Free Adjunction of Morphisms

Lutz Schröder and Horst Herrlich
Department of Mathematics, University of Bremen, Pf. 330440, 28334 Bremen, Germany

Abstract. We develop a general setting for the treatment of extensions of categories by means of freely adjoined morphisms. To this end, we study what we call composition graphs, i.e. large graphs with a partial binary operation on which we impose only rudimentary requirements. The quasicategory thus obtained contains the quasicategory of all categories as a full reflective subquasicategory; we characterize composition graphs for which this reflexion is of a particularly simple nature.

This leads to the concept of semicategory; we apply semicategories to solve characterization problems concerning absolutely initial sources, absolute monosources and potential sections. For instance, we show that in any category, the absolutely initial sources are precisely the sources that contain a section.

Keywords: (absolutely) initial morphism, (absolute) monomorphism, graph, category, section.

Mathematics Subject Classifications (1991): 18A10, 18A20, 18A22.

Unserem geschätzten Freund und Kollegen Nico Pumplün zu seiner Emeritierung gewidmet

Introduction

The search for certain extensions of abstract categories intuitively gives rise to the idea of a "free adjunction of morphisms" subject to certain equations. As an example, consider the problem of extending a category in such a way that a given monomorphism $s : A \to B$ becomes a section: The natural approach consists in "freely" adding a morphism $t : B \to A$, which leads to morphisms of type $f_1tf_2t\dots tf_n$, and then factoring out the congruence generated by $ts \cong id_A$. The proof that this does indeed give an embedding will somehow require a normal form for the resulting morphisms. Problems of this type can often be solved by the general method developed here.

In section 1 we introduce the concept of a composition graph. The idea is essentially to generalize the concept of category by dropping most axioms, including the requirement that two morphisms are composable whenever the domain of one coincides with the codomain of the other. Similar structures have been considered by Ehresmann and his students (cf. [2]–[6] and [8]–[10]); our concept is even weaker. In this setting, the process of adding new morphisms becomes trivial; equations can in many cases be expressed by defining suitable compositions. Since every composition graph freely generates a category, such constructions can be used to implicitly describe extensions of categories.

In this way, problems of the initially mentioned type are reduced to understanding the free category over a given composition graph. In section 2, we characterize composition graphs that allow a particularly simple normal form for morphisms of the free category; we call composition graphs of this type semicategories.

In the last section, we apply semicategories to the treatment of several extension problems: We characterize absolutely initial sources and reobtain a coinciding characterization of absolute monosources; the proofs are by contraposition and involve the construction of extensions that serve to destroy the properties in question. Moreover, we solve the initially stated problem.

Terminology generally follows [1], except that categories are *not* in general assumed to be *locally small* (i.e. we do not require hom-classes $hom(A, B)$ to be sets); however, the results exposed in section 3 do hold in the quasicategory of locally small categories.

1. Composition Graphs

In this context, we regard a *graph* as a class G with two unary total operations d and c, subject to the equations $cc = dc = c$ and $dd = cd = d$. The elements of the underlying class of a graph will be called *morphisms*, the nodes (i.e. the elements of $d[G] = c[G]$) *objects* or *identities*. Observe that in a composition graph as defined below, identities need not act as identities are supposed to. Objects will be denoted either by capital letters A, B etc. or — when considered as "identities" — in the form id_A, id_B etc.

DEFINITION 1.1. *A graph G with a binary partial operation $\circ$ (i.e. one defined on an arbitrary subset of $G \times G$) is called a* composition graph *if, whenever $f \circ g$ is defined, $c(f \circ g) = cf$, $df = cg$ and $d(f \circ g) = dg$. Instead of $f \circ g$ we also write fg. A graph morphism F between two composition graphs is called a* functor *if, whenever fg is defined, $FfFg$ is defined and equal to $F(fg)$.*

All categories are, of course, composition graphs. In fact, the quasicategory of all categories is a (full) reflective subquasicategory of the quasicategory of all composition graphs and functors. The free category

