Coalgebra and Coalgebraic Logics

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Sichere Kognitive Systeme

Introduction: DL

- Description logics (DLs) are a core formalism of KR and the Semantic Web
  - Underlying logic of OWL-DL
- Notational variant of modal logic
- Relational semantics
  - Binary relations between individuals
  - Universal and existential quantification

Relational Semantics of $\mathcal{ALC}$ (= Multi-K)

Concepts

$$C ::= \bot \mid A \mid \neg C \mid C_1 \sqcap C_2 \mid \forall R.\, C$$

Interpretations $\mathcal{I}$:

- $(\Delta^{\mathcal{I}}, (A^{\mathcal{I}}), (R^{\mathcal{I}}))$ where
  - $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$
  - $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$
- Extension $C^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ of concepts C:

$$(\forall R.\, C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \forall y \in \Delta^{\mathcal{I}}.\, xR^{\mathcal{I}}y \Rightarrow y \in C^{\mathcal{I}}\}$$

E.g.

$$\mathsf{ChessFanatic} = \mathsf{ChessPlayer} \sqcap \forall \mathsf{hasFriend}.\, \mathsf{ChessFanatic}$$

Introduction: Real-Life Reasoning

Many modes of expression need more than relational semantics, e.g.

Uncertainty	(Probabilities)
Vagueness	(Fuzzy truth values)
Defeasibility	(Preference orderings)
Causation and agency	(Games)

Large variety of domain-specific DLs

+ Suitable expressive means for every purpose

– Multiplied need for tools and algorithms

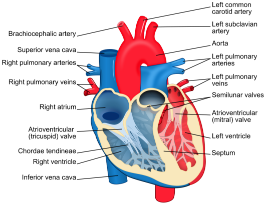
Enter Coalgebra

Coalgebra acts as a unified framework for real-life DLs

- semantically
- logically
  - generic complete axiomatizations
- algorithmically
  - generic decidability results
  - generic algorithms and complexity analysis

OWL in the Representation of CPGs

From the German CPG for coronary heart disease:



> ***7-13*** *In presence of medium prior probability and inconclusive ergometry, an exercise test with imaging should be carried out.*

Approximation in relational DL:

$$\forall \mathsf{hasPriorRiskCHD}.\mathsf{Medium} \sqcap \forall \mathsf{hasDiagnostics}.(\mathsf{Ergometry} \to \mathsf{Inconclusive}) \sqsubseteq \exists \mathsf{hasRecommendedDiagnostics}.(\mathsf{ExerciseTest} \sqcap \exists \mathsf{hasObservation}.\mathsf{Imaging})$$

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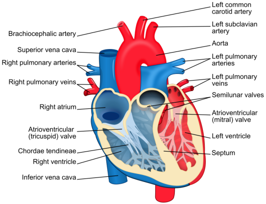

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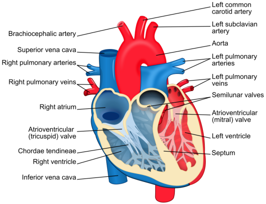

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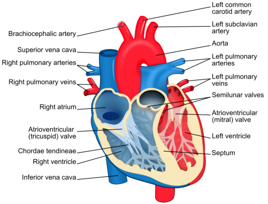

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CDL in the Representation of CPGs

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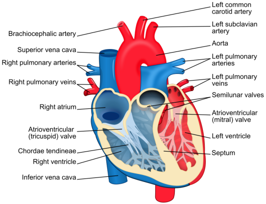

Better approximation in coalgebraic description logic:

$$\begin{aligned}&\mathsf{moderately}(\mathsf{probably}\ (\exists \mathsf{hasDisorder}.\,\mathsf{CHD})) \sqcap \\ &\qquad \forall \mathsf{hasDiagnostics}.\,(\mathsf{Ergometry} \to \mathsf{Inconclusive}) \\ &\qquad\quad \Rightarrow \exists \mathsf{hasRecommendedDiagnostics}. \\ &\qquad\qquad (\mathsf{ExerciseTest} \sqcap \exists \mathsf{hasObservation}.\,\mathsf{Imaging})\end{aligned}$$

More examples from CPGs

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Nested defeasible implication:

Units normally seeing at least 100 new cases of cancer per annum should be able to maintain their expertise.

