# A Semantic PSPACE Criterion for the Next 700 Rank-0-1 Modal Logics

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## Abstract

*Upper complexity bounds for modal logics are often a complex issue treated with a wide range of frequently ad-hoc techniques. As domain-specific modal logics (often non-normal) abound in the literature and new ones appear at regular intervals, it is therefore desirable to develop a generic algorithmic framework for deriving such bounds systematically. Here, we present a semantics-based criterion for modal logics whose axioms do not nest modal operators to be decidable in* $PSPACE$*, typically a tight upper bound; the generality of our approach is based on a coalgebraic semantics. This result complements an earlier tableau-based method and extends the class of logics covered by generic techniques. In some cases, e.g. conditional logics, the semantic criterion is established very easily, and in others, notably various logics of quantitative uncertainty, it follows by dissecting off-the-shelf results. Thus, our method allows establishing* $PSPACE$*-completeness of a wide range of logics with only moderate effort.*

## 1. Introduction

One of the attractive aspects of modal logic is that purpose-built modal logics combine the right level of expressivity with decidability, often in low complexity classes. Modal logics without fixed point operators tend to be decidable in $PSPACE$, i.e. are not dramatically harder than propositional logic. In particular, Ladner showed in seminal work that various normal modal logics including $K$, $KD$, and $T$ are $PSPACE$-complete, with the upper bound proved using shallow tree models [17]. Slightly uncharacteristically from the point of view of complexity theory (but fairly typically for the complexity analysis of logics), lower $PSPACE$ bounds for modal logics are frequently obtained more or less for free from Ladner's initial results, namely by embedding a known $PSPACE$-hard logic such as $K$ or $KD$. Contrastingly, establishing upper bounds often remains a non-trivial task. Instances of the latter problem, in particular non-normal cases, have received much attention in the recent literature; we list only a few examples:

- A $PSPACE$ upper bound for graded modal logic [8] is obtained using a constraint set algorithm in [31]. This corrects a previously published incorrect algorithm and refutes a previous $EXPTIME$ hardness conjecture.
- More recently, a $PSPACE$ upper bound for Presburger modal logic (which contains both graded modal logic and majority logic [22]) has been established using a Ladner-type algorithm [5] .
- Using a variant of a shallow neighbourhood frame construction from [32], a $PSPACE$ upper bound for coalition logic is established in [25].
- $PSPACE$ upper bounds for $CK$ and related conditional logics [3] are obtained in [21, 20] by a detailed proof-theoretic analysis of a labelled sequent calculus.

These examples motivate the search for broadly applicable generic algorithms for modal logics. We aim in particular at genericity w.r.t. the branching type of the systems underlying the semantics — while normal modal logics are usually interpreted over Kripke frames, the semantics of non-normal modal logics involves a wide variety of system types, e.g. probabilistic systems [18, 14, 6], frames with ordered branching [5], game frames [25], or conditional frames [3]. We achieve this degree of generality by working in the framework of *coalgebraic modal logic* [24], which encapsulates the branching type in the choice of a signature functor whose coalgebras play the role of models.

The class of modal logics interpreted over the class of *all* coalgebras for a given signature functor, in analogy to the interpretation of $K$ over all Kripke frames, coincides precisely with the class of of *rank-1 logics* [28, 30], where a (possibly polyadic) modal logic has rank 1 if its axioms are propositional combinations of atoms containing exactly one modal operator (thus including the normal modal logics $K$ and $KD$, but excluding e.g. $T$ and $S4$). Existing generic decidability and completeness results for coalgebraic modal logic [23, 4, 28, 29] have been limited to this class. In the present work, we restrict most of the exposition to rank-1 logics as well, but conclude with a generalization of our method to *rank-0-1 logics*, defined by arbitrary axioms that avoid nesting of modal operators (thus including e.g. the normal modal logic $T$, but still excluding $S4$).

Already the class of rank-1 logics, while certainly restrictive, covers a large number of important logics, including besides the above-mentioned examples (graded modal logic, majority logic, Presburger modal logic, coalition logic, $CK$) e.g. probabilistic modal logics [18, 14, 6], the conditional logic $CK + ID$ [3], and various deontic logics (e.g. [9]). The seemingly minute generalization step to rank-0-1 logics is not only technically non-trivial, but also further extends the range of candidates for our method quite substantially. E.g. the class of rank-0-1 logics includes, besides $T$, all 16 conditional logics covered in [20] (of which only 4 are rank-1), in particular $CK + MP$ [3], and Elgesem's logic of agency (discussed e.g. in [10]).

A tableau-based generic $PSPACE$-algorithm for rank-1 logics is described in [29]. Here, we complement this result by a semantics-based method for rank-0-1 logics which relies on a small model property at the single-step level, the *one-step polysize model property (OSPMP)*. This property leads to a $PSPACE$ algorithm which traverses a polynomially branching shallow model, thus providing an alternative route to $PSPACE$ upper bounds which, unlike the tableau method, does not require tractable axiomatizations. A further consequence is an $NP$ upper bound for satisfiability of formulas with bounded nesting of modal operators, generalizing a result for $K$ and $T$ from [11] (where similar results are also proved for various other normal modal logics).

The OSPMP is, in some cases, very easy to establish. E.g. in order to reprove Ladner's $PSPACE$ upper bound for $K$, one essentially just has to note that constructing a set that meets $n$ given sets requires at most $n$ elements; the case of the conditional logics $CK$, $CK{+}ID$, and $CK{+}MP$ is similarly simple. In other cases, the OSPMP can be obtained by sharpening off-the-shelf results. This holds in particular for various logics of quantitative uncertainty. Such logics, concerned e.g. with probability, upper probability, Dempster-Shafer belief, possibility, and expectation, come in two flavours: a *one-step* version without nesting of modal operators, typically $NP$-complete, and a 'properly' *modal* version with nested modalities, typically $PSPACE$-complete. E.g. the logic of probability was first introduced as a one-step logic [7] and later extended to a modal logic in the above sense [6]. Using material from proofs of $NP$ upper bounds for the one-step versions, we obtain tight $PSPACE$ upper bounds for the modal versions, some of which are, to our knowledge, novel.

The material is organized as follows. We recall the basic definitions of coalgebraic modal logic in Section 2. The $PSPACE$ criterion for rank-1 logics is presented in Section 3, and illustrated by means of extensive example applications. The generalization to rank-0-1 is covered in Section 4, along with corresponding additional examples. We refer to [19] for unexplained categorical terminology.

## 2. Coalgebraic Modal Logic

We recapitulate the basics of the coalgebraic interpretation of modal logic, and discuss a number of examples.

A *modal signature* $\Lambda$ is a set of modal operators with associated finite arity. The signature $\Lambda$ determines two languages: firstly, the *one-step logic* of $\Lambda$, whose formulas $\psi, \dots$ (the *one-step formulas*) over a set $V$ of propositional variables are defined be the grammar

$$\psi ::= \bot \mid \psi_1 \wedge \psi_2 \mid \neg\psi \mid L(\phi_1, \dots, \phi_n),$$

where $L \in \Lambda$ is $n$-ary and the $\phi_i$ are propositional formulas over $V$; and secondly, the *modal logic* of $\Lambda$, whose set $\mathcal{F}(\Lambda)$ of $\Lambda$*-formulas* $\psi, \dots$ over $V$ is defined by the grammar

$$\psi ::= a \mid \bot \mid \psi_1 \wedge \psi_2 \mid \neg\psi \mid L(\psi_1, \dots, \psi_n),$$

where $a \in V$. Thus, the one-step logic excludes top-level variables and nesting of modal operators; e.g. the $K$-axiom $\Box(a \to b) \to \Box a \to \Box b$ is a one-step formula, but not the axioms $T$ ($\Box a \to a$) and 4 ($\Box a \to \Box\Box a$). The boolean operations $\vee, \to, \leftrightarrow, \top$ are defined as usual. The *rank* $\mathrm{rank}(\phi)$ of $\phi \in \mathcal{F}(\Lambda)$ is the maximal nesting depth of modal operators in $\phi$ (note however that the notion of rank-1 logic defined below is stricter than suggested by this definition, as it excludes top-level propositional variables in axioms; the latter are allowed only in rank-0-1 logics). We denote by $\mathcal{F}_n(\Lambda)$ the set of formulas of rank at most $n$; we refer to the languages $\mathcal{F}_n(\Lambda)$ as *bounded-rank fragments*.

For our purposes, the modal logic is the actual object of study, while the one-step logic is a technical tool; however, one-step logics also appear as logics of independent interest in the literature [7, 12, 13]. One of the central ideas of coalgebraic modal logic is that properties of the full modal logic, such as soundness, completeness, and decidability, can be reduced to properties of the much simpler one-step logic. This is also the spirit of the present work, whose core is a construction of polynomially branching shallow models for the modal logic assuming a small model property for the one-step logic.

The semantics of both the one-step logic and the modal logic of $\Lambda$ are parametrized coalgebraically, as follows.

**Definition 2.1.** Let $T : \mathsf{Set} \to \mathsf{Set}$ be a functor, referred to as the *signature functor*, where $\mathsf{Set}$ is the category of sets. A $T$*-coalgebra* $A = (X, \xi)$ consists of a set $X$ of *states* and a *transition* function $\xi : X \to TX$. A $\mathcal{P}X$*-valuation* $\pi$ for a set $V$ of propositional variables assigns to each $a \in V$ a set $\pi(a) \subseteq X$ of states satisfying $a$. A $T$*-model* $M = (A, \pi)$ over $V$ consists of a $T$-coalgebra $A = (X, \xi)$ and a $\mathcal{P}X$-valuation $\pi$ for $V$.