$\mathbf{A}^*$ over a composition graph $\mathbf{A}$ is constructed as follows: First, take the category $\mathbf{W}(\mathbf{A})$ of all words $\mathbf{f} = (f_n, \ldots, f_1)_A$, where the f_i are morphisms in $\mathbf{A}$, A an object of $\mathbf{A}$, $n \geq 0$, $df_{i+1} = cf_i$ for $i = 1, \ldots, n-1$ and $df_1 = A$ if $n > 0$; let $d\mathbf{f} = ()_A$, $c\mathbf{f} = ()_{cf_n}$ for $n > 0$, $c\mathbf{f} = ()_A$ for $n = 0$, and define composition as concatenation of words (This is *not* the free category over the underlying graph of $\mathbf{A}$.). The index object serves only as a distinction between empty words, i.e. identities, and will be omitted in case $n > 0$ if no confusion is possible. Now $\mathbf{A}^*$ is obtained from $\mathbf{W}(\mathbf{A})$ by factoring out the congruence generated by

$$\begin{aligned} (f,g)_A &\cong (fg)_A, && \text{whenever } fg \text{ is defined, and} \\ (id_A)_A &\cong ()_A && \text{for all objects } A. \end{aligned}$$

We use square brackets to denote equivalence classes under $\cong$. It is easy to see that we have a functor $R_{\mathbf{A}} : \mathbf{A} \to \mathbf{A}^*$, defined by $R_{\mathbf{A}} f = [(f)_{df}]$, and that every functor from $\mathbf{A}$ into a category factors uniquely through $R_{\mathbf{A}}$.

We write $\mathbf{f} \succeq \mathbf{g}$ for words $\mathbf{f}$ and $\mathbf{g}$ if $\mathbf{f} = \mathbf{g}$ or $\mathbf{g}$ is obtained from $\mathbf{f}$ by either composing two letters or by removing a letter that is an identity. Then $\cong$ is the equivalence relation generated by $\succeq$. We denote the transitive closure of $\succeq$ by $\sqsupseteq$; obviously, $\sqsupseteq$ is already a partial order.

In general, $R_{\mathbf{A}}$ need not be faithful; we will provide an example in the next section. Also, if $\mathbf{A}$ is locally small, $\mathbf{A}^*$ may fail to be so:

EXAMPLE 1.2. *Let T be a proper class and let $\mathbf{A}$ be the locally small composition graph*

$$A \xrightarrow{\ a_t\ } B_t \xrightarrow{\ b_t\ } C,$$

where t ranges over T and the B_t are pairwise distinct; composition is nowhere defined. Then

$$\{(b_t, a_t) | t \in T\}$$

is a proper class of pairwise inequivalent words in $hom_{\mathbf{W}(\mathbf{A})}(A, C)$.

We do however have the following local smallness criterion:

PROPOSITION 1.3. *Let $\mathbf{A}$ be a locally small composition graph, and let P denote the class of all objects A in $\mathbf{A}$ such that there exist morphisms f and g with $cg = df = A$ and fg not defined. If P is a set, then $\mathbf{A}^*$ is locally small.*

Proof. It suffices to show that for each pair of objects (A, B), the class M of words in $hom_{\mathbf{W}(\mathbf{A})}(A, B)$ that are minimal with respect to

$\sqsupseteq$ is a set. Now if $(f_n, \ldots, f_1)_A$ is minimal, cf_i must belong to P for $i = 1, \ldots, n-1$, i.e. we have

$$M \subset \bigcup_{n \geq 2} \bigcup_{\substack{(A_1,\ldots,A_n)\in \\ \{A\}\times P^{n-2}\times\{B\}}} \prod_{i=1}^{n-1} hom_{\mathbf{A}}(A_i, A_{i+1}),$$

which is a set.

Composition graphs and functors form a concrete quasicategory over the quasicategory of graphs; we call a functor *initial* if it is an initial morphism in this concrete quasicategory. It is easy to see that a functor F is initial iff, whenever $FfFg = Fh$ is defined for morphisms f, g and h with $cf = ch$, $df = cg$ and $dg = dh$, then $fg = h$ is defined. Observe that a functor between categories is initial iff it is faithful.

The following properties, although seemingly trivial, play a crucial role in the study of composition graphs:

DEFINITION 1.4. *A composition graph is called* weakly identitive *if, whenever one of the terms* $(cf)f$ *and* fdf *is defined, it equals* f, *and* strongly identitive *if, in addition, these terms are always defined.*

It is easy to see that $\mathbf{A}$ is weakly identitive if $R_{\mathbf{A}}$ is faithful and strongly identitive if $R_{\mathbf{A}}$ is initial; furthermore, any initial functor with strongly identitive codomain is faithful.

Ehresmann's *multiplicative graphs* or *neocategories* (cf. [8], [9]) are, in our terminology, the strongly identitive composition graphs; the reasons for our dropping any a priori requirements of this nature will become apparent in the next section. (The possibility of a similar weakening of axioms is mentioned, but not pursued in [5].) A decription of the free category over a multiplicative graph can be found in [8]; the construction given here is, of course, slightly different.