Comparison of probabilities:

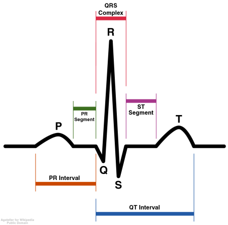
Radiotherapy should be given following mastectomy or breast conserving surgery [. . .] where the benefit to the individual is likely to outweigh risks of radiation related morbidity.

(SIGN breast cancer CPG)

Combined vague temporality, belief, and uncertainty:

Aspirin should be given to all patients with a STEMI as soon as possible after the diagnosis is deemed probable.

(European CPG for acute ST-segment elevated myocardial infarction)



Incomplete Overview of Non-relational Logics

Logic	Systems	Syntax	Reading
Probabilistic logics	Markov chains	$L_p\phi$	With Prob. $\geq p$, ϕ
Graded logics	Multigraphs	$\geq nR.\phi$	$\geq n$ R-successors satisfy ϕ
Conditional logics	Preference models	$\phi \Rightarrow \psi$	If ϕ then normally ψ
Alternating-time logic	Concurrent game struct.	$[C]\phi$	Coalition C can force ϕ
Game logic	Game models	$\langle\gamma\rangle\phi$	Angel can force ϕ in game γ

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KR

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Reactive systems

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Multi-agent systems

Tool Support

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…for such logics has been notoriously limited:

- CondLean: weak conditional logics
- Pronto: $P\text{-}\mathcal{SHIQ}(D)$.

Start Over: Coalgebra

(Here: set-based)

T : **Set** $\to$ **Set** functor — think: TX structured collection over X

Coalgebra: map $\xi : X \to TX$

- X set of states
- $\xi(x)$ structured collection of successors of x

Morphism of coalgebras:

$$
\begin{array}{ccc}
X & \xrightarrow{\;f\;} & Y \\
{\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \zeta} \\
TX & \xrightarrow[\;Tf\;]{} & TY
\end{array}
$$

Coalgebra: Examples

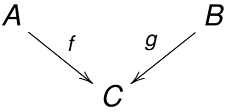
- $TX = X$ (identity): deterministic transitions $X \to X$.
- $TX = O$ (constant): output $X \to O$
  - Special case: $O = \mathcal{P}(V)$ — valuation for variables in V
- $TX = I \to X$ (exponential): input $X \to (I \to X)$
- $TX = \mathcal{P}X$ (powerset): non-deterministic transitions $X \to \mathcal{P}(X)$.
- $TX = D(X) = \{\mu \mid \mu$ discrete prob. meas. on $X\}$ (distributions): Markov chains $X \to D(X)$
- $TX = \mathcal{B}X = X \to \mathbb{N} \cup \{\infty\}$ (multisets): Integer weighted transitions $X \to \mathcal{B}X$

Coalgebra: Yet More Examples

- $TX = 2^{2^X}$ (neighbourhoods): neighbourhood frames $X \to 2^{2^X}$
- $$TX = \Big\{(k_1, \dots, k_n, f) \mid f : \big(\prod_{i \in N} \{1, \dots, k_i\}\big) \to X\Big\}$$
  (games): concurrent game structures $X \to TX$.

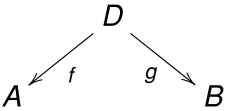
Behavioural Equivalence

A, B coalgebras: $x \in A$, $y \in B$ behaviourally equivalent if $\exists$



s.t. $f(x) = g(y)$.

x, y bisimilar if $\exists$



s.t. $x = f(z), y = g(z)$ for some z.

Bisimilarity is a proof principle for behavioural equivalence (coinduction)

- Always sound (pushouts)
- Complete if T preserves weak pullbacks

Final Coalgebras

C final if $\forall B. \exists! f : B \to A$.