**Remark 2.2.** A $T$-model over $V$ may be regarded as a coalgebra for the functor $T_V$ defined by $T_V X = TX \times \mathcal{P}V$, so that propositional variables in the modal logic could be subjected to Occam's razor. We have chosen to treat them explicitly in order to facilitate presentation of the examples.

We view coalgebras as generalized transition systems: the transition function maps a state to a structured set of successors and observations, with the precise nature of the structure determined by the signature functor. Thus, the signature functor encapsulates the branching type of the transition systems playing the role of frames in the semantics.

**Assumption 2.3.** We can assume w.l.o.g. that $T$ preserves injective maps [1]. For ease of notation, we will in fact sometimes assume that $TX \subseteq TY$ in case $X \subseteq Y$. Moreover, $T$ is w.l.o.g. *non-trivial*, i.e. $TX = \emptyset \implies X = \emptyset$.

Coalgebraic modal logic in the form employed here abstractly captures the interpretation of modal operators using predicate liftings [24], generalizing earlier approaches [15, 26, 16]. Here, we use polyadic predicate liftings [27].

**Definition 2.4.** An *$n$-ary predicate lifting* ($n \in \mathbb{N}$) for a functor $T$ is a natural transformation

$$\lambda : \mathcal{Q}^n \to \mathcal{Q} \circ T^{op},$$

where $\mathcal{Q}$ denotes the contravariant powerset functor $\mathsf{Set}^{op} \to \mathsf{Set}$ (i.e. $\mathcal{Q}X$ is the powerset $\mathcal{P}X$, and $\mathcal{Q}f(A) = f^{-1}[A]$), and $\mathcal{Q}^n$ refers to its $n$-fold cartesian product given by $\mathcal{Q}^n X = (\mathcal{Q}X)^n$.

A coalgebraic semantics for $\Lambda$, shortly a *coalgebraic modal logic*, is formally defined as a *$\Lambda$-structure $\mathcal{M}$* (*over $T$*) consisting of a signature functor $T$ and an assignment of an $n$-ary predicate lifting $\mathcal{M}[\![L]\!]$ for $T$ to every $n$-ary modal operator $L \in \Lambda$. We omit the mention of $\mathcal{M}$ when this is unlikely to cause confusion. The semantics of the modal language $\mathcal{F}(\Lambda)$ is then given in terms of a satisfaction relation $\models_M$ between states $x$ of $T$-models $M = ((X, \xi), \pi)$ and $\mathcal{F}(\Lambda)$-formulas over $V$. The relation $\models_M$ is defined inductively, with the usual clauses for variables and boolean operations. The clause for an $n$-ary modal operator $L$ is

$$x \models_M L(\phi_1, \dots, \phi_n) \;\Leftrightarrow\; \xi(x) \in [\![L]\!]([\![\phi_1]\!]_M, \dots, [\![\phi_n]\!]_M)$$

where $[\![\phi]\!]_M = \{x \in X \mid x \models_M \phi\}$. We drop the subscripts $M$ when $M$ is clear from the context. Our main interest is in the (local) *satisfiability problem* over $\mathcal{M}$:

**Definition 2.5.** An $\mathcal{F}(\Lambda)$-formula $\phi$ is *satisfiable* if there exist a $T$-model $M$ and a state $x$ in $M$ such that $x \models_M \phi$. Moreover, $\phi$ is *valid* if $x \models_M \phi$ for all $M, x$.

Contrastingly, the semantics of the one-step logic is given in terms of satisfaction relations $\models^1_{X,\tau}$ between elements $t \in TX$ and one-step formulas over $V$, where $X$ is a set and $\tau$ is a $\mathcal{P}X$-valuation for $V$. The valuation $\tau$ canonically induces an interpretation of $[\![\phi]\!]^0\tau \subseteq X$ of propositional formulas $\phi$ over $V$. We write $X, \tau \models^0 \phi$ if $[\![\phi]\!]^0\tau = X$. The relation $\models^1_{X,\tau}$ is then defined by the usual clauses for boolean operators and

$$t \models^1_{X,\tau} L(\phi_1, \dots, \phi_n) \;\Leftrightarrow\; t \in [\![L]\!]([\![\phi_1]\!]^0\tau, \dots, [\![\phi_n]\!]^0\tau).$$

Note in particular that the semantics of the one-step logic does not involve a notion of state transition. For a one-step formula $\psi$, we put $[\![\psi]\!]^1\tau = \{t \in TX \mid t \models^1_{X,\tau} \psi\}$.

**Definition 2.6.** A *one-step model* $(X, \tau, t)$ over $V$ consists of a set $X$, a $\mathcal{P}X$-valuation $\tau$ for $V$, and $t \in TX$. For a one-step formula $\psi$ over $V$, $(X, \tau, t)$ is a *one-step model of* $\psi$ if $t \models^1_{X,\tau} \psi$. In this case, $\psi$ is *(one-step) satisfiable*.

**Remark 2.7.** We say that a modal logic is of *rank* 1 if it is axiomatisable by one-step formulas. It turns out that the rank-1 modal logics are precisely the coalgebraic modal logics in the above sense [28, 30]. This is due to the interpretation of coalgebraic modal logic over the class of *all* $T$-coalgebras, and thus is by no means an inherent limitation of the coalgebraic approach: in order to interpret modal logics of arbitrary rank over a functor $T$, one considers covarieties of $T$-coalgebras. Research on generic methods for such logics is under way. A first step in this direction is the generalization of our method to rank-0-1 logics in Section 4.

*For the remainder of this section and the next, we fix a $\Lambda$-structure $\mathcal{M}$ over $T$.* We recall some basic notions from propositional logic, as well as notation for coalgebraic modal logic following [4]:

**Definition 2.8.** We denote the set of propositional formulas over a set $S$, generated by the basic connectives $\neg$ and $\wedge$, by $\mathsf{Prop}(S)$. We use variables $\epsilon$ etc. to denote either nothing or $\neg$. Thus, a *literal* over $S$ is a formula of the form $\epsilon a$, with $a \in S$. A *(conjunctive) clause* is a finite, possibly empty, disjunction (conjunction) of literals. The set of all (conjunctive) clauses over $S$ is denoted by $\mathsf{Cl}(S)$ ($\mathsf{cCl}(S)$). We denote by $\mathsf{Up}_\Lambda(S)$ the set $\{L(a_1, \dots, a_n) \mid L \in \Lambda \text{ } n\text{-ary}, a_1, \dots, a_n \in S\}$.

Given a set $V$ of propositional variables, a *$Z$-substitution* is a substitution $\sigma$ of the propositional variables $a \in V$ by elements $\sigma(a)$ of a set $Z$; we denote the result of applying $\sigma$ to a formula $\phi$ over $V$ by $\phi\sigma$.

Note that in the above notation, the set of one-step formulas over $V$ is $\mathsf{Prop}(\mathsf{Up}_\Lambda(\mathsf{Prop}(V)))$. The one-step logic may alternatively be presented in terms of pairs of formulas separating out the lower propositional layer:

**Definition 2.9.** A *one-step pair* $(\phi, \psi)$ over $V$ consists of formulas $\psi \in \mathsf{cCl}(\mathsf{Up}_\Lambda(V))$ and $\phi \in \mathsf{Prop}(V)$. A one-step model $(X, \tau, t)$ is a *one-step model of* $(\phi, \psi)$ if $X, \tau \models^0 \phi$ and $t \models^1_{X,\tau} \psi$. In this case, we write $(X, \tau, t) \models^1 (\phi, \psi)$, and we say that $(\phi, \psi)$ is *(one-step) satisfiable*.

Essentially, one-step satisfiable pairs correspond, via negation of the conclusion, to unsound one-step rules in the sense of [28]. In analogy to the equivalence between axioms and one-step rules described in [28], one-step pairs and one-step formulas may replace each other for purposes of satisfiability:

**Proposition 2.10.** *For every one-step pair* $(\phi, \psi)$ *over* $V$ *with* $\phi$ *satisfiable, there exists a* $\mathsf{Prop}(V)$*-substitution* $\sigma$ *such that the one-step formula* $\psi\sigma$ *is equivalent to* $(\phi, \psi)$ *in the sense that if* $t \models^1_{X,\tau} \psi\sigma$ *then* $(X, \sigma\tau, t) \models^1 (\phi, \psi)$*, and if* $(X, \tau, t) \models^1 (\phi, \psi)$ *then* $t \models^1_{X,\tau} \psi\sigma$ *(and* $\sigma\tau = \tau$*). Here,* $\sigma\tau$ *denotes the* $\mathcal{P}X$*-valuation taking* $a$ *to* $[\![\sigma(a)]\!]^0\tau$*.*

*Conversely, we have, for* $\psi \in \mathsf{cCl}(\mathsf{Up}_\Lambda(\mathsf{Prop}(V)))$*, an equivalent one-step pair* $(\phi, \psi_1)$ *over* $V \cup W$*, where* $\psi$ *decomposes as* $\psi \equiv \psi_1\sigma$*, with* $\psi_1 \in \mathsf{cCl}(\mathsf{Up}_\Lambda(W))$*,* $\sigma$ *a* $\mathsf{Prop}(V)$*-substitution, and* $V \cap W = \emptyset$*, and where* $\phi$ *is the conjunction of the formulas* $a \leftrightarrow \sigma(a)$*,* $a \in W$*. Here, restricting valuations to* $V$ *induces a bijection between one-step models of* $(\phi, \psi_1)$ *and one-step models of* $\psi$*.*

The above fact is crucial for the extraction of the *PSPACE*-criterion from the proofs of small model results in the literature (cf. Examples 3.15.4–6 below).