2. Semicategories

In dealing with the word problem in the construction of the free category, words that are minimal with respect to $\sqsupseteq$ will play an important role. Obviously, every equivalence class of words contains at least one minimal word; in this section, we characterize those composition graphs for which such minimal representatives are unique.

The following convention simplifies the statement of the main result of this section:

DEFINITION 2.1. *Let* $\mathbf{A}$ *be a weakly identitive composition graph. We say that* $fg = h$ *is* pseudodefined *if* $fg = h$ *is defined in* $\mathbf{A}$*, or* $f = cg$ *and* $h = g$*, or* $g = df$ *and* $h = f$*.*

LEMMA 2.2. *Let* $\mathbf{A}$ *be a weakly (strongly) identitive composition graph. For words* $\mathbf{f} = (f_n, \ldots, f_1)$ *and* $\mathbf{g}$ *in* $\mathbf{W}(\mathbf{A})$ *with* $n \geq 2$*, we have* $\mathbf{f} \succeq \mathbf{g}$ *iff there exists* i *such that* $f_{i+1}f_i$ *is pseudodefined (defined) and* $\mathbf{g} = (f_n, \ldots, f_{i+1}f_i, \ldots, f_1)$*.*

THEOREM 2.3. *Let* $\mathbf{A}$ *be a composition graph; equivalent are*

(i) Each equivalence class of words in $\mathbf{A}^*$ *has exactly one minimal representative;*

(ii) for each word $\mathbf{f}$ *in* $\mathbf{W}(\mathbf{A})$*, there exists exactly one minimal word* $\varepsilon(\mathbf{f})$ *with* $\mathbf{f} \sqsupseteq \varepsilon(\mathbf{f})$*;*

(iii) $\mathbf{A}$ *is weakly identitive, and whenever* fg *and* gh *are defined in* $\mathbf{A}$*, then* $fg = f$ *and* $gh = h$*, or* $f(gh)$ *and* $(fg)h$ *are pseudodefined and equal.*

Proof. (ii) $\Rightarrow$ *(i):* For words $\mathbf{f}$ and $\mathbf{g}$ in $\mathbf{W}(\mathbf{A})$, $\varepsilon(\mathbf{g}) = \varepsilon(\mathbf{f})$ is implied by $\mathbf{f} \succeq \mathbf{g}$, hence also by $\mathbf{f} \cong \mathbf{g}$. Now if $\mathbf{g}$ and $\mathbf{f}$ are both minimal, $\mathbf{f} \cong \mathbf{g}$ implies $\mathbf{f} = \varepsilon(\mathbf{f}) = \varepsilon(\mathbf{g}) = \mathbf{g}$.

(i) $\Rightarrow$ *(iii):* Apply (i) to words of type (f, df), (cf, f) and (f, g, h), respectively.

(iii) $\Rightarrow$ *(ii):* We show that, whenever $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{g}'$ are words with $\mathbf{f} \succeq \mathbf{g}$ and $\mathbf{f} \succeq \mathbf{g}'$, there exists $\mathbf{h}$ with $\mathbf{g} \succeq \mathbf{h}$ and $\mathbf{g}' \succeq \mathbf{h}$. Double induction then yields the analogous statement for $\sqsupseteq$. If $\mathbf{f} \sqsupseteq \mathbf{g}, \mathbf{g}'$, where $\mathbf{g}$ and $\mathbf{g}'$ are both minimal, $\mathbf{g}, \mathbf{g}' \sqsupseteq \mathbf{h}$ implies $\mathbf{g} = \mathbf{h} = \mathbf{g}'$.

Now let $\mathbf{f} = (f_n, \ldots, f_1)$ and $\mathbf{f} \succeq \mathbf{g}, \mathbf{g}'$; assume w.l.o.g. that $n \geq 2$. By lemma 2.2, there exist i and j such that $f_{i+1}f_i$ and $f_{j+1}f_j$ are pseudodefined,

$$\begin{aligned} \mathbf{g} &= (f_n, \ldots, f_{i+1}f_i, \ldots, f_1) \quad \text{and} \\ \mathbf{g}' &= (f_n, \ldots, f_{j+1}f_j, \ldots, f_1); \end{aligned}$$

w.l.o.g. let $j > i$. If $j > i + 1$, we have

$$\mathbf{g}, \mathbf{g}' \succeq (f_n, \ldots, f_{j+1}f_j, \ldots, f_{i+1}f_i, \ldots, f_1).$$

