $\mathsf{Coalg}(L^\star)$ to $\mathsf{Coalg}(T)$ provided by Lemmas 17 and 21 in particular preserves validity of additional frame conditions, we can summarise as follows:

Corollary 22. *Let $\mathcal{L} = (\Lambda, \mathcal{A}, \Theta)$ be an extended rank-1 logic, and let L be the functorial presentation of $(\Lambda, \mathcal{A})$. If $\Gamma \subseteq \Lambda$ is finite and $\Theta \subseteq \mathcal{F}(\Gamma)$, then the following are equivalent for $\phi \in \mathcal{F}(\Gamma)$:*

1. *ϕ is finitely satisfiable in* $\mathsf{Coalg}(T, \Theta)$
2. *ϕ is finitely satisfiable in* $\mathsf{Coalg}(L^\star, \Theta)$

Together with Corollary 12 this means that we can reduce satisfiability over $\mathsf{Coalg}(T)$ to satisfiability over $\mathsf{Alg}(L)$; we pursue this theme in the subsequent sections.

5 The Finite Model Property

We will now exploit Corollary 22 to reduce the question of $\mathcal{L}$ having the finite model property to the question of an associated equational logic having the finite algebra property. We begin by associating an algebraic theory to every (extended) rank-1 logic $\mathcal{L}$.

Definition 23. Let $\mathcal{L} = (\Lambda, \mathcal{A})$ be a rank-1 logic. The algebraic theory *associated with* $\mathcal{L}$ is the pair (Σ, E), where

- $\Sigma = \Lambda \cup \{\bot, \to\}$ is the signature of boolean algebra augmented with a unary operator for every modality of Λ (silently assuming $\to, \bot \notin \Lambda$)
- $E = E_{\mathsf{BA}} \cup \{\phi = \top \mid \phi \in \mathcal{A}\}$ consists of an equational axiomatisation E_{BA} of boolean algebra, together with the (equational form of) the axioms of $\mathcal{L}$.

If $\mathcal{T} = (\Sigma, E)$ is the algebraic theory associated with a rank-1 logic, we write $\mathsf{Alg}(\mathcal{T})$ for the category of (Σ, E)-algebras in the sense of universal algebra [21] and adopt the standard interpretation of terms $\phi \in F(\Lambda)$ given a valuation of the (propositional) variables in V. As every (Σ, E)-algebra in particular carries a boolean algebra structure, we present (Σ, E)-algebras as $(A, (f_M)_{M \in \Lambda})$ where $A \in \mathsf{BA}$ and $f_M : A \to A$ for all $M \in \Lambda$.

If $\phi \in \mathcal{F}(\Lambda)$ and $\mathbb{A} \in \mathsf{Alg}(\mathcal{T})$, we abbreviate $\mathbb{A} \models \phi$ iff $\mathbb{A} \models \phi = \top$, i.e. $[\![\phi]\!]^{\pi}_{\mathbb{A}} = \top$ for all valuations $\pi : V \to A$. As previously, if $\Theta \subseteq \mathcal{F}(\Lambda)$, we write $\mathsf{Alg}(\mathcal{T}, \Theta)$ for the full subcategory of all $\mathcal{T}$-algebras $\mathbb{A}$ for which $\mathbb{A} \models \phi$ for all $\phi \in \Theta$; note that $\mathsf{Alg}(\mathcal{T}, \Theta) = \mathsf{Alg}(\Sigma, E \cup \Theta)$. If $\phi, \psi \in \mathcal{F}(\Lambda)$, we write $\mathcal{T}, \Theta \vdash \phi = \psi$ if $\phi = \psi$ can be derived equationally from $E \cup \Theta$. A formula $\phi \in \mathcal{F}(\Lambda)$ is $(\mathcal{T}, \Theta)$-*inconsistent*, if $\mathcal{T}, \Theta \vdash \phi = \bot$ and ϕ is $(\mathcal{T}, \Theta)$-*consistent*, otherwise.

Finally, a formula $\phi \in \mathcal{F}(\Lambda)$ is *(finitely) satisfiable* in $\mathsf{Alg}(\mathcal{T}, \Theta)$ if there exists a (finite) $\mathbb{A} \in \mathsf{Alg}(\mathcal{T}, \Theta)$ such that $\mathbb{A} \not\models \phi = \bot$.