States in final coalgebras = behaviours = (all states)/beh. eq.

- Domain for process algebra semantics

T κ-accessible for some $\kappa \implies T$ has a final coalgebra.

E.g.

- $TX = A \times X$: $C = (X, \xi)$ with
  - $X =$ streams over A
  - $\xi(s) = (\text{hd}(s), \text{tl}(s))$.
- $TX = \mathcal{P}_\omega X$: $C =$ finite or infinite trees with unordered finite branching modulo bisimilarity.

Coalgebraic Modal Logic

Syntactic parameter: modal similarity type Σ (set of polyadic operators);

$$\mathcal{F}(\Sigma) \ni \phi ::= \bot \mid \neg\phi \mid \phi \wedge \psi \mid L\phi \qquad (L \in \Sigma)$$

Semantic parameters ($\to$ logic $\mathcal{L}$):

- Functor T
- For each $L \in \Sigma$ predicate lifting

$$[\![L]\!] : 2^{-} \to 2^{T^{op}}$$

(i.e. $[\![L]\!]_X(f^{-1}[A]) = (Tf)^{-1}[[\![L]\!]_Y(A)]$ for $A \subseteq Y$).

Then $[\![\phi]\!]_C \subseteq X$ for $C = (X, \xi)$, with

$$[\![L\phi]\!]_C := \xi^{-1}[[\![L]\!][\![\phi]\!]].$$

Nearly Everything is Coalgebraic

Logic	Systems	Syntax	Functor
K / $\mathcal{ALC}$	Kripke models	$\Box$, $\forall R$	Powerset $\mathcal{P}(X)$
Probabilistic logics	Markov chains	$L_p\phi$ $\sum a_i P(\phi_i) \geq b$	Distributions $D(X)$
Graded logics	Multigraphs	$\geq nR.\phi$ $\sum a_i \#(\phi_i) \geq b$	Multisets $\mathcal{B}(X) = X \to \mathbb{N}_\infty$
Conditional logics	Preference models	$\phi \Rightarrow \psi$	Preference orders $\exists(S, \preceq).S \to X$
Alternating-time logic	Concurrent game struct.	$[C]\phi$	Games $\exists(S_i).(\prod S_i \to X)$
Game logic	Game models	$\langle\gamma\rangle\phi$	Monotone nbhd systems

(Schröder/Pattinson/Cirstea/Kurz/Venema et al. 2004–2010)

Example: Probabilistic Modal Logic

(Fagin/Halpern JACM 1994)

Functor $D(X)$ = distributions on X

Coalgebras $X \to DX$ = Markov chains

Operators L_p 'with probability $\geq p$'

$$[\![L_p]\!]_X(A) = \{\mu \in D(X) \mid \mu(A) \geq p\}$$

Example: Alternating Time

(Alur et al. JACM 2002)

$N = \{1, \dots, n\}$ set of agents, $C \subseteq N$ coalition

Functor:

$$T(X) = \Big\{(k_1, \dots, k_n, f) \mid f : \big(\prod_{i \in N} \{1, \dots, k_i\}\big) \to X\Big\}$$

Coalgebras $X \to TX$ = concurrent game structures

Operators $[C]$ 'C can force ... in the next step'

$$[\![[C]]\!]_X(A) = \{f \in T(X) \mid \exists \sigma_C . \forall \sigma_{N-C} . f(\sigma_C, \sigma_{N-C}) \in A\}$$

CML and Behavioural Equivalence

- Coalgebraic modal logic is invariant under behavioural equivalence
- If T is finitary, then T admits an expressive CML (i.e. logical implies behavioural equivalence) (Schröder, Pattinson)
  - Needs polyadic operators
  - Final coalgebra consists of maximally satisfiable sets
- CML is the behavioural-equivalence-invariant fragment of „Coalgebraic FOL“

(Schröder/Pattinson FOSSACS 2010)

Generic Deduction Systems

Parametrized Systems:

- Fixed propositional part
- Further fixed parts depending on orthogonal features (nominals, fixed points)
- Parameter: Axiomatization of the functor through (cutfree complete) local rules
  (Schröder/Pattinson LICS 06, ACM TOCL 09; Pattinson/Schröder I&C 2010)
- Complexity analysis reduced to the local level

The Local Logic

V set of prop. var.,

$$\Sigma V = \{La \mid a \in V, L \in \Sigma\}.$$

Given $\tau : V \to \mathcal{P}(X)$, interpret

- propositional formulas ϕ over V as $[\![\phi]\!]\tau \subseteq X$
- propositional formulas ψ over ΣV as $[\![\psi]\!]\tau \subseteq TX$ by

$$[\![La]\!]\tau = [\![L]\!]_X \tau(a)$$

Local constraints (ϕ, ψ) over V: $\phi \in \mathrm{Prop}(V)$, $\psi \in \mathrm{Prop}(\Sigma(V))$

Local models: $M = (X, \tau, t)$ with $t \in TX$;

$$M \models (\phi, \psi) :\Longleftrightarrow \begin{array}{l}(i)\ \ [\![\phi]\!]\tau = X \\ (ii)\ t \in [\![\psi]\!]\tau\end{array}$$

Top-level decomposition:

$$\phi = \bar{\phi}\sigma_\phi$$

with $\bar{\phi} \in \mathrm{Prop}(\Sigma(W))$, $\sigma_\phi : W \to \mathcal{F}(\Sigma)$.

Theory $\mathrm{Th}(\sigma) \in \mathrm{Prop}(W)$ of a substitution $\sigma : W \to \mathcal{F}(\Sigma)$:

$$\mathrm{Th}(\sigma)\kappa = \top \iff \bigwedge_{w \in W} \sigma(w) = \kappa(w) \text{ satisfiable}$$

for $\kappa : W \to \{\top, \bot\}$.

Theorem (Local reduction): ϕ is satisfiable iff $(\mathrm{Th}(\sigma_\phi), \bar{\phi})$ is satisfiable.

Corollary: $\mathcal{L}$ is in PSPACE if satisfiability of local constraints (ϕ, ψ) is strictly in PSPACE (i.e. discounting ϕ).

The Local Exponential Model Property

For $\phi \in \mathsf{Prop}(V)$, the canonical model is

$$(X_\phi, \tau_\phi) = (\{\kappa \in 2^V \mid \phi\kappa = \top\}, \lambda v.\{\kappa \mid \kappa(v) = \top\}).$$

Lemma (Local exponential model property): A local constraint (ϕ, ψ) is satisfiable iff it is satisfiable over (X_ϕ, τ_ϕ).

Example: K is in PSPACE

(See Ladner 1977)

Decide satisfiability of (ϕ, ψ) over V, w.l.o.g. ψ conjunctive clause:

For all $\neg\Box a \in \psi$, check that there is $\kappa : V \to \{\top, \bot\}$ s.t.

1. $\phi\kappa = \top$ (in LOGSPACE, Lynch 1977)
2. $\kappa(a) = \bot$
3. $\kappa(b) = \top$ whenever $\Box b \in \psi$.

Example: Presburger ML is in PSPACE

(See Demri/Lugiez IJCAR 2006)

Decide satisfiability of (ϕ, ψ) over V, w.l.o.g. ψ conjunctive clause:

Global variables c_a for all $a \in V$, initially 0.

For all $\kappa \in 2^V$,

1. Check that $\phi\kappa = \top$
2. Guess multiplicity n
3. For all $a \in V$ s.t. $\kappa(a) = \top$, $c_a := c_a + n$

Implicitly, this constructs $t \in \mathcal{B}X_\phi$.
Check that $t \in [\![\psi]\!]\tau_\phi$ using the c_a.