Now observe that in condition (iii), definedness of fg and gh may be replaced by pseudodefinedness; applying the modified condition to the remaining case $j = i + 1$, we obtain $\mathbf{g} = \mathbf{g}'$ or

$$\mathbf{g}, \mathbf{g}' \succeq (f_n, \ldots, f_{i+2}f_{i+1}f_i, \ldots, f_1).$$

COROLLARY 2.4. *If* **A** *fulfills the equivalent conditions in the above theorem, then* $R_{\mathbf{A}}$ *is an embedding (i.e. injective on objects and faithful); if, in addition,* **A** *is strongly identitive, then* $R_{\mathbf{A}}$ *is initial.*

Proof. Faithfulness of $R_{\mathbf{A}}$ translates into the statement that, whenever $(f) \cong (g)$ for morphisms f and g in **A**, then $f = g$. Initiality of $R_{\mathbf{A}}$ just means that $(f, g) \cong (h)$ implies $fg = h$; recall lemma 2.2.

The full generality of the above theorem is not needed in this context; we introduce a more easily stated condition that is properly stronger than the ones given in the theorem:

DEFINITION 2.5. *A composition graph* **A** *is called* associative *if, whenever fg and gh are defined in* **A**, *then $(fg)h$ and $f(gh)$ are defined and equal. A weakly identitive and associative composition graph is called a* semicategory.

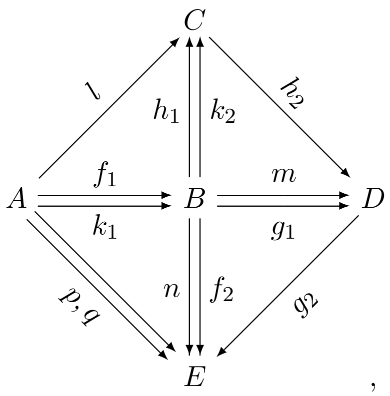
PROPOSITION 2.6. *If* **A** *is a semicategory, then* $R_{\mathbf{A}}$ *is an embedding; if, in addition,* **A** *is strongly identitive, then* **A** *is a category.*

While strong identitivity of **A** is a necessary condition for initiality of $R_{\mathbf{A}}$, associativity quite clearly is not. However, even for a seemingly slight weakening of the associativity requirement, the above result breaks down completely:

DEFINITION 2.7. *We call a composition graph* weakly associative *if, whenever fg and gh are defined, then the equation $f(gh) = (fg)h$ holds in the strong sense (i.e. whenever any one side is defined, then so is the other, and both sides are equal).*

REMARK 2.8. *It is easy to check that the weak associative law implies an analogous law for terms of length four; surprisingly, this is not true for terms of length five or more. Note that initiality of* $R_{\mathbf{A}}$ *implies that a weak associative law holds for terms of arbitrary length in* **A**.

EXAMPLE 2.9. *Let* **A** *be the strongly identitive composition graph*



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where $h_2h_1 = g_1$*,* $g_2g_1 = f_2$*,* $f_2f_1 = p$*,* $k_2k_1 = h_1f_1 = l$*,* $h_2k_2 = m$*,* $g_2m = n$ *and* $nk_1 = q$*.* $\mathbf{A}$ *is weakly associative; to see this without too much actual calculation, note that any nontrivial instance of the weak associative law would require the existence of pairwise distinct objects* $K_1, \dots, K_4$ *with* $hom(K_i, K_j) \neq \emptyset$ *whenever* $i < j$*. Since nontrivial terms in* $\mathbf{A}$ *have length at most four, a weak associative law holds for terms of arbitrary length in* $\mathbf{A}$*.*

However, $R_{\mathbf{A}}$ *is not injective: In* $\mathbf{W}(\mathbf{A})$*, we have*

$$\begin{aligned}(p) &\cong (f_2, f_1)\\ &\cong (g_2, g_1, f_1)\\ &\cong (g_2, h_2, h_1, f_1)\\ &\cong (g_2, h_2, l)\\ &\cong (g_2, h_2, k_2, k_1)\\ &\cong (g_2, m, k_1)\\ &\cong (n, k_1)\\ &\cong (q).\end{aligned}$$

3. Free Adjunction of Morphisms

As already mentioned in the introduction, the results of the preceding sections suggest the following recipe for extending categories by free adjunction of morphisms with equations: Add new arrows with the desired domains and codomains, and define compositions to guarantee the equations in question; then define just enough additional compositions to obtain a semicategory and pass to the — now well understood — free category over the latter. The following theorems make use of this procedure. Proposition 1.3 grants that all these results remain true in the quasicategory of locally small categories.