In essence, if $\mathcal{T}$ is the algebraic theory associated with a rank-1 logic, $\mathcal{T}$-algebras are boolean algebras with operators in the sense of Jónnson and Tarski [11] but without the requirement that the operators preserve either joins or meets. We now relate the categories $\mathsf{Alg}(L)$ and $\mathsf{Alg}(\mathcal{T})$; this is in essence Theorem 15 of [2].

Lemma 24. *Let $\mathcal{L}$ be a rank-1 logic with functorial presentation L and algebraic theory $\mathcal{T}$. Then there exists a concrete isomorphism $C : \mathsf{Alg}(L) \to \mathsf{Alg}(\mathcal{T})$ that commutes with the respective forgetful functors, i.e. $U_L \circ C = U_{\mathcal{T}}$ where $U_L : \mathsf{Alg}(L) \to \mathsf{Set}$ and $U_{\mathcal{T}} : \mathsf{Alg}(\mathcal{T}) \to \mathsf{Set}$. Moreover, this isomorphism is compatible with the (algebraic) semantics of modal formulas: If $\phi \in \mathcal{F}(\Lambda)$, $\mathbb{A} = (A, \alpha) \in \mathsf{Alg}(L)$ and $\pi : V \to A$ is a valuation, then $[\![\phi]\!]^{\pi}_{A} = [\![\phi]\!]^{\pi}_{C(\mathbb{A})}$.*

Together with Corollary 22, we obtain:

Corollary 25. *Let $\Gamma \subseteq \Lambda$ be finite, and let $\mathcal{L} = (\Lambda, \mathcal{A}, \Theta)$ be an extended rank-1 logic with $\Theta \subseteq \mathcal{F}(\Gamma)$. Then the following are equivalent for $\phi \in \mathcal{F}(\Gamma)$:*

1. *ϕ is finitely satisfiable in $\mathsf{Coalg}(T, \Theta)$*
2. *ϕ is finitely satisfiable in $\mathsf{Alg}(\mathcal{T}, \Theta)$.*

We are now in the position to relate the finite model property of a modal logic with the finite algebra property of the associated algebraic theory.

Definition 26. Let $\mathcal{L} = (\Lambda, \mathcal{A}, \Theta)$ be an extended rank-1 logic, and let $\mathcal{T}$ be the algebraic theory associated with $(\Lambda, \mathcal{A})$. If T is a structure for $\mathcal{L}$, we say that $\mathcal{L}$ has the finite model property w.r.t. $\mathsf{Coalg}(T, \Theta)$ if every $\mathcal{L}$-consistent formula is finitely satisfiable in $\mathsf{Coalg}(T, \Theta)$. Dually, $\mathcal{T}$ has the finite algebra property w.r.t. $\mathsf{Alg}(\mathcal{T}, \Theta)$ if every $(\mathcal{T}, \Theta)$-consistent formula $\phi \in \mathcal{F}(\Lambda)$ is finitely satisfiable in $\mathsf{Alg}(\mathcal{T}, \Theta)$.

It is easy to see that the above definition of the finite algebra property is equivalent to the standard definition (validity over $\mathsf{Alg}(\mathcal{T}, \Theta)_f$ implies validity over $\mathsf{Alg}(\mathcal{T}, \Theta)$) as we are dealing with extensions of boolean algebras.

The only ingredient that is missing for our first main theorem is the following (standard) lemma that relates equational and modal deduction.

Lemma 27. *Let $\mathcal{L} = (\Lambda, \mathcal{A}, \Theta)$ be an extended rank-1 logic, and let $\mathcal{T}$ be the algebraic theory associated with $(\Lambda, \mathcal{A})$. Then*

$$\mathcal{T}, \Theta \vdash \phi = \psi \text{ iff } \mathcal{L} \vdash \phi \leftrightarrow \psi \text{ and } \mathcal{L} \vdash \phi \text{ iff } \mathcal{T}, \Theta \vdash \phi = \top$$

for all $\phi, \psi \in \mathcal{F}(\Lambda)$.