The coalgebraic approach subsumes many interesting modal logics, including e.g. graded and probabilistic modal logics and coalition logic [29]. Below, we present the most basic example, the modal logic $K$, as well as various conditional logics and logics of quantitative uncertainty.

**Example 2.11.** 1. *Modal logic* $K$*:* Let $\Lambda = \{\Box\}$, with $\Box$ a unary modal operator. We define a $\Lambda$-structure over the covariant powerset functor $\mathcal{P}$ (i.e. $\mathcal{P}X$ is powerset, and $\mathcal{P}f(A) = f[A]$) by putting $[\![\Box]\!]_X(A) = \{B \in \mathcal{P}X \mid B \subseteq A\}$. Naturality of $[\![\Box]\!]$ is just the equivalence $f[B] \subseteq A \iff B \subseteq f^{-1}[A]$.

$\mathcal{P}$-coalgebras are Kripke frames, and $\mathcal{P}$-models are Kripke models. The modal logic of $\Lambda$ is precisely the modal logic $K$, equipped with its standard Kripke semantics. Contrastingly, a one-step formula over $V$ is a propositional combination of atoms of the form $\Box\phi$, where $\phi \in \mathsf{Prop}(V)$. For $A \in \mathcal{P}X$, we have $A \models^1_{X,\tau} \Box\phi$ iff $A \subseteq [\![\phi]\!]^0\tau$. One easily checks that the one-step logic is *NP*-complete (see also below), while the modal logic $K$ is *PSPACE*-complete [17].

2. *Conditional logic* $CK$*:* Let $T$ be the functor given by $TX = (\mathcal{Q}X \to \mathcal{P}X)$, with $A \to B$ denoting the set of functions from $A$ to $B$ (and $\mathcal{Q}$ contravariant powerset, cf. Definition 2.9). Then $T$-coalgebras are *standard conditional models* [3]. The signature of conditional logic has a single binary modal operator $\Rightarrow$, interpreted over $T$ by

$$[\![\Rightarrow]\!]_X(A, B) = \{f : \mathcal{Q}X \to \mathcal{P}X \mid f(A) \subseteq B\}.$$

The modal logic of this structure is the conditional logic $CK$ [3]. Formulas $\Rightarrow (\phi, \psi)$, usually written $\phi \Rightarrow \psi$, are non-monotonic conditionals.

3. *Conditional logic* $CK + ID$*:* An important extension of $CK$ is the logic $CK + ID$ [3] obtained by imposing the rank 1 axiom

$$(ID) \quad a \Rightarrow a.$$

The semantics of $CK + ID$ is captured coalgebraically by passing from the functor $T$ of the previous example to the subfunctor $T_{ID}$ defined by

$$T_{ID}X = \{f \in TX \mid \forall A \in \mathcal{Q}X.\, f(A) \subseteq A\},$$

with $\Rightarrow$ interpreted analogously as for $CK$.

4. *Modal logic of probability [6]:* Let $D_\omega$ denote the (finite) distribution functor; i.e. $D_\omega X$ is the set of finitely supported probability distributions on $X$, and $D_\omega f$ acts as image measure formation. Coalgebras for $D_\omega$ are probabilistic transition systems (i.e. Markov chains).

The modal logic of probability has $n$-ary modal operators $L(a_1, \dots, a_n; b)$ for $a_1, \dots, a_1, b \in \mathbb{Q}$, whose application to formulas $\phi_1, \dots, \phi_n$ is written

$$\textstyle\sum_{i=1}^n a_i w(\phi_i) \geq b.$$

The terms $w(\phi)$ are called *weights*. Weights are interpreted as probabilities; i.e.

$$[\![L(a_1, \dots, a_n; b)]\!]_X(A_1, \dots, A_n) = \{P \in D_\omega X \mid \textstyle\sum_{i=1}^n a_i P(A_i) \geq b\}.$$

The one-step logic of this structure, i.e. a logic without nesting of weights that talks only about a single probability distribution, is introduced in [7], and the modal logic in [6] (in the context of a logic additionally featuring classical knowledge operators).

5. *Modal logic of upper probability [12]:* This logic has the same signature as the modal logic of probability, but interprets weights as *upper probabilities*: Let $T$ be the composite $\mathcal{P} \circ D_\omega$ of the powerset functor and the finite distribution functor. We put

$$[\![L(a_1, \dots, a_n; b)]\!]_X(A_1, \dots, A_n) = \{\mathfrak{P} \in TX \mid \textstyle\sum_{i=1}^n a_i \mathfrak{P}^*(A_i) \geq b\},$$

where $\mathfrak{P}^*(A)$ denotes the *upper probability* $\mathfrak{P}^*(A) = \sup\,\{PA \mid P \in \mathfrak{P}\}$. From the point of view of knowledge representation, $T$-coalgebras model situations where agents are unsure about probability distributions, e.g. when it is unclear whether a tossed coin is fair. Only the one-step logic of this structure is considered in [12]; however, the modal logic of this structure is of interest as a natural variation of the modal logic of probability.

6. *Modal logic of possibilistic expectation [13]:* The notions of expectation considered in [13] are parametrized over an underlying notion of likelihood, such as probability, upper probability, Dempster-Shafer belief functions, or

Dubois-Prade possibility measures. All these cases can be cast in the coalgebraic framework; here, we focus on possibility measures. Let Ps denote the functor that takes a set $X$ to the set of *possibility measures* on $X$, i.e. functions $p : \mathcal{P}X \to [0,1]$ such that $p(\emptyset) = 0$, $p(X) = 1$, and $p(A \cup B) = \max(p(A), p(B))$ if $A \cap B = \emptyset$. For a function $f : X \to Y$, $\mathsf{Ps}(f)(p)(A) = p(f^{-1}[A])$. Coalgebras $(X, \xi)$ for Ps are transition systems which assign to every state $x \in X$ a possibility measure on $X$ describing the successors of $x$.

A possibility measure $p$ induces an expectation function $E_p$ for *gambles*, i.e. real valued functions $f$ on $X$ with finite range $\{y_1 < \ldots < y_n\}$, defined by

$$E_p(f) = y_1 + \textstyle\sum_{i=1}^{n-1}(y_{i+1} - y_i)p(\{x \mid f(x) > y\}).$$

The modal logic of expectation is concerned with expected values of linear combinations of formulas, regarded as gambles given by the corresponding characteristic functions. It has, for $k, n_1, \ldots, n_k \in \mathbb{N}$ and $a_i, b_{ij}, c \in \mathbb{Z}$ ($i = 1, \ldots, k$, $j = 1, \ldots, n_i$), modal operators $L((a_i); (b_{ij}); c)$ of arity $n = \sum n_k$, whose application to formulas $\phi_{ij}$ is written

$$\textstyle\sum_{i=1}^{k} a_i e(\sum_{j=1}^{n_k} b_{ij}\phi_{ij}) \geq c.$$

The terms $e(\dots)$ represent *expectations*. The interpretation over Ps is given by

$$[\![L(c;(a_i);(b_{ij}))]\!]_X((A_{ij})_{i=1,\ldots,k;j=1,\ldots,n_i}) = \{p \in \mathsf{Ps}(X) \mid \textstyle\sum_{i=1}^{k} a_i E_p(\sum_{j=1}^{n_i} b_{ij}\mathbf{1}_{A_{ij}}) \geq c\},$$

where $\mathbf{1}_A : X \to \{0,1\}$ denotes the characteristic function of $A \subseteq X$. Expectations subsume weights, i.e. $w(\phi)$ can be coded as $e(\phi)$.

Again, only the one-step logics of expectation are explicitly considered in [13]; the modal versions are of interest as natural extensions of the corresponding modal logics involving only weights. The modal logic of expectation admits formulas such as

$$e(1/2(e(u + 3v) \geq 2) + 1/3u) \geq 1/2$$

(where $u$, $v$ are propositional variables), while the one-step logic excludes nested expectations.