In what follows, phrases like "all functors" etc. always refer to functors into categories (rather than just composition graphs). For the sake of readability, we treat absoluteness properties of single morphisms separately; the corresponding statements for sources then arise as straightforward generalizations.

First we characterize various forms of absolute initiality (cf. also [1], 10P/Q.):

DEFINITION 3.1. *A morphism* $f : A \to B$ *in a category* $\mathbf{A}$ *is called* initial *w.r.t. a faithful functor* $F : \mathbf{A} \to \mathbf{B}$ *if, whenever* $g : C \to B$ *is a morphism in* $\mathbf{A}$ *and* $h : FC \to FA$ *is a morphism in* $\mathbf{B}$ *with*

$(Ff)h = Fg$, then there exists a morphism $\bar{h} : C \to A$ with $F\bar{h} = h$. f is called absolutely initial *if it is initial with respect to all faithful functors.*

THEOREM 3.2. *Let $\mathbf{A}$ be a category and $f : A \to B$ a morphism in $\mathbf{A}$. Equivalent are*

(i) f is absolutely initial;

(ii) f is initial with respect to all object-full embeddings;

(iii) f is a section.

Proof. (i)⇒(ii): Trivial.

(ii)⇒(iii): Assume that f is not a section. Extend $\mathbf{A}$ to a composition graph $\mathbf{B}$ by adding a morphism $x : A \to A$ and defining $(hf)x = hf$ whenever hf is defined in $\mathbf{A}$. Since, by assumption, $id_A x$ is undefined, $\mathbf{B}$ is weakly identitive. $\mathbf{B}$ is associative: If $\alpha\beta$ and $\beta\gamma$ are defined in $\mathbf{B}$, α and β belong to $\mathbf{A}$. If $\gamma = x$, we have $\beta = hf$, where h is in $\mathbf{A}$; thus $\alpha(\beta\gamma) = \alpha hf = (\alpha hf)x = (\alpha\beta)\gamma$. The other case is trivial.

Thus, $\mathbf{B}$ is a semicategory, and by Corollary 2.4, the obvious functor $F : \mathbf{A} \to \mathbf{B}^*$ is an embedding. However, f is not F-initial: We have $Ff[(x)] = [(f, x)] = [(f)] = Ff$; but $[(x)]$ is not in the image of F, since $R_{\mathbf{B}}$ is faithful.

(iii)⇒(i): Trivial.

The notion of initiality can be extended to arbitrary functors:

DEFINITION 3.3. *Let $\mathbf{A}$ be a category, $F : \mathbf{A} \to \mathbf{B}$ a functor and $f : A \to B$ a morphism in $\mathbf{A}$. We say that f is* initial *w.r.t. F if, whenever we have $g : C \to B$ and $h : FC \to FA$ with $(Ff)h = Fg$, there exists a unique $\bar{h} : C \to A$ with $f\bar{h} = g$ and $F\bar{h} = h$.*

For faithful functors, this definition coincides with the one already given.

For each category $\mathbf{A}$, we denote the unique functor from $\mathbf{A}$ to the terminal category by $T_{\mathbf{A}}$. The "extended" version of the above theorem is trivial:

PROPOSITION 3.4. *Let $f : A \to B$ be a morphism in a category $\mathbf{A}$. Equivalent are*

(i) f is initial w.r.t. all functors;

(ii) f is initial w.r.t. $T_{\mathbf{A}}$;

(iii) f is an isomorphism.

Proof. (i)⇒(ii): Trivial.

(ii)⇒(iii): Since $T_{\mathbf{A}} fid = T_{\mathbf{A}} id_B$, there exists g with $fg = id_B$. By the uniqueness requirement in the definition of initiality, f is a monomorphism; thus, f is an isomorphism.

(iii)⇒(i): Trivial.

Next we generalize and extend a characterization of absolute monomorphisms, which can in part be extracted from [12], theorem 3.2. (including proof) and proposition 1.1.. The generalized statement is concerned with the following preorder on the class of morphisms of a category:

DEFINITION 3.5. *Let f and g be morphisms of a category; f is said to be* more monic than g *if, for any pair of morphisms (r, s), $fr = fs$ implies $gr = gs$.*

In [7], this preorder is denoted by $f \Rightarrow g$. Obviously, $f \Rightarrow g$ implies that the domains of f and g coincide. Observe that f is a monomorphism iff f is more monic than the identity.