Our main conceptual contribution can now be formulated as follows:

Theorem 28. *Let $\mathcal{L} = (\Lambda, \mathcal{A}, \Theta)$ be a rank-1 logic, let $\Gamma \subseteq \Lambda$ be finite, and let $\Theta \subseteq \mathcal{F}(\Gamma)$. For $\Gamma \subseteq \Delta \subseteq \Lambda$, let $\mathcal{T}_\Delta$ denote the algebraic theory associated with the Δ-reduct of $(\Lambda, \mathcal{A})$. Then $\mathcal{L}$ has the finite model property w.r.t.* $\mathsf{Coalg}(T, \Theta)$ *if all $\mathcal{T}_\Delta$, for Δ finite with $\Gamma \subseteq \Delta \subseteq \Lambda$, have the finite algebra property w.r.t* $\mathsf{Alg}(\mathcal{T}_\Delta, \Theta)$. *In this case $\mathcal{L}$ is moreover complete w.r.t.* $\mathsf{Coalg}(T, \Theta)$*, i.e. $\mathcal{L} \vdash \phi$ if* $\mathsf{Coalg}(T, \Theta) \models \phi$.

Some remarks are in order: First we can only allow frame conditions Θ over finitely many modalities, as our techniques rely on the duality between FBA and Fin, cf Lemma 9. Second, to establish the finite model property for $\mathcal{L}$, we need to establish the finite algebra property for all $\mathcal{T}_\Delta$ in the terminology of the previous theorem where $\Delta \subseteq \Gamma$ contains the modalities occurring in the frame conditions Θ together with those occurring in a particular consistent formula that we seek to satisfy. This restriction is however not problematic as we will see in the following two sections.

6 The Finite Algebra Property

Theorem 28 leaves an important question unanswered: which algebraic theories enjoy the finite algebra property? This question is partially answered in the present section, where we describe a general mechanism for constructing finite algebras that covers a large variety of cases and generalises [14] to a non-normal setting and – modulo the duality between neighbourhood frames and modal algebras [6] – also [3] to combinations of rank 0/1 formulas and the (4) axiom.

For the whole section, suppose that $\mathcal{L} = (\Lambda, \mathcal{A}, \Theta)$ is an extended rank-1 logic and $\mathcal{T} = (\Sigma, E)$ is the algebraic theory associated with $(\Lambda, \mathcal{A})$. Our goal in this section is to find conditions that ensure that $\mathcal{T}$ has the finite algebra property w.r.t $\mathsf{Alg}(\mathcal{T}, \Theta)$, which we pursue by constructing finite algebras over maximally consistent subsets of closed sets of formulas.

Definition 29. The *normalised negation* $\sim \phi$ of a formula is the formula $\neg\phi$, if ϕ is not of the form $\neg\psi$, and $\sim \phi = \psi$ if $\phi = \neg\psi$. A set $\Delta \subseteq \mathcal{F}(\Lambda)$ is *closed*, if Δ contains ψ whenever ψ is a subformula of some $\phi \in \Delta$ and Δ contains $\sim \phi$ if $\phi \in \Delta$. We write $\mathbf{cl}(\phi)$ for closure (the smallest closed set containing ϕ) of a single formula $\phi \in \mathcal{F}(\Lambda)$. If Δ is a closed set and $\Gamma \subseteq \Lambda$ is a set of modal operators, the *Γ-extension* of Δ is the set

$$\Delta_\Gamma = \{M\phi \mid \phi \in \Delta, M \in \Gamma\} \cup \{\neg M\phi \mid \phi \in \Delta, M \in \Gamma\} \cup \Delta$$

We write $\mathcal{M}_\Delta$ for the set of maximally $\mathcal{L}$-consistent subsets of a closed set Δ and drop the subscript if there is no danger of confusion.

Note that $\Delta_\Gamma = \Delta$ if $\Gamma = \emptyset$. The construction of a satisfying model for a consistent formula ϕ that we describe will be based on the closure $\mathbf{cl}(\phi)$ or its Γ-extension for a finite set Γ of modalities. The construction uses the map σ introduced below.