**Convention 2.12.** For complexity considerations, we assume that $\Lambda$ is equipped with some reasonable size measure, thus inducing a size measure for $\mathcal{F}(\Lambda)$. For formulas $\phi$ over $V$, we assume w.l.o.g. that $|V| \leq \mathit{size}(\phi)$ (dropping unused variables if necessary). Moreover, we need to be precise about representation sizes of elements of $TX$ for a functor $T$. For finite $X$, we assume given a representation of elements of $TX$ as strings over some finite alphabet, with $\mathit{size}_X(t)$ denoting the size of the representation of $t \in TX$. We do not require that all elements of $TX$ are representable, nor that all strings denote elements of $TX$. We assume, however, that it is decidable in polynomial time whether a given string represents an element of $TX$ (while this assumption is easily invalidated in contrived examples, it does not present a problem in typical applications, and we will mostly omit the discussion of this issue in our examples). We require that representations are well-behaved under inclusions $TX \subseteq TY$ induced according to Assumption 2.3 by inclusions $X \subseteq Y$ into a finite base set $Y$ whose elements are represented in size $\lceil \log |Y| \rceil$ (where $\lceil r \rceil = \min\{n \in \mathbb{N} \mid n \geq r\}$), namely that representable elements of $TX$ remain representable in $TY$, and that

$$\mathit{size}_Y(t) \leq \lceil \log |Y| \rceil \mathit{size}_X(t) \text{ for } t \in TX \subseteq TY. \qquad (*)$$

This assumption is usually unproblematic: we may typically think of elements of $TY$ as terms built over elements of $Y$, so that reinterpreting $t \in TX$ as $t \in TY$ amounts to replacing elements of $X$ by the corresponding elements of $Y$, possibly increasing their size to $\lceil \log |Y| \rceil$.

We make these issues explicit for the above examples:

**Example 2.13.** 1. *Modal logic $K$:* For $X$ finite, elements of $\mathcal{P}X$ are represented as lists. For $X \subseteq Y$, reinterpreting an element of $\mathcal{P}X$ as an element of $\mathcal{P}Y$ amounts to replacing a list over $X$ by the list of the corresponding elements of $Y$, so that Condition $(*)$ of Convention 2.12 holds; analogous arguments work for the remaining examples.

2. *Conditional logic $CK$:* For $X$ finite, elements of $TX = \mathcal{Q}X \to \mathcal{P}X$ are represented as lists of maplets $A \mapsto B$, with $A$ and $B$ represented as lists; such a list represents the map $f$ with $f(A) = B$ if there is a maplet $A \mapsto B$ in the list, and $f(A) = \emptyset$ otherwise. (It is crucial that we do not require explicit images for all $A$, since otherwise, elements of $TX$ would always be of exponential size.)

3. *Conditional logic $CK + ID$:* The representation is as for $CK$; deciding whether a given string represents an element of $T_{ID}X$ essentially amounts to checking a list of subset inclusions, which is clearly in $P$.

4. *Modal logic of probability:* Elements of $D_\omega X$ with all probabilities rational are represented as formal sums $\sum p_i x_i$ (thus denoting the probability distribution assigning to $A$ the probability $\sum_{x_i \in A} p_i$), with the $p_i$ represented in binary. Similarly, rational indices of modal operators $L(a_1, \ldots, a_n; b)$ are assumed to be coded in binary.

5. *Modal logic of upper probability:* The representable elements of $TX = \mathcal{P}D_\omega X$ are the *finite* sets of representable probability distributions, represented as lists.

6. *Modal logic of possibilistic expectation:* For $X$ finite, a possibility measure $p \in \mathsf{Ps}(X)$ is representable if $p(A) \in \mathbb{Q}$ for all $A$. In this case, $p$ is represented in the form $p(A) = \sum_{U \subseteq A} m(U)$ by a *consonant mass function* $m$, i.e. a function $m : \mathcal{P}A \to [0,1] \cap \mathbb{Q}$ such that

$m(\emptyset) = 0$, $\sum_{U \in \mathcal{P}A} m(U) = 1$, and $\{U \subseteq X \mid m(U) > 0\}$ is linearly ordered under inclusion. We represent $m$ as a list of maplets $U \mapsto x$, listing only sets of positive mass. If $p$ is represented by the mass function $m$, then $E_p(f) = \sum_{U \in \mathcal{P}X} m(U) \min_{x \in U} f(x)$ ([13], Lemma A.11).

## 3. Polynomially branching shallow models

We now turn to the announced construction of polynomially branching shallow models for modal logics whose one-step logic has a small model property; this construction leads to a *PSPACE* decision procedure.

**Definition 3.1.** We say that $\mathcal{M}$ has the *one-step polysize model property (OSPMP)* if there exist polynomials $p$ and $q$ such that, whenever a one-step pair $(\phi, \psi)$ is one-step satisfiable, then it has a one-step model $(X, \tau, t)$ such that $|X| \leq p(|\psi|)$ and $t$ is representable with $size(t) \leq q(|\psi|)$.

Due to Proposition 2.10, the OSPMP can also be expressed in terms of one-step formulas:

**Proposition 3.2.** *The $\Lambda$-structure $\mathcal{M}$ has the OSPMP iff there exist polynomials $p$, $q$ such that, whenever $\psi \in \mathsf{cCl}(\mathsf{Up}_\Lambda(\mathsf{Prop}(V)))$ is one-step satisfiable, then $\psi$ has a one-step model $(X, \tau, t)$ with $t$ representable, $|X| \leq p(|\psi_1|)$, and $size(t) \leq q(|\psi_1|)$, where $\psi \equiv \psi_1\sigma$ with $\psi_1 \in \mathsf{cCl}(\mathsf{Up}_\Lambda(V))$ and $\sigma$ a $\mathsf{Prop}(V)$-substitution.*

In both formulations of the OSPMP, we may replace conjunctive clauses by arbitrary propositional formulas.

**Remark 3.3.** While it is to be expected that the construction of polynomially branching models depends on a condition like the OSPMP, it does not seem to be the case that the precise formulation of this condition is implicit in the literature. Note in particular that, in the notation of Proposition 3.2, the polynomial bound depends only on $|\psi_1|$. When we say in the introduction that the OSPMP can be obtained from off-the-shelf results (e.g. [7, 12, 13]), we refer to polynomial-size model theorems in which the polynomial bound depends on $|\psi|$, which may be exponentially bigger than $|\psi_1|$; typically, only an inspection of the given proofs shows that the bound can be sharpened to be polynomial in $|\psi_1|$.

**Remark 3.4.** It is shown in [28] that the one-step logic always has an exponential-size model property: a one-step formula $\psi$ over $V$ is satisfiable iff it has a one-step model of the form $(\mathcal{P}V, \tau, t)$, with $\tau(a) = \{B \in \mathcal{P}V \mid a \in V\}$.

We are now ready to prove the shallow model theorem.

**Definition 3.5.** A *supporting Kripke frame* of a $T$-coalgebra $(X, \xi)$ (or of a $T$-model based on $(X, \xi)$) is a Kripke frame $(X, R)$ such that for each $x \in X$,

$$\xi(x) \in T\{y \mid xRy\} \subseteq TX.$$

A state $x \in X$ is a *loop* if $xRx$.

**Theorem 3.6** (Shallow model property)**.** *If $\mathcal{M}$ has the OSPMP, then $\mathcal{F}(\Lambda)$ has the* polynomially branching shallow model property*: There exist polynomials $p$, $q$ such that every satisfiable $\mathcal{F}(\Lambda)$-formula $\psi$ is satisfiable at the root $x_0$ of a $T$-model $((X, \xi), \pi)$ which has a supporting Kripke frame $(X, R)$ such that $(X, R)$ is, up to possible loops at the leaves, a tree of depth at most* $\mathrm{rank}(\psi)$ *and branching degree at most $p(|\psi|)$, and the size of $\xi(x)$ as an element of $T\{y \mid xRy\}$ is at most $q(|\psi|)$.*

**Definition 3.7.** For $x \in X$ and a $\mathcal{P}X$-valuation $\tau$, we put

$$\mathsf{Th}_\tau(x) \equiv \bigwedge_{x \in \tau(a)} a \wedge \bigwedge_{x \notin \tau(a)} \neg a.$$

*Proof (Theorem 3.6).* Induction over the rank of $\psi$. Let $U$ be the set of propositional variables for $\psi$. If $\mathrm{rank}(\psi) = 0$, i.e. $\psi \in \mathsf{Prop}(U)$, then we let $X$ consist of a single point $x$, we choose a $\mathcal{P}X$-valuation $\pi$ such that $\psi$ is satisfied, and we pick an arbitrary $\xi(x) \in TX$, where $TX \neq \emptyset$ by non-triviality of $T$.

If $\mathrm{rank}(\psi) = n + 1$, then we can assume that the top level of $\psi$ is a conjunctive clause, i.e. that $\psi \in \mathsf{cCl}(\mathsf{Up}_\Lambda(\mathcal{F}_n(\Lambda)) \cup U)$. We thus have a decomposition $\psi \equiv \psi_1\sigma \wedge \psi_0$, where $\psi_1 \in \mathsf{cCl}(\mathsf{Up}_\Lambda(V))$ contains every variable in a suitably chosen set $V$, $V \cap U = \emptyset$, $\sigma$ is an $\mathcal{F}_n(\Lambda)$-substitution, and $\psi_0 \in \mathsf{cCl}(U)$. Let $\phi$ denote the propositional theory of $\sigma$, i.e. the conjunction of all clauses $\chi$ over $V$ such that $\chi\sigma$ is valid. Let $z_0 \models_{((Z,\zeta),\pi_0)} \psi$. Then $(Z, \kappa, \zeta(z_0)) \models^1 (\phi, \psi_1)$, where $\kappa(a) = [\![\sigma(a)]\!]_{((Z,\zeta),\pi_0)}$. By the OSPMP, it follows that $(\phi, \psi_1)$ has a one-step model $(Y, \tau, t)$ of polynomial size in $|\psi_1| \leq |\psi|$.