THEOREM 3.6. *Let $\mathbf{A}$ be a category and let $f : A \to B$ and g be morphisms in $\mathbf{A}$. Equivalent are:*

(i) f is absolutely more monic *than g (i.e. $Ff \Rightarrow Fg$ for all functors F with domain $\mathbf{A}$);*

(ii) $Ef \Rightarrow Eg$ for all full embeddings E with domain $\mathbf{A}$;

(iii) $Ef \Rightarrow Eg$ for all object-full embeddings E with domain $\mathbf{A}$;

(iv) g factors through f (i.e. there exists h such that $g = hf$).

Proof. (i)⇒(ii),(iii): Trivial.

(ii)⇒(iv): Assume w.l.o.g. that f is not a section (otherwise, $mf = id$ implies $g = gmf$). Extend $\mathbf{A}$ to a composition graph $\mathbf{B}$ by adding an object X, a pair of morphisms $x, y : X \to A$ and, for each $\mathbf{A}$-morphism $h : A \to C$ that factors through f in the sense indicated above, a distinct morphism $z_h : X \to C$; for all such h define $kz_h = z_{kh}$, whenever kh is defined in $\mathbf{A}$, and $hx = hy = z_h$. $\mathbf{B}$ is weakly identitive, since $id_B z_h = z_{id_B h} = z_h$ and since $id_A x$ and $id_A y$ are undefined by assumption on f. As in the proof of theorem 3.2, one easily checks that $\mathbf{B}$ is associative, i.e. a semicategory.

Thus, the obvious functor $F : \mathbf{A} \to \mathbf{B}^*$ is an embedding. All morphisms in $\mathbf{B}^*$ that do not belong to $\mathbf{A}$ have domain X, i.e. F is full; thus, $Ff \Rightarrow Fg$ by assumption. By construction, we have $Ff[(x)] = Ff[(y)]$, hence also $Fg[(x)] = Fg[(y)]$, i.e. the words (g, x) and (g, y) are equivalent. By theorem 2.3, this implies that gx is defined, which means that g factors through f.

(iii)⇒(iv): Analogous to part (ii)⇒(iv); this time, however, extend $\mathbf{A}$ by a single morphism $x : A \to A$ as carried out in the proof of theorem 3.2, part (ii)⇒(iii).

(iv)⇒(i): Trivial.

COROLLARY 3.7. *A monomorphism is absolute, respectively preserved by all (object-) full embeddings, iff it is a section.*

Note the coincidence of the corollary with theorem 3.2.

Finally we characterize "potential sections" (cf. also [1], 7P.), using the obvious extension mentioned in the introduction:

THEOREM 3.8. *Let $\mathbf{A}$ be a category and S a set of morphisms in $\mathbf{A}$. Equivalent are*

(i) S consists of monomorphisms;

(ii) there exists a category $\mathbf{B}$ and an embedding $E : \mathbf{A} \to \mathbf{B}$ such that Es is a section for each $s \in S$.

Proof. (ii)⇒(i): Trivial.

(i)⇒(ii): Assume w.l.o.g. that S does not contain any retractions (i.e. isomorphisms). Extend $\mathbf{A}$ to a composition graph $\mathbf{C}$ by adding (pairwise distinct) new morphisms $t_s : B \to A$ for all $s : A \to B$ in S and defining $t_s(sf) = f$ whenever sf is defined in $\mathbf{A}$; since s is monic, $t_s(sf)$ is welldefined. By assumption, $t_s id_B$ is undefined; thus, $\mathbf{C}$ is weakly identitive. $\mathbf{C}$ is associative: If $\alpha\beta$ and $\beta\gamma$ are defined in $\mathbf{C}$, we can assume w.l.o.g. that $\alpha = t_s$ for some $s \in S$, $\beta = sf$ for some f in $\mathbf{A}$ and γ belongs to $\mathbf{A}$; thus $(\alpha\beta)\gamma = f\gamma = t_s(sf\gamma) = \alpha(\beta\gamma)$.

Now by corollary 2.4, the obvious functor $E : \mathbf{A} \to \mathbf{C}^*$ is an embedding, and by construction, Es is a section for each $s \in S$.