Lemma 30. *Let Δ be closed and finite. The assignment*

$$\sigma : \mathcal{P}(\mathcal{M}_\Delta) \to \mathcal{F}(\Lambda), A \mapsto \bigvee_{\Phi \in A} \bigwedge \Phi$$

satisfies $\mathcal{T}, \Theta \vdash \sigma(A \cup B) = \sigma(A) \vee \sigma(B)$ and $\mathcal{T}, \Theta \vdash \sigma(\neg A) = \neg\sigma(A)$.

Conceptually speaking, σ induces a morphism of boolean algebras $\mathcal{P}(\mathcal{M}) \to \mathcal{F}(\Lambda)/\sim$ where $\sim$ is logical equivalence. We now introduce (algebraic) filtrations that we employ to witness the finite algebra property.

Definition 31. Let $\Gamma \subseteq \Lambda$, and let Δ be a closed and finite subset of $\mathcal{F}(\Lambda)$. Let $\mathcal{M}$ denote the collection of maximally consistent subsets of Δ_Γ. The *natural valuation* induced by Δ and Γ is the mapping $\tau : V \to \mathcal{P}(\mathcal{M})$, $\tau(p) = \{\Phi \in \mathcal{M} \mid p \in \Phi\}$. A Σ-algebra $\mathbb{F} = (\mathcal{P}(\mathcal{M}), (f_M)_{M\in\Lambda})$ is a $\Delta(\Gamma)$-*filtration* if it satisfies

$$f_M(\{\Phi \in \mathcal{M} \mid \phi \in \Phi\}) = \{\Phi \in \mathcal{M} \mid M\phi \in \Phi\}$$

for all $M\phi \in \Delta$. If $\Gamma = \emptyset$ we will simply speak of a Δ-filtration. We call a $\Delta(\Gamma)$-filtration *safe* for a set $\Phi \subseteq \mathcal{F}(\Lambda)$ of axioms if $\mathbb{F} \models \phi = \top$ for all $\phi \in \Phi$.

Informally, a $\Delta(\Gamma)$-filtration puts a Σ-algebra structure on the set $\mathcal{P}(\mathcal{M})$ of sets of maximally consistent subsets of Δ_Γ such that the truth lemma is satisfied for all $\phi \in \Delta$. This will be applied to the case where $\Delta = \mathbf{cl}(\phi)$ is the closure of a single formula and we will usually choose $\Gamma = \emptyset$; however in some cases (notably in presence of the (5) axiom) we need to rely on the additional structure provided by formulas $\psi \in \Delta_\Gamma \setminus \Delta$ to prove the truth lemma for formulas of Δ. The structure present in $\Delta(\Gamma)$-filtrations clearly suffices to prove the (algebraic) truth lemma and the finite algebra property follows if all axioms are safe.

Proposition 32. *Let $\Delta \subseteq \mathcal{F}(\Lambda)$ be closed, let $\Gamma \subseteq \Lambda$, and let $\mathbb{F}$ be a $\Delta(\Gamma)$ filtration. Then*

$$[\![\phi]\!]^\tau_\mathbb{F} = \{\Phi \in \mathcal{M}_{\Delta_\Gamma} \mid \phi \in \Phi\}$$

for all $\phi \in \Delta$. If moreover every closed and finite set Δ admits a $\Delta(\Gamma)$-filtration for a finite subset $\Gamma \subseteq \Lambda$ that is safe for $\mathcal{A} \cup \Theta$, then $\mathcal{T}$ has the finite algebra property w.r.t. $\mathsf{Alg}(\mathcal{T}, \Theta)$.

The remainder of this section is devoted to showing that every modal logic $\mathcal{L} = (\Lambda, \mathcal{A}, \Theta)$ where Θ is rank-0/1 admits a safe Δ-filtration for every closed set $\Delta \subseteq \mathcal{F}(\Lambda)$; we refer the reader to Definition 3 for the notions of rank-0/1 axioms and logics.

Definition 33. Let Δ be closed and finite. A *sieve* over Δ is a mapping $\nu : \mathcal{M} \to \mathcal{P}(\mathcal{F}(\Lambda))$ such that $\Phi \subseteq \nu(\Phi)$ for all $\Phi \in \mathcal{M}$ and $\nu(\Phi)$ is maximally $\mathcal{L}$-consistent. The Σ-algebra $(\mathcal{P}(\mathcal{M}_\Delta), (f_M)_{M\in\Lambda})$ where

$$f_M : \mathcal{P}(\mathcal{M}) \to \mathcal{P}(\mathcal{M}), \quad A \mapsto \{\Phi \in \mathcal{M} \mid \nu(\Phi) \vdash M(\sigma(A))\}$$

is called the *standard Δ-filtration* defined by the sieve ν.