From this model, we now construct a polynomially branching shallow model $((X, \xi), \pi)$ for $\psi$. To begin, note that $\mathsf{Th}_\tau(y)\sigma$ is satisfiable for every $y \in Y$. For suppose not; then $\neg\mathsf{Th}_\tau(y)\sigma$ is valid, hence $\neg\mathsf{Th}_\tau(y)$ is a conjunct of $\phi$. Thus, $Y, \tau \models^0 \neg\mathsf{Th}_\tau(y)$, in contradiction to the fact that $y \in [\![\mathsf{Th}_\tau(y)]\!]\tau$ by construction. By induction, we thus have, for every $y \in Y$, a shallow model $((X_y, \xi_y), \pi_y)$ of $\mathsf{Th}_\tau(y)\sigma$, where we may assume $y \in X_y$ and $y \models_{((X_y,\xi_y),\pi_y)} \mathsf{Th}_\tau(y)\sigma$, with depth at most $\mathrm{rank}(\mathsf{Th}_\tau(y)\sigma) = n$. We take $((X, \xi), \pi)$ as the disjoint union of the $((X_y, \xi_y), \pi_y)$, extended by a fresh state $x_0$, for which we put $\xi(x_0) = t \in TY \subseteq TX$, and $x_0 \in \pi(a)$ iff $z_0 \in \pi_0(a)$ for $a \in U$.

Then obviously $x_0 \models_{((X,\xi),\pi)} \psi_0$; it remains to be checked that $x_0 \models_{((X,\xi),\pi)} \psi_1\sigma$, i.e. that

$$t \models^1_{X,\theta} \psi_1, \tag{$*$}$$

where $\theta(a) = [\![\sigma(a)]\!]_{((X,\xi),\pi)}$. By induction over the formula structure and naturality of predicate liftings, $y \models_{((X,\xi),\pi)} \rho$ iff $y \models_{((X_y,\xi_y),\pi_y)} \rho$ for every formula $\rho$. In particular, $y \models_{((X,\xi),\pi)} \mathsf{Th}_\tau(y)\sigma$ for all $y$, so that $[\![\sigma(a)]\!]_{((X,\xi),\pi)} \cap Y = \tau(a)$. Thus, our goal $(*)$ above reduces by naturality of predicate liftings to $t \models^1_{Y,\tau} \psi_1$, which holds by construction.

Finally, we have to establish that the overall branching degree of the model is polynomial in $|\psi|$. The model is recursively constructed from models for formulas $\rho$ which are conjunctive clauses over the set of subformulas of $\psi$, hence of at most quadratic size in $|\psi|$, and the branching degree at the root of a model for $\rho$ is polynomially bounded in $|\rho|$; this proves the claim. □

**Remark 3.8.** The leaves $x$ of the shallow model constructed above, i.e. the satisfying states for formulas of rank 0, are loops in the supporting Kripke frame in case the element $\xi(x) \in T\{x\}$ chosen in the base step fails to be in $T\emptyset \subseteq T\{x\}$. In particular, this is necessarily the case if $T\emptyset = \emptyset$ as e.g. in the case $T = D_\omega$. Note moreover that the construction relies crucially on a polynomial bound for one-step models depending only on the second component of a one-step pair, as the first component of the pair $(\phi, \psi_1)$ constructed in the proof may be of exponential size.

Theorem 3.6 immediately leads to the following decision procedure on an alternating Turing machine [2]: Let $\mathcal{M}$ have the OSPMP, and let $p$, $q$ be polynomial bounds on the branching degree as in Theorem 3.6.

**Algorithm 3.9.** (Decide satisfiability of an $\mathcal{F}(\Lambda)$-formula $\psi$ over $U$)

1. If $\mathrm{rank}(\psi) = 0$, check that $\psi$ is propositionally satisfiable and terminate correspondingly. Otherwise:
2. Decompose $\psi$ as $\psi_1\sigma$, with $\psi_1 \in \mathsf{Prop}(\mathsf{Up}_\Lambda(V) \cup U)$, $V \cap U = \emptyset$, and $\sigma$ an $\mathcal{F}(\Lambda)$-substitution for $V$.
3. Put $Y = \{1, \dots, p(|\psi|)\}$.
4. (Existential) Guess $t \in TY$ with $size(t) \leq q(|\psi|)$.
5. (Existential) Guess a $\mathcal{P}Y$-valuation $\tau$.
6. (Existential) Guess a $\{\top, \bot\}$-substitution $\kappa$ for $U$.
7. Check that $t \models^1_{Y,\tau} \psi_1\kappa$.
8. (Universal) Nondeterministically choose $y \in Y$.
9. Check recursively that $\mathsf{Th}_\tau(y)\sigma$ is satisfiable.

In Step 7, $\psi_1\kappa$ refers to the $\mathsf{Prop}(\mathsf{Up}_\Lambda(V))$-formula obtained by substituting $\bot$ or $\top$ for $a \in U$ in $\psi_1$ according to $\kappa$. In Step 4, a polynomial-size element of $TY$ can be guessed in polynomial time by Convention 2.12. Since the rank decreases with each recursive call, the above algorithm runs in polynomial time, thus establishing an upper bound in $APTIME = PSPACE$ [2], provided that Step 7 can be performed in polynomial time.

**Definition 3.10.** The *one-step model checking problem* for $\mathcal{M}$ is to check whether $t \in [\![L]\!](A_1, \dots, A_n)$ for $L \in \Lambda$ $n$-ary, $X$ a finite set, $A_1, \dots, A_n \subseteq X$, and $t \in TX$ representable. The input size of this problem is $1 + size(t) + size(L) + n|X|$.

**Corollary 3.11.** *If $\mathcal{M}$ has the OSPMP and the one-step model checking problem for $\mathcal{M}$ is in P, then the satisfiability problem of $\mathcal{F}(\Lambda)$ over $\mathcal{M}$ is in PSPACE.*

**Remark 3.12.** A more refined analysis shows that it suffices to require the one-step model checking problem to be in the polynomial hierarchy; however, we do not see practical applications of this extra generality.

We remark that the same conditions as in Corollary 3.11 yield also a typical complexity bound for one-step logics:

**Proposition 3.13.** *If $\mathcal{M}$ has the OSPMP and the one-step model checking problem for $\mathcal{M}$ is in P, then the satisfiability problem of the one-step logic of $\Lambda$ over $\mathcal{M}$ is in NP.*

For bounded-rank fragments, the polynomially branching shallow model property becomes a polynomial size model property. We therefore have, as an immediate consequence of Theorem 3.6, the following generalization of a result for the modal logic $K$ established in [11] (where corresponding results are proved also for several other normal modal logics outside rank 1, in particular $T$):

**Corollary 3.14.** *If $\mathcal{M}$ has the OSPMP and the one-step model checking problem for $\mathcal{M}$ is in P, then the satisfiability problem of $\mathcal{F}_n(\Lambda)$ over $\mathcal{M}$ is in NP for every $n \in \mathbb{N}$.*

We now illustrate how the above applies to our running examples:

**Example 3.15.** 1. *Modal logic $K$:* One-step model checking for $K$ is just subset inclusion, which is clearly in $P$. To verify the OSPMP, let $(X, \tau, A)$ be a one-step model of a one-step pair $(\phi, \psi)$ over $V$; i.e. $\psi$ is a conjunctive clause over atoms $\Box a$, where $a \in V$. For $\neg\Box a$ in $\psi$, there exists $x_a \in A$ such that $x_a \notin \tau(a)$. Taking $Y$ to be the set of these $x_a$, we obtain a polynomial-size one-step model $(Y, \tau_Y, Y)$ of $(\phi, \psi)$, where $\tau_Y(a) = \tau(a) \cap Y$ for all $a$. This reproves Ladner's $PSPACE$ upper bound for $K$ [17] by Corollary 3.11, as well as Halpern's $NP$ upper bound for bounded-rank fragments [11] by Corollary 3.14.

2. *Conditional logic $CK$:* To prove that $CK$ has the OSPMP, let $(X, \tau, f) \models^1 (\phi, \psi)$, where $\psi \equiv \bigwedge_{i=1}^n \epsilon_i(a_i \Rightarrow b_i)$. If $\tau(a_i) \neq \tau(a_j)$, fix an element $y_{ij}$ in the symmetric difference of $\tau(a_i)$ and $\tau(a_j)$. Moreover, if $\epsilon_i$ is negation, fix $z_i \in f(\tau(a_i)) \setminus \tau(b_i)$. Let $Y$ be the set of all $y_{ij}$ and all $z_i$. Let $\tau_Y$ be the $\mathcal{P}Y$-valuation defined by $\tau_Y(v) = \tau(v) \cap Y$, and let $f_Y \in TY$ be represented by the maplets $\tau_Y(a_i) \mapsto f(\tau(a_i)) \cap Y$. Note that $f_Y$ is well-defined, as $\tau_Y(a_i) \neq \tau_Y(a_j)$ whenever $\tau(a_i) \neq \tau(a_j)$. Then $(Y, \tau_Y, f_Y)$ is a one-step model of $(\phi, \psi)$. The cardinality of $Y$ is quadradic in $\psi$, and the representation size of $f_Y$ is polynomial.