REMARK 3.9. *The embedding constructed above preserves monomorphisms: Let m be a monomorphism in $\mathbf{A}$, and let $\mathbf{f} = (f_n, \ldots, f_1)$, $\mathbf{g} = (g_k, \ldots, g_1)$ be words in $\mathbf{W}(\mathbf{C})$ such that $(m)\mathbf{f} \cong (m)\mathbf{g}$. Normalize so that f_n, but not f_{n-1}, belongs to $\mathbf{A}$ and $(f_{n-1}, \ldots, f_1)$ is minimal, correspondingly for $\mathbf{g}$. Then we have $(mf_n, f_{n-1}, \ldots, f_1) \cong (mg_k, g_{k-1}, \ldots, g_1)$; assuming w.l.o.g. that m is not a retraction, these words are both minimal, hence equal. Since m is monic, it follows that $\mathbf{f} = \mathbf{g}$.*

The same is not true about epimorphisms: For example, let $s \in S$ be an epimorphism; we have $(s, t_s, s) \cong (s)$, but (s, t_s) and $()$ are both minimal, hence inequivalent.

REMARK 3.10. *In a similar manner, one can add factorizations* $ts = r$ *iff* $s \Rightarrow r$*. Concerning the proof, note that we can again assume that* s *is not a retraction, since* $sf = id$ *implies* $sfs = s$ *and thus* $rfs = r$*, i.e. the desired factoriziation already exists (and since* s *is an absolute epimorphism, we cannot freely add another one). Furthermore, one checks as in the above remark that the resulting embedding preserves the relation* $\Rightarrow$*.*

REMARK 3.11. *For an arbitrary set* S *of morphisms in a category* $\mathbf{A}$*, one can construct a functor which is universal for the property of mapping* S *to a set of sections. This is done by freely adding left inverses as in the construction above, requiring only* $ts = id$ *to be defined in the intermediate composition graph; in general, the resulting functor* F *is not an embedding. Theorem 3.8 implies that the codomain restriction of* F *to its image is universal for the property of mapping* S *to a set of monomorphisms.*

REMARK 3.12. *The constructions carried out in the proofs of theorem 3.6 and theorem 3.8 can be regarded as adjunctions of a "free coequalized pair" and "free retractions", respectively. Ideas of this type can be applied to similar problems. For example, the proof of the characterization theorem for absolute weak colimits given in [12] (theorem 3.2.) can be interpreted as the adjunction of a "free natural sink"; the corresponding construction can be made explicit as exemplified here (but does not involve use of theorem 2.3).*

4. Sources and Cones

We call families of morphisms with common domain *sources*. A source $\mathcal{S} = (f_i : A \to A_i)_I$ in a category $\mathbf{A}$ is called *initial* w.r.t. a faithful functor $F : \mathbf{A} \to \mathbf{B}$ if, whenever $(g_i : B \to A_i)_I$ is a source in $\mathbf{A}$ and h is a morphism in $\mathbf{B}$ such that $(Ff_i)h = Fg_i$ for all $i \in I$, then there exists a morphism $\bar{h} : B \to A$ in $\mathbf{A}$ such that $F\bar{h} = h$. The notions of absolute initiality and initiality w.r.t. an arbitrary functor are generalized analogously.

The proofs of the above results on absolutely initial morphisms, absolute monomorphisms and the like are easily adapted to sources; one obtains the following statements:

DEFINITION 4.1. *Let* $\mathcal{S}$ *and* $\mathcal{T}$ *be sources in a category* $\mathbf{A}$*; we call* $\mathcal{S}$ more monic than $\mathcal{T}$ *(*$\mathcal{S} \Rightarrow \mathcal{T}$*) if* $\mathcal{S}f = \mathcal{S}g$ *implies* $\mathcal{T}f = \mathcal{T}g$ *for each pair of morphisms* (f, g)*.*

THEOREM 4.2. *Let* $\mathbf{A}$ *be a category and let* $\mathcal{S} = (f_i : A \to B_i)_I$ *and* $\mathcal{T} = (g_j : A \to C_j)_J$ *be sources in* $\mathbf{A}$*. Equivalent are*

(i) $\mathcal{S}$ *is absolutely more monic than* $\mathcal{T}$*;*

(ii) $E\mathcal{S} \Rightarrow E\mathcal{T}$ *for all full embeddings* E *with domain* $\mathbf{A}$*;*

(iii) $E\mathcal{S} \Rightarrow E\mathcal{T}$ *for all object-full embeddings* E *with domain* $\mathbf{A}$*;*

(iv) $\mathcal{T}$ *factors through* $\mathcal{S}$ *(i.e. for every* $j \in J$ *there exists* $i \in I$ *such that* g_j *factors as* $g_j = hf_i$*).*