We usually omit the explicit mention of the operators and just use $\mathcal{P}(\mathcal{M})$ to refer to the standard filtration. Note that the standard filtration implicitly depends on a choice of maximally consistent extension $\nu(\Phi)$ of $\Phi \in \mathcal{M}$, but the choice of $\nu(\Phi)$ is immaterial.

Lemma 34. *Let $\mathcal{P}(\mathcal{M})$ be the standard Δ-filtration for a closed and finite set $\Delta \subseteq \mathcal{F}(\Lambda)$ given by a sieve ν. Then $f_M(\{\Phi \in \mathcal{M} \mid \phi \in \Phi\}) = \{\Phi \in \mathcal{M} \mid M\phi \in \Phi\}$ for all $M \in \Lambda$.*

We are now in the position to prove that the algebraic theory associated to a rank-0/1 logic has the finite algebra property if we can show that the standard filtration is safe for all rank-0/1 axioms.

Proposition 35. *Let* $\mathcal{L} = (\Lambda, \mathcal{A}, \Theta)$ *be rank-0/1 and assume that* Δ *is closed and finite. Then the standard* Δ*-filtration defined by any sieve is safe for* $\mathcal{A} \cup \Theta$*.*

Consequently, we have the finite algebra property for all rank 0/1 logics.

Theorem 36. *Let* $\mathcal{L} = (\Lambda, \mathcal{A}, \Theta)$ *be rank-0/1, and let* $\mathcal{T}$ *be the algebraic theory associated with* $(\Lambda, \mathcal{A})$*. Then* $\mathcal{T}$ *has the finite algebra property w.r.t.* $\mathsf{Alg}(\mathcal{T}, \Theta)$.

The finite model property for $\mathcal{L}$ now follows at once from Theorem 28.

Corollary 37. *Let* $\mathcal{L} = (\Lambda, \mathcal{A}, \Theta)$ *be rank-0/1, and let* $\Theta \subseteq \mathcal{F}(\Gamma)$ *for a finite subset* $\Gamma \subseteq \Lambda$*. Then* $\mathcal{L}$ *has the finite model property w.r.t.* $\mathsf{Coalg}(T, \Theta)$*; in particular* $\mathcal{L}$ *is complete w.r.t.* $\mathsf{Coalg}(T, \Theta)$.

7 Frame Conditions Beyond Rank 0/1

We now extend the ideas developed in the previous section to logics beyond rank 0/1. As the standard filtration does not necessarily satisfy the frame conditions imposed by axioms beyond rank 0/1, we have to adapt the standard filtration accordingly. We treat three instances of frame conditions beyond rank-1 in detail: instances of the (4) axiom, generalised to not necessarily normal operators, and instances of the (B) and (5) axioms for normal operators, the latter being primarily of interest in the context of logics that additionally feature non-normal operators.

Definition 38. Let ν be a sieve over a finite closed set Δ. For $M \in \Lambda$ we write $M^\star$ for M's dual $\neg M \neg$. If $\Phi, \Psi \in \mathcal{M}_\Delta$ we say that Φ *evolves* to Ψ along M ($\Phi \rightsquigarrow_M \Psi$) iff $\nu(\Phi) \vdash M^\star \bigwedge \Psi$. Similarly Φ *zigzags* to Ψ along M ($\Phi \leftrightsquigarrow_M \Psi$) iff there are $\Omega_0, \dots, \Omega_n$ such that

- $\Omega_0 \rightsquigarrow_M \Phi$ and $\Omega_n \rightsquigarrow_M \Psi$
- $\Omega_i \rightsquigarrow_M \Omega_{i-1}$ or $\Omega_{i-1} \rightsquigarrow_M \Omega_i$ for all $i = 1, \dots, n$.