It is easy to see that one-step model checking for $CK$ is in $P$, so that we obtain that $CK$ is in $PSPACE$ (hence

$PSPACE$-complete, as it contains $K$ as a sublogic). This has previously been proved using a detailed analysis of a labelled sequent calculus [21, 20] (it must be said that the method of [21, 20] yields an explicit polynomial bound on space usage which is not matched by the generic algorithm). The $NP$ upper bound for bounded-rank fragments of $CK$ arising from Corollary 3.14 is, to our knowledge, new.

3. *Conditional Logic $CK+ID$:* The above construction of polynomial-size one-step models works without changes for $CK+ID$; to see that for $f \in T_{ID}X$, the small one-step model $(Y, \tau_Y, f_Y)$ indeed satisfies $f_Y \in T_{ID}Y$ note that $f(A) \subseteq A$ implies $f(A) \cap Y \subseteq A \cap Y$, and that $\emptyset \subseteq A$ for all $A$ (recall that $f_Y(A) = \emptyset$ if no image of $A$ is explicitly listed in the representation of $f_Y$). One-step model checking is as for $CK$.

We thus obtain a $PSPACE$ upper bound for $CK+ID$, as well as an $NP$ upper bound for bounded-rank fragments. The $PSPACE$ bound is conjectured in [21] and proved in [20]; the same further remarks apply as for $CK$.

4. *Modal logic of probability [6]:* In [7], Thm. 2.6, a polynomial size model property is stated for the one-step logic of probability. Inspection of the proof shows that the polynomial bound depends only on the number of modal operators in a one-step formula and the representation size of the largest coefficient. By Proposition 3.2, it follows that the modal logic of probability has the OSPMP. One-step model checking amounts to checking linear inequations between given rational numbers, which is clearly in $P$. By Corollary 3.11, it follows that the modal logic of probability is in $PSPACE$ (hence $PSPACE$-complete, as one can embed $KD$ by mapping $\Diamond$ to $w(\_) > 0$). A proof of this upper bound is sketched in [6].

5. *Modal logic of upper probability [12]:* Analogously as in the previous example, one sees that the proof of [12], Theorem 5.1 actually establishes that the modal logic of upper probability has the OSPMP. It is easy to see that one-step model checking (which by definition applies only to *finite* sets of probability distributions) is in $P$, so that *the modal logic of upper probability is in $PSPACE$* (and hence $PSPACE$-complete analogously as the modal logic of probability). To our knowledge, this result is new. (In [12], it is proved that the *one-step* logic of upper probability is in $NP$.)

6. *Modal logic of possibilistic expectation:* One extracts a proof of the OSPMP from the proof of [13], Theorem 6.2, analogously to the previous examples, noting this time that the polynomial bound on models depends only on the number of literals in a formula, the number of expectations, and the size of the largest coefficient. Using the characterization of expectation in terms of mass functions, it is easy to see that one-step model checking is in $P$. Thus, we have shown that *the modal logic of possibilistic expectation is in $PSPACE$* (and hence $PSPACE$-complete analogously as the modal logic of probability), to our knowledge a new result. (In [13], it is proved that the *one-step* logic of possibilistic expectation is in $NP$.)

## 4. Rank-0-1 Logics

We now proceed to generalize the above arguments from rank-1 logics to *rank-0-1 logics*, whose axioms may have propositional variables at the top level. The paradigmatic rank-0-1 logic is the modal logic $T$, obtained from $K$ by adding the axiom $\Box a \to a$. The formal definitions are as follows.

**Definition 4.1.** A *rank-0-1 formula* over $V$ is an $\mathcal{F}_1(\Lambda)$-formula, i.e. an element of $\mathsf{Prop}(\mathsf{Up}_\Lambda(\mathsf{Prop}(V)) \cup V)$ (in contrast to a one-step formula, i.e. an element of $\mathsf{Prop}(\mathsf{Up}_\Lambda(\mathsf{Prop}(V)))$). A *rank-0-1 logic* $\mathcal{L}$ consists of a modal signature $\Lambda$, a $\Lambda$-structure $\mathcal{M}$ over a functor $T$, and a set $\mathcal{A}$ of rank-0-1 axioms. An *$\mathcal{L}$-frame* is a $T$-coalgebra $C = (X, \xi)$ such that $(C, \pi) \models \rho$ for all valuations $\pi$ and all $\rho \in \mathcal{A}$, where $(C, \pi) \models \rho$ if $x \models_{(C,\pi)} \rho$ for all $x \in X$. An *$\mathcal{L}$-model* is a $T$-model $(C, \pi)$ such that $C$ is an $\mathcal{L}$-frame. An $\mathcal{F}(\Lambda)$-formula $\psi$ is *$\mathcal{L}$-satisfiable* if there exists a state $x$ in some $\mathcal{L}$-model such that $x \models \psi$.

We fix the data $\mathcal{L}$, $\mathcal{M}$, $\mathcal{A}$ etc. as above for the remainder of this section. The shallow model result for rank-0-1 logics relies on a generalization of the notion of one-step model which incorporates a present state:

**Definition 4.2.** A *one-step $\mathcal{L}$-model* $(X, \tau, t, x_0)$ over $V$ consists of a one-step model $(X, \tau, t)$ over $V$ and a point $x_0 \in X$ such that for all $\rho \in \mathcal{A}$ and all $\mathcal{P}X$-valuations $\kappa$ for the set $U$ of propositional variables of $\rho$, $(X, \kappa, t, x_0) \models^1 \rho$. Here, the satisfaction relation $\models^1$ is defined by the obvious clauses for top-level boolean operators; $(X, \kappa, t, x_0) \models^1 L(\phi_1, \dots, \phi_n)$ iff $t \models^1_{(X,\kappa)} L(\phi_1, \dots, \phi_n)$ in the one-step logic, and for $a \in U$, $(X, \kappa, t, x_0) \models^1 a$ iff $x_0 \in \kappa(a)$. We say that $(X, \tau, t, x_0)$ is a *one-step $\mathcal{L}$-model* of a one-step pair $(\phi, \psi)$ if $(X, \tau, t) \models^1 (\phi, \psi)$ in the sense of Definition 2.6; in this case, $(\phi, \psi)$ is *one-step $\mathcal{L}$-satisfiable*.

$\mathcal{L}$ has the *one-step polysize model property (OSPMP)* if there exist polynomials $p$, $q$ such that for every one-step $\mathcal{L}$-model $(X, \tau, t, x_0)$ of a one-step pair $(\phi, \psi)$ over $V$, there exists a one-step $\mathcal{L}$-model $(Y, \sigma, s, y_0)$ of $(\phi, \psi)$ such that $|Y| \le p(|\psi|)$, $s$ is representable with $size(s) \le q(|\psi|)$, and $x_0 \in \tau(a)$ iff $y_0 \in \sigma(a)$ for all $a \in V$.

One can now show

**Theorem 4.3** (Shallow model property for rank-0-1)**.** *If $\mathcal{L}$ has the OSPMP, then every $\mathcal{L}$-satisfiable $\mathcal{F}(\Lambda)$-formula is satisfiable in a polynomially branching shallow $\mathcal{L}$-model.*

The precise definition of shallow $\mathcal{L}$-models is analogous to Theorem 3.6, except that now all states in supporting Kripke frames may be loops. The main adaptation to the proof is to consider, in the branches of the shallow model, satisfaction of all subformulas occurring in the scope of modal operators in an $\mathcal{L}$-satisfiable formula, rather than only the arguments of top-level modal operators as in the case of rank-1 logics.

The complexity analysis of rank-0-1 logics based on Theorem 4.3 works with representations of pairs $(t, x) \in TX \times X$ rather than just of elements of $TX$ in order to represent one-step $\mathcal{L}$-models $(X, \tau, t, x)$. One obtains

**Corollary 4.4.** *If $\mathcal{L}$ has the OSPMP, one-step model checking is in P, and it is decidable in polynomial time whether a quadruple $(X, \tau, t, x_0)$ is a one-step $\mathcal{L}$-model, then $\mathcal{L}$-satisfiability of $\mathcal{F}(\Lambda)$-formulas is in PSPACE, and $\mathcal{L}$-satisfiability of $\mathcal{F}_n(\Lambda)$-formulas is in NP.*

**Example 4.5.** 1. *Modal logic $T$:* This logic is obtained from $K$ by imposing the rank-0-1 axiom $\Box a \to a$; a frame $(X, R)$ is a $T$-frame iff $R$ is reflexive. A quadruple $(X, \tau, A, x_0)$, $A \in \mathcal{P}X$, is a one-step $T$-model iff $x_0 \in A$, which is clearly decidable in polynomial time. Polynomial-size one-step models are constructed as for $K$ (Example 3.15.1), except that the point $x_0$ of the original model is retained in the carrier set $Y$, and becomes the point of the small model. By Corollary 4.4, this reproves both Ladner's $PSPACE$ bound for $T$ [17] and Halpern's $NP$ upper bound for bounded-rank fragments of $T$ [11].

2. *Conditional Logic $CK{+}MP$:* The logic $CK{+}MP$ [3] is obtained from $CK$ by imposing the rank-0-1 axiom

$$(MP) \quad (a \Rightarrow b) \to (a \to b).$$

(This axiom is undesirable in default logics, but reasonable in relevance logics.) For purposes of $CK + MP$, in pairs $(f, x) \in (\mathcal{Q}X \to \mathcal{P}X) \times X$ the map $f$ is represented by lists of maplets $A \mapsto B$ as in the case of $CK$ (Example 2.13.2), but with default value $f(A) = A \cap \{x\}$ in case there is no maplet for $A$ in the list. A quadruple $(X, \tau, f, x_0)$ is a one-step $CK{+}MP$-model iff for $A \in \mathcal{Q}X$, $x_0 \in f(A)$ whenever $x_0 \in A$. As thanks to the choice of default value, one only has to check the maplets in the list representing $f$, this condition is decidable in polynomial time. Polynomial-size one-step $\mathcal{L}$-models are constructed as in the case of $CK$ (Example 3.15.2), except that the point $x_0$ of the original model is retained in the carrier set $Y$.