Observing that $\mathcal{S}$ is a monosource iff $\mathcal{S}$ is more monic than the identity, one has

COROLLARY 4.3. *A monosource is absolute, respectively preserved by all (object-) full embeddings, iff it contains a section.*

REMARK 4.4. *Stretching terminology a bit, theorem 3.8 can be restated as "all monomorphisms are potentially absolute". By way of contrast, the above corollary and theorem 3.8 can be put together to yield that a monosource is "potentially absolute" iff it contains a monomorphism; corresponding statements hold for the relation "more monic than". At this point, this type of game necessarily stops, since for any property that is reflected by embeddings, the notions "absolute" and "absolutely potentially absolute" obviously coincide.*

THEOREM 4.5. *Let* $\mathbf{A}$ *be a category and* $\mathcal{S}$ *a source in* $\mathbf{A}$*. Equivalent are*

(i) $\mathcal{S}$ *is absolutely initial;*

(ii) $\mathcal{S}$ *is initial with respect to all object-full embeddings;*

(iii) $\mathcal{S}$ *contains a section.*

(Recall that the meaning of the word "absolute" in this theorem is different from the one in the preceding statement.) There is a slight change in the picture when we shift attention to initiality w.r.t. arbitrary functors:

PROPOSITION 4.6. *Let* $\mathcal{S}$ *be a source in a category* $\mathbf{A}$*.* $\mathcal{S}$ *is initial w.r.t.* $T_{\mathbf{A}}$ *iff* $\mathcal{S}$ *is a product.*

Proof. Straightforward.

COROLLARY 4.7. *Let* $\mathbf{A}$ *be a category and let* $\mathcal{S} = (f_i : A \to A_i)_I$ *be a source in* $\mathbf{A}$*. Equivalent are* *(i) $\mathcal{S}$ is initial with respect to all functors;*

(ii) $\mathcal{S}$ is initial with respect to $T_{\mathbf{A}}$ and all embeddings;

(iii) $\mathcal{S}$ is a product and contains an isomorphism.

Proof. (i)⇒(ii): Trivial.

(ii)⇒(iii): By proposition 4.6 and the above theorem, $\mathcal{S}$ is a product and contains a section, i.e. there exist $i_0 \in I$ and g such that $gf_{i_0} = id_A$. Define a source $(h_i : A_{i_0} \to A_i)_I$ by

$$h_i = \begin{cases} id_{A_{i_0}}, & \text{if } i = i_0, \\ f_i g & \text{otherwise.} \end{cases}$$

By the product property, we have h such that $f_i h = h_i$ for all i; thus, f_{i_0} is a retraction, i.e. an isomorphism.

(iii)⇒(i): Let $F : \mathbf{A} \to \mathbf{B}$ be a functor, $(g_i : B \to A_i)_I$ a source in $\mathbf{A}$ and $h : FB \to FA$ such that $(Ff_i)h = Fg_i$ for all $i \in I$. By the product property, there exists a unique g with $f_i g = g_i$ for all i. Since $\mathcal{S}$ contains a section, the source $(Ff_i)_I$ is monic; hence $Fg = h$.

REMARK 4.8. *The notion of initiality can be extended to* natural sources *or* cones *in a category $\mathbf{A}$, i.e. natural transformations $\mu : A \to D$ from the constant functor associated with an object A of $\mathbf{A}$ to a diagram $D : \mathbf{I} \to \mathbf{A}$: a cone $\mu : A \to D$ is called* initial *with respect to a functor $F : \mathbf{A} \to \mathbf{B}$ if, whenever $\nu : B \to D$ is a cone and f is a morphism in $\mathbf{B}$ such that $F\mu f = F\nu$, then there exists a unique morphism $\bar{f} : B \to A$ with $\mu\bar{f} = \nu$ and $F\bar{f} = f$. If F is faithful, initiality of μ is equivalent to initiality of the associated source $(\mu_i : A \to D(i))_{i \in \mathrm{Ob}\mathbf{I}}$; furthermore, one easily checks that a cone in $\mathbf{A}$ is initial w.r.t. $T_{\mathbf{A}}$ iff it is a limit cone.*

The adaptation of the above corollary to initiality of cones leads to a characterization that, unlike the one for sources, includes cases that are neither trivial nor grossly unfamiliar: A cone is initial w.r.t. all functors iff it is a limit cone and contains a section; this holds e.g. for all nonempty equalizers in the category of sets. This observation correlates with the fact that all absolute limits are connected (cf. [12], 5.); note that absolute limits are initial w.r.t. all functors.

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