We consider the following operators of type $\mathcal{P}(\mathcal{M}) \to \mathcal{P}(\mathcal{M})$ (see Definition 33 for the definition of f_M)

$$\begin{aligned} f^B_M(A) &= \{\Psi \in f_M(A) \mid \forall \Phi \in \mathcal{M}(\Phi \rightsquigarrow_M \Psi \Rightarrow \Psi \in A)\} \\ f^5_M(A) &= \{\Psi \in f_M(A) \mid \forall \Phi \in \mathcal{M}(\Phi \leftrightsquigarrow_M \Psi \Rightarrow \Phi \in A)\} \\ f^{\bullet 4}_M(A) &= \mathrm{gfp}(X \mapsto f^\bullet_M(A) \cap f^\bullet_M(X)) \end{aligned}$$

where gfp denotes the greatest fixpoint and in the last line, $\bullet$ stands for nothing or one of B, 5.

As we will see later, each of the above operators can be used to force the validity of the corresponding axioms in a filtration so that e.g. f^{45} forces the validity of the axioms (4) and (5) if the corresponding axioms are derivable in $\mathcal{L}$.

Definition 39. Every modality $M \in \Lambda$ is of type E. The operator M is normal, if $\mathcal{L} \vdash M(p \to q) \to Mp \to Mq$ and $\mathcal{L} \vdash M\top$. It is *monotone* if $\mathcal{L} \vdash M(p \wedge q) \to Mp$. Moreover M is of type

- 4, if M is monotone and $\mathcal{L} \vdash Mp \to MMp$
- B, if M is normal and $\mathcal{L} \vdash p \to MM^\star p$
- 5 if M is normal and $\mathcal{L} \vdash M^\star p \vdash MM^\star p$

where again $M^\star = \neg M \neg$ denotes M's dual. We say that an operator is of type $4B$ (of type 45) if it is both of type 4 and of type B (of type 5).

We now show that the operators defined above give rise to a $\Delta(\Gamma)$ filtration provided the types of modalities match the associated operators.

Proposition 40. *Let $t(M) \in \{E, 4, 5, B, 4B, 45\}$, and let M be of type $t(M)$ for all $M \in \Lambda$. If $\Delta \subseteq \mathcal{F}(\Lambda)$ is closed then the Σ-algebra $(\mathcal{P}(\mathcal{M}_{\Delta_\Gamma}), (f_M^{t(M)})_{M \in \Lambda})$ is a $\Delta(\Gamma)$-filtration if $\Gamma \supseteq \{M \in \Lambda \mid t(M) = 5$ or $t(M) = 45\}$.*

This last proposition needs the extra generality provided by a $\Delta(\Gamma)$ filtration for $\Gamma \neq \emptyset$ to prove the truth lemma for modalities of type $5, 45$ which is not needed to show safety of axioms:

Proposition 41. *Let $t(M) \in \{E, 4, 5, B, 4B, 45\}$, and let M be of type $t(M)$ for all $M \in \Lambda$. If $\Delta \subseteq \mathcal{F}(\Lambda)$ is closed then the Σ-algebra $(\mathcal{P}(\mathcal{M}_\Delta), (f_M^{t(M)})_{M \in \Lambda})$ satisfies*

- *$Mp \to MMp$ if $t(M)$ is one of 4, 4B, 45*
- *$M^\star p \to MM^\star p$ if $t(M) = 5$ or $t(M) = 45$*
- *$p \to MM^\star p$ if $t(M) = B$ or $t(M) = 4B$.*

where again $M^\star$ is the dual of M.

The next proposition collects additional safe axioms:

Proposition 42. *Let $t(M) \in \{E, B, 5, 4, 4B, 45\}$, and let M be of type $t(M)$ for all $M \in \Lambda$. Then the following axioms are valid in the filtration $(\mathcal{P}(\mathcal{M}), (f_M^{t(M)})_{M \in \Lambda})$*

1. *any clause whose right side only mentions operators of type E, that is all clauses $\bigwedge_i M_i\phi_i \wedge \bigwedge_j p_j \to \bigvee_j M_k\phi_k \vee \bigvee_l p_l$ where $t(M_k) = E$ for all k*
2. *truth and falsity preservation, i.e. the axioms $M\top$ and $\neg M\bot$ for all $M \in \Lambda$*
3. *preservation of conjunctions, i.e. the axiom $M(p \wedge q) = Mp \wedge Mq$*
4. *the (S)-axiom $p \to Mp$*

As T-coalgebras are just Kripke frames for $T = \mathcal{P}$, the preceding proposition immediately gives the finite algebra property (and hence the finite model property) for a large number of well-studied systems, including $K, KT, KB, KB4, S4, S5$. We discuss two examples that apply our techniques outside the realm of normal logics in the next section.