By Corollary 4.4, this reproves the $PSPACE$ bound for $CK + MP$ conjectured in [21] and proved in [20] and establishes a novel $NP$ bound for bounded-rank fragments of $CK{+}MP$. The same further comments apply as for $CK$.

## 5. Conclusion

We have described a polynomially branching shallow model construction for rank-0-1 modal logics, i.e. logics axiomatized without nested modal operators. This leads to a generic $PSPACE$ algorithm, and moreover shows that bounding the nesting depth of modalities brings the complexity down to $NP$. In example applications, we have

- recovered known tight $PSPACE$ upper bounds for the normal modal logics $K$ and $T$ and the conditional logics $CK$, $CK{+}ID$, and $CK{+}MP$ in a concise fashion;
- expanded the sketch of the tight $PSPACE$ upper bound for the modal logic of probability given in [6] to an explicit proof by sharpening results from [7];
- obtained (to our knowledge: new) tight $PSPACE$ upper bounds for the modal logics of upper probability and possibilistic expectation (which extend the corresponding one-step logics of [12, 13]);
- obtained (to our knowledge: new) tight $NP$ upper bounds for bounded-rank fragments of the conditional logics $CK$, $CK{+}ID$, and $CK{+}MP$.

A central topic for future investigation is the extension of these and related results [28, 29, 30] to a setting where modal logics are interpreted over covarieties of coalgebras for a functor rather than over the class of all coalgebras. Such a framework provides a coalgebraic semantics also for logics axiomatized outside rank 1. A further central point of interest is the study of coalgebraic modal logics with iteration, i.e. generic CTL.

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# A. Appendix: Omitted proofs

### Proposition 2.10

*Proof.* The second claim is clear. The first claim is proved as follows. As in [28], let $\kappa$ be a satisfying truth valuation for $\phi$ and put $\sigma(a) = a \wedge \phi$ if $\kappa(a) = \bot$, and $\sigma(a) = \phi \to a$ otherwise. Then $\phi\sigma$ and the formulas $\phi \to (a \leftrightarrow \sigma(a))$, for $a \in V$, are tautologies [28]. Both directions of the first claim now follow straightforwardly. □

### Proposition 3.13

*Proof.* The reduction from one-step formulas to one-step pairs described in Proposition 2.10 can be performed in polynomial time. Under the given conditions, one-step satisfiability of one-step pairs is decidable in *NP* by first guessing and then checking a polynomial-size one-step model. □

### Proposition 3.2

*Proof. Only if:* Let $\psi \in \mathsf{cCl}(\mathsf{Up}_\Lambda(\mathsf{Prop}(V)))$ be one-step satisfiable. By Proposition 2.10, $\psi$ is equivalent to a one-step pair of the form $(\phi, \psi_1)$, with $\psi_1$ as in the statement. By the OSPMP, $(\phi, \psi_1)$ has a one-step model $(X, \tau, t)$ such that $|X| \leq p(|\psi_1|)$ and $\mathit{size}(t) \leq q(|\psi_1|)$; by Proposition 2.10, this model gives rise to a one-step model of $\psi$ with the components $X$ and $t$ unchanged.

*If:* Let $(\phi, \psi)$ be a one-step satisfiable pair over $V$. By Proposition 2.10, $(\phi, \psi)$ is equivalent to a one-step formula of the form $\psi\sigma$, where $\sigma$ is a $\mathsf{Prop}(V)$-substitution. By assumption, $\psi\sigma$ has a one-step model $(X, \tau, t)$ such that $|X| \leq p(|\psi|)$ and $\mathit{size}(t) \leq q(|\psi|)$. By Proposition 2.10, this model gives rise to a one-step model of $(\phi, \psi)$ with the components $X$ and $t$ unchanged. □

### Corollary 3.14

*Proof.* Let $p$ and $q$ be polynomial bounds on the branching degree of supporting Kripke frames and on the size of successor structures $\xi(x)$ as guaranteed by Theorem 3.6. Let $n \in \mathbb{N}$. Then by Theorem 3.6, every satisfiable formula $\psi \in \mathcal{F}_n(\Lambda)$ is satisfiable in a model $((X, \xi), \pi)$ such that $|X| \leq \sum_{i=0}^{n} p(|\psi|)^i =: N$ and $\mathit{size}(\xi(x)) \leq \log(N) q(|\psi|)$ for all $x \in X$, where the second inequality relies also on Convention 2.12. Thus, the entire representation size of the model $((X, \xi), \pi)$ is bounded by $M := N \log(N) q(|\psi|)$, which is polynomial in $|\psi|$. Thus, the following non-deterministic algorithm decides satisfiability of $\psi$ in polynomial time:

1. Guess a model $((X, \xi), \pi)$ of size at most $M$
2. Check that $[\![\psi]\!]_{((X,\xi),\pi)} \neq \emptyset$.

The second step can be performed in polynomial time by recursively computing extensions $[\![\phi]\!]_{((X,\xi),\pi)}$, since the one-step model checking problem is in $P$. □

### Precise Statement and Proof of Theorem 4.3

**Theorem A.1.** *If $\mathcal{L}$ has the OSPMP, then $\mathcal{L}$ has the* polynomially branching shallow model property*: There exist polynomials $p$, $q$ such that every $\mathcal{L}$-satisfiable $\mathcal{F}(\Lambda)$-formula $\psi$ is satisfiable in at the root of an $\mathcal{L}$-model $((X, \xi), \pi)$ which has a supporting Kripke frame $(X, R)$ such that removing all loops from $(X, R)$ yields a tree of depth at most* $\mathrm{rank}(\psi)$ *and branching degree at most $p(|\psi|)$, and the size of $\xi(x)$ as an element of $T\{y \mid xRy\}$ is at most $q(|\psi|)$.*

**Lemma A.2.** *If there exists a non-empty $\mathcal{L}$-frame, then there exists an $\mathcal{L}$-frame with singleton carrier.*

*Proof.* Let $(X, \xi)$ be a non-empty $\mathcal{L}$-frame, and let $x_0 \in X$. Let $Y = \{x_0\}$, and let $f$ denote the unique map $X \to Y$. We claim that $(Y, \zeta)$ with $\zeta(x_0) = Tf\xi(x_0)$ is an $\mathcal{L}$-frame. For a $\mathcal{P}Y$-valuation $\pi$, let $\bar{\pi}$ denote the $\mathcal{P}X$-valuation given by $\bar{\pi}(a) = \emptyset$ if $\pi(a) = \emptyset$, and $\bar{\pi}(a) = X$ otherwise. We will show that

$$x_0 \models_{(Y,\zeta,\pi)} \rho \text{ iff } x_0 \models_{(X,\xi,\bar{\pi})} \rho \qquad (*)$$

for every rank-0-1 formula $\rho$ over $U$ and every $\mathcal{P}Y$-valuation $\pi$ for $U$; the claim then follows immediately.

We can assume that $\rho$ is a clause, and hence of the form $\rho \equiv \rho_1 \vee \rho_0$, where $\rho_1 \in \mathsf{Cl}(\mathsf{Up}(\mathsf{Prop}(U)))$ and $\rho_0 \in \mathsf{Cl}(U)$. It is clear that for $\phi \in \mathsf{Prop}(U)$,

$$[\![\phi]\!]_{(X,\xi,\bar{\pi})} = Tf^{-1}[\![\phi]\!]_{(Y,\zeta,\pi)}. \qquad (+)$$

Therefore, $(*)$ holds for $\rho_0$; it remains to prove $(*)$ for $\rho_1$. This reduces immediately to the case that $\rho_1$ is of the form $L(\phi_1, \dots, \phi_n)$, where $\phi_1, \dots, \phi_n \in \mathsf{Prop}(U)$. We have

$$\begin{aligned} & x_0 \models_{(Y,\zeta,\pi)} L(\phi_1, \dots, \phi_n) \\ \iff & Tf(\xi(x_0)) \in [\![L]\!]([\![\phi_1]\!]_{(Y,\zeta,\pi)}, \dots, [\![\phi_n]\!]_{(Y,\zeta,\pi)}) \\ \iff & \xi(x_0) \in [\![L]\!]([\![\phi_1]\!]_{(X,\xi,\bar{\pi})}, \dots, [\![\phi_n]\!]_{(X,\xi,\bar{\pi})}) \\ \iff & x_0 \models_{(X,\xi,\bar{\pi})} L(\phi_1, \dots, \phi_n), \end{aligned}$$

using $(+)$ and naturality of $[\![L]\!]$ in the second step. □

**Definition A.3.** An $\mathcal{F}(\Lambda)$-formula $\psi$ is $\mathcal{L}$-*valid* if $x \models_M \psi$ for all $\mathcal{L}$-models $M$ and all states $x$ in $M$.

*Proof (Theorem A.1).* Induction over the rank of $\psi$. Let $U$ be the set of propositional variables for $\psi$. If $\mathrm{rank}(\psi) = 0$, i.e. $\psi \in \mathsf{Prop}(U)$, then we let $(X, \xi)$ be a singleton $\mathcal{L}$-frame with $X = \{x\}$, which exists according to the above lemma, and we choose a $\mathcal{P}X$-valuation $\pi$ for $U$ such that $\psi$ is satisfied.

Now let $\mathrm{rank}(\psi) = n + 1$. We can assume that $\psi$ is a conjunctive clause, hence of the form $\psi_1 \wedge \psi_0$, where $\psi_1 \in \mathsf{cCl}(\mathsf{Up}_\Lambda(\mathcal{F}_n(\Lambda)))$ and $\psi_0 \in \mathsf{cCl}(U)$. Let $z_0$ be a state in an $\mathcal{L}$-model $((Z, \zeta), \pi_0)$ such that $z_0 \models_{((Z,\zeta),\pi_0)} \psi$. Let $MSub(\psi_1)$ denote the set of subformulas of $\psi_1$ occuring in $\psi_1$ within the scope of a modal operator, let $V$ be the set of variables $a_\rho$, indexed over $\rho \in MSub(\psi_1)$, and let $\sigma$ denote the substitution taking $a_\rho$ to $\rho$ for all $\rho$. Let $\bar{\psi} \in \mathsf{cCl}(\mathsf{Up}_\Lambda(V))$ be the conjunction of all literals $\epsilon L(a_{\rho_1}, \dots, a_{\rho_n})$ such that $L(\rho_1, \dots, \rho_n)$ is a subformula of $\psi_1$ and $z_0 \models_{((Z,\zeta),\pi_0)} \epsilon L(\rho_1, \dots, \rho_n)$. (Recall that $\epsilon$ denotes either nothing or negation.) Moreover, let $\phi$ denote the propositional theory of $\sigma$, i.e. the conjunction of all clauses $\chi$ over $V$ such that $\chi\sigma$ is $\mathcal{L}$-valid.

Then $(Z, \kappa, \zeta(z_0), z_0)$ is a one-step $\mathcal{L}$-model of $(\phi, \bar{\psi})$, where $\kappa(a) = [\![\sigma(a)]\!]_{(Z,\zeta)}$. By the OSPMP, it follows that $(\phi, \psi_1)$ has a one-step $\mathcal{L}$-model $(Y, \tau, t, x_0)$ of polynomial size in $|\bar{\psi}|$ such that for all $\rho \in MSub(\psi_1)$, $x_0 \in \tau(a_\rho)$ iff $z_0 \in \kappa(\rho)$, which in turn is equivalent to $z_0 \models_{((Z,\zeta),\pi_0)} \rho$.

From this model, we now construct a shallow $\mathcal{L}$-model $((X, \xi), \pi)$ for $\psi$. To begin, note that $\mathsf{Th}_\tau(y)\sigma$ is $\mathcal{L}$-satisfiable for every $y \in Y$. For suppose not; then $\neg\mathsf{Th}_\tau(y)\sigma$ is $\mathcal{L}$-valid, hence $\neg\mathsf{Th}_\tau(y)$ is a conjunct of $\phi$. Thus, $Y, \tau \models^0 \neg\mathsf{Th}_\tau(y)$, in contradiction to the fact that $y \in [\![\mathsf{Th}_\tau(y)]\!]\tau$ by construction. By induction, we thus have, for every $y \in Y$, a shallow model $((X_y, \xi_y), \pi_y)$ of $\mathsf{Th}_\tau(y)\sigma$, where we may assume $y \in X_y$ and $y \models_{((X_y,\xi_y),\pi_y)} \mathsf{Th}_\tau(y)\sigma$, with depth at most $\mathrm{rank}(\mathsf{Th}_\tau(y)\sigma) = n$. We take $((X, \xi), \pi)$ as the disjoint union of the $((X_y, \xi_y), \pi_y)$ over $y \in Y - \{x_0\}$, extended by the state $x_0$, for which we put $\xi(x_0) = t \in TY \subseteq TX$ and $x_0 \in \pi(a)$ iff $z_0 \in \pi_0(a)$ for $a \in U$.

We have to prove that $(X, \xi)$ is an $\mathcal{L}$-frame. Satisfaction of $\mathcal{A}$ in $(X, \xi)$ by $x \in X_y$ follows by induction over the formula structure and naturality of predicate liftings from satisfaction in $(X_y, \xi_y)$; satisfaction of $\mathcal{A}$ by $x_0$ follows directly from the fact that $(Y, \tau, t, x_0)$ is a one-step $\mathcal{L}$-model.

Next, we have to verify that $x_0 \models_{((X,\xi),\pi)} \psi$. Obviously $x_0 \models_{(X,\xi,\pi)} \psi_0$; it remains to be checked that $x_0 \models_{((X,\xi),\pi)} \psi_1$. We will prove the stronger statement $x_0 \models_{((X,\xi),\pi)} \bar{\psi}\sigma$, i.e.

$$t \models^1_{X,\theta} \bar{\psi}, \tag{1}$$

where $\theta(a_\rho) = [\![\rho]\!]_{((X,\xi),\pi)}$ for $\rho \in MSub(\psi_1)$.

By induction over $\chi$ and naturality of predicate liftings, $y \models_{((X,\xi),\pi)} \chi$ iff $y \models_{((X_y,\xi_y),\pi_y)} \chi$ for $y \in Y - \{x_0\}$ and for every formula $\chi$. In particular, $y \models_{((X,\xi),\pi)} \mathsf{Th}_\tau(y)\sigma$ for all $y \in Y - \{x_0\}$, i.e.

$$y \models_{((X,\xi),\pi)} \rho \iff y \in \tau(a_\rho) \tag{2}$$

for all $\rho \in MSub(\psi_1)$. We prove by induction over $\rho \in MSub(\psi_1)$ that

$$x_0 \models_{((X,\xi),\pi)} \rho \iff x_0 \in \tau(a_\rho), \tag{3}$$

which in connection with (2) yields

$$[\![\rho]\!]_{((X,\xi),\pi)} \cap Y = \tau(a_\rho). \tag{4}$$

The steps for boolean operations are straightforward. For $L(\rho_1, \dots, \rho_n) \in MSub(\psi_1)$, we have

$$\begin{aligned} & x_0 \models_{((X,\xi),\pi)} L(\rho_1, \dots, \rho_n) \\ \iff & t \in [\![L]\!]_Y([\![\rho_i]\!]_{((X,\xi),\pi)} \cap Y)_{i=1,\dots,n} \\ & = [\![L]\!]_Y(\tau(a_{\rho_1}), \dots, \tau(a_{\rho_n})) \\ \iff & t \models^1_{(Y,\tau)} L(a_{\rho_1}, \dots, a_{\rho_n}), \end{aligned}$$

using naturality of $[\![L]\!]$ in the first step and the inductive hypothesis in the shape of (4) in the second step. Since $t \models_{(Y,\tau)} \bar{\psi}$, the latter statement is equivalent to $z_0 \models_{((Z,\zeta),\pi)} L(\rho_1, \dots, \rho_n)$. By the definition of $\kappa$, this is equivalent to $z_0 \in \kappa(a_{L(\rho_1,\dots,\rho_n)})$, which in turn is equivalent to $x_0 \in \tau(a_{L(\rho_1,\dots,\rho_n)})$ by construction of $(Y, \tau, t, x_0)$. Finally, the base case $\rho \equiv a \in U$ is similar to the above but simpler.

By (4) and naturality of predicate liftings, our remaining goal (1) reduces to $t \models^1_{Y,\tau} \bar{\psi}$, which holds by construction.

Finally, we have to establish that the overall branching degree of the model is polynomial in $|\psi|$. The model is recursively constructed from polynomial-size one-step models for pairs whose second components are conjunctive clauses over atoms $L(a_{\rho_1}, \dots, a_{\rho_n})$, where $L(\rho_1, \dots, \rho_n)$ is a subformula of $\psi$. Such conjunctive clauses are of at most quadratic size in $|\psi|$ (even $O(|\psi| \log |\psi|)$ if subformulas of $\psi$ are represented by pointers into $\psi$); this proves the claim. □

**Corollary 4.4**

*Proof.* The $NP$ upper bound for bounded-rank fragments is obtained by the same algorithm as in the proof of Corollary 3.14, except that one additionally has to check that the putative model is an $\mathcal{L}$-model. This can be done in polynomial time because one-step $\mathcal{L}$-models can be recognized in polynomial time.

The $PSPACE$ upper bound for the full logic is obtained by a variation of Algorithm 3.9, where instead of decomposing $\psi$ as $\psi_1\sigma$ in Step 2, one guesses a conjunctive clause $\tilde{\psi}$ from the DNF of $\psi$, decomposes $\tilde{\psi}$ as $\psi_1 \wedge \psi_0$ and takes $V$ and $\sigma$ as in the proof of Theorem 4.3, and then guesses $\bar{\psi} \in \mathsf{cCl}(V)$. Propositional satisfiability of $\psi_0$ is checked separately, replacing Step 6. In Step 7, it is checked that $t \models^1_{Y,\tau} \bar{\psi}$, and additionally that $(Y, \tau, t, 1)$ is a one-step $\mathcal{L}$-model. □