8 Reasoning about Uncertainty and Knowledge

This section shows how the theory developed in the preceding sections can be instantiated to obtain synthetic proofs of completeness and the finite model property for logics of uncertainty and belief discussed in [7]. We fix a finite set N of agents and the modal signature $\Lambda = \{\mathsf{K}_n \mid n \in N\} \cup \{L^i_p \mid p \in [0,1] \cap \mathbb{Q}, i \in N\}$. We read the formula $\mathsf{K}_i\phi$ as "agent i knows phi" and $L^i_p\phi$ as "according to agent i, the formula ϕ hods with probability at least p". Formulas of $\mathcal{F}(\Lambda)$ are interpreted over T-coalgebras where $TX = \prod_{i \in N} \mathcal{D}(X) \times \mathcal{P}(X)$ and $\mathcal{D}(X)$ is the probability distributions functor $\mathcal{D}(X) = \{\mu : X \to [0,1] \mid \{x \in X \mid \mu(x) > 0\}$ is finite and $\sum_{x \in X} \mu(x) = 1\}$. That is, a T-coalgebra consists of a carrier set C and additionally, for each agent $i \in N$, a relation $R_i \subseteq C \times C$ and a function $f_i : C \to \mathcal{D}(C)$ that assigns probabilities to events in C. We turn T into a structure for Λ by virtue of the predicate liftings

$$[\![\mathsf{K}_i]\!]_X(A) = \{(\mu_i, S_i)_{i \in N} \mid S_i \subseteq A\}$$
$$[\![L^i_p]\!]_X(A) = \{(\mu_i, S_i)_{i \in N} \mid \textstyle\sum_{x \in A} \mu_i(x) \geq p\}.$$

The semantics of $\mathcal{F}(\Lambda)$ differs slightly from the semantics discussed in [7] as probability distributions are supposed to have finite support; however this difference is immaterial in the light of the finite model property.

It follows from [5] that a one-step complete axiomatisation of each K_i together with a one-step complete axiomatisation of the L^i_p is one-step complete for T; we denote the full set of axioms by $\mathcal{A}$. We refer to [5] for the precise form of the axioms and note that the one-step rule in *loc.cit.* can be equivalently presented as a rank-1 axiom by virtue of Proposition 15 in [17]. However note that any such one-step complete axiomatisation only prescribes the K-axioms $\mathsf{K}_i\top = \top$ and $\mathsf{K}_i(a \wedge b) \leftrightarrow \mathsf{K}_i a \wedge \mathsf{K}_i b$ for the knowledge operators K_i, and no interaction of knowledge and quantitative uncertainty on the logical level. On top of the basic (one-step complete) axiomatisation $\mathcal{A}$, we consider the axioms ($\mathsf{K}^\star_i = \neg\mathsf{K}_i\neg$)

(4)	$\mathsf{K}_i\mathsf{K}_i p \to \mathsf{K}_i p$	(C)	$\mathsf{K}_i p \to L^i_1 p$
(5)	$\mathsf{K}^\star_i p \to \mathsf{K}_i\mathsf{K}^\star_i p$	(U)	$p \to L^i_1 p$
(B)	$p \to \mathsf{K}_i\mathsf{K}^\star_i p$	(S)	$p \to \mathsf{K}_i p$

where the names of the axioms on the right abbreviate the corresponding frame properties of [7]. Theorems 28 together with Propositions 40 and 42 now show that any extension of $\mathcal{A}$ with one or more of the above axioms has the finite model property. We can even equip different agents with different reasoning facilities, i.e. allow positive introspection for some but not for others. This generalises the corresponding result in [7] in that we establish the finite model property for all logics $\mathcal{L} = (\Lambda, \mathcal{A}, \Theta)$, where Θ is a sub-collection of the above frame conditions. By understanding every axiom (A) as the collection of instances of (A) for all agents $i \in N$ we can restrict the validity of axioms to specific sets of agents in the next theorem.

Theorem 43. *Let $\mathcal{L} = (\Lambda, \mathcal{A}, \Theta)$ and suppose that $\Theta \subseteq \{(4), (5), (B), (C), (U), (S)\}$ not containing both (B) and (5) for the same agent. Then $\mathcal{L}$ has the finite model property w.r.t.* $\mathsf{Coalg}(T, \Theta)$.

9 A Logic for Coalitions and Filibusters

As a second example, we apply our techniques to prove completeness and the finite model property for an extension of Pauly's coalition logic [16]. We consider a fixed set $N = \{1, \dots, n\}$ of *agents*. Subsets of N are called *coalitions*. The signature Λ of coalition logic consists of modal operators $[C]$, where C ranges over coalitions, read 'coalition C has a collaborative strategy to ensure that ...'. A coalgebraic semantics for coalition logic is based on the class-valued signature functor T defined by

$$TX = \{(S_1, \dots, S_n, f) \mid \emptyset \neq S_i \in \mathsf{Set}, f : \textstyle\prod_{i \in N} S_i \to X\}.$$

The elements of TX are understood as *strategic games* with set X of states, i.e. tuples consisting of nonempty sets S_i of *strategies* for all agents i, and an *outcome function* $(\prod S_i) \to X$. A T-coalgebra is a *game frame* [16]. We denote the set $\prod_{i \in C} S_i$ by S_C, and for $\sigma_C \in S_C, \sigma_{\bar{C}} \in S_{\bar{C}}$, where $\bar{C} = N - C$, $(\sigma_C, \sigma_{\bar{C}})$ denotes the obvious element of $\prod_{i \in N} S_i$. A Λ_{C}-structure over T is defined by

$$[\![[C]]\!]_X(A) = \{(S_1, \dots, S_n, f) \in TX \mid \exists \sigma_C \in S_C. \forall \sigma_{\bar{C}} \in S_{\bar{C}}. f(\sigma_C, \sigma_{\bar{C}}) \in A\}.$$

A one-step complete axiomatisation $\mathcal{A}$ of coalition logic consists of the axioms $(\top)$, $(\bot)$, (N) and (S) below

$(\bot)$	$\neg[C]\bot$	(C)	$[C]\phi \to \phi$
$(\top)$	$[C]\top$	(F)	$[C][C]\phi \to [C]\phi$
(N)	$[\emptyset]\phi \vee [N]\neg\phi$	(P)	$\phi \to [C]\phi$
(S)	$[C_1]\phi \wedge [C_2]\phi \to [C_1 \cup C_2](\phi \wedge \psi)$		

where $C_1 \cap C_2 = \emptyset$ in the superadditivity axiom (S). One-step completeness of $\mathcal{A}$ is proved in [18] using the rule format of the axioms above. Additionally, we consider a subset $\Theta \subseteq \{(C), (F), (P)\}$, again with axioms in Θ possibly restricted to specific coalitions.

The (C) axiom expresses that the power of a coalition is limited to forcing things that are already valid; we think of such coalitions as conservative. In a similar vein, if ϕ expresses the act of blocking a motion and a coalition (of senators) has the power to achieve ϕ, then the (F)-axiom (together with monotonicity) expresses that C can block this motion indefinitely. Accordingly we refer to (F) as the filibuster axiom. Finally, by virtue of axiom (P), a coalition can perpetuate properties of a strategic game. Using the same convention as in the previous example, we understand each axiom (A) as the collection of instances of (A) for all coalitions. Hence a subset of $\{(C), (F), (P)\}$ in general only contains instances of each axiom for a specific set of coalitions. Again, the combination of Theorem 28 and Propositions 40 and 42 shows:

Theorem 44. *Let $\mathcal{L} = (\Lambda, \mathcal{A}, \Theta)$ where $\Theta \subseteq \{(C), (F), (P)\}$. Then $\mathcal{L}$ has the finite model property w.r.t* $\mathsf{Coalg}(T, \Theta)$*; in particular $\mathcal{L}$ is complete w.r.t.* $\mathsf{Coalg}(T, \Theta)$.

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