Local Rules

Sequents = multisets (disjunctive)

Local rule: $\dfrac{\Gamma_1 \dots \Gamma_n}{\Gamma_0}$ sequents over V / sequent over ΣV

$\frac{\varphi}{\psi}$ locally sound if $(\forall i = 1, \dots, n. [\![\Gamma_i]\!]\tau = X) \implies [\![\Gamma_0]\!]\tau = TX$

(i.e. if local constraint $(\bigwedge_{i=1}^{n} \Gamma_i, \neg\Gamma_0)$ unsatisfiable)

Local Cut-Free Completeness

Set $\mathcal{R}$ of local rules locally cut-free complete
if for sequents Δ over ΣV

$$[\![\Delta]\!]\tau = TX \implies \exists \Gamma_1 \dots \Gamma_n / \Gamma_0 \in \mathcal{R}, \sigma : V \to V. \\ [\![\Gamma_i \sigma]\!]\tau = X \ (i = 1, \dots, n),\ \Gamma_0 \sigma \subseteq \Delta.$$

$$\cfrac{\cfrac{\Gamma_1\sigma \dots \Gamma_n\sigma}{\Gamma_0\sigma}}{\Delta}$$

Local completeness: Same with several rules and substitutions $V \to \mathrm{Prop}(V)$.

Theorem: If $\mathcal{R}$ is locally complete, then $\mathcal{R}$ is locally cut-free complete iff $\mathcal{R}$ absorbs cut and contraction.

Theorem: If $\mathcal{R}$ is locally cut-free complete, then $\mathcal{R}$ is cut-free complete.

Cutting Local Rules

E.g. K:

$$\frac{\neg a_1, \neg a_2, b_1}{\neg\Box a_1, \neg\Box a_2, \Box b_1} \qquad \frac{\neg b_1, \neg b_2, c}{\neg\Box b_1, \neg\Box b_2, \Box c}$$

Cut $\Box b_1$ vs. $\neg\Box b_1$:

$$\frac{\exists b_1 . (\neg a_1, \neg a_2, b_1 \quad \neg b_1, \neg b_2, c)}{\neg\Box a_1, \neg\Box a_2, \neg\Box b_2, \Box c}$$

Simplify:

$$\frac{\neg a_1, \neg a_2, \neg b_2, c}{\neg\Box a_1, \neg\Box a_2, \neg\Box b_2, \Box c}$$

Local Cut-free Completeness: Examples

K:

$$\frac{\neg a_1, \dots, \neg a_n, b}{\neg \Box a_i, \dots, \Box a_n, \Box b} \quad (n \geq 0)$$

Probabilistic Modal Logic:

$$\frac{\overbrace{\textstyle\sum_{i=0}^{n} r_i a_i \sqsupset \sum_{i=0}^{n} r_i p_i}^{\text{Arithmetic of characteristic functions}}}{\bigsqcup_{0 \leq i \leq n} sgn(r_i) L_{p_i} a_i}$$

where $n \geq 0$, $r_i \in \mathbb{Z} - \{0\}$, $\sqsupset = \begin{cases} > & \text{if } r_i < 0 \text{ for all } i \\ \geq & \text{otherwise} \end{cases}$

Local Constraints Via Local Rules

Absorption of cut: need to match only one modal rule to a target sequent Δ.

Absorption of contraction: need to match only rules $\Gamma_1 \dots \Gamma_n / \Gamma_0$ with $|\Gamma_0| \leq |\Delta|$.

Still have doubly exponential number of candidate rules (e.g. PML).

Definition: $\mathcal{R}$ is PSPACE tractable if every rule match is dominated by one with a polynomial-size code (plus some technicalities).

E.g. PML is PSPACE tractable:
by size estimates from LP, can restrict to polynomial size r_i.

Local Constraints Via Local Rules (Cont'd)

Theorem: If $\mathcal{R}$ is locally cut-free complete and PSPACE tractable, then satisfiability of local constraints is strictly in PSPACE.

Proof: Check that (ϕ, ψ) is consistent (w.l.o.g. ψ conjunctive clause):

For all rules $R = \Gamma_1 \dots \Gamma_n / \Gamma_0$ with polynomial-size codes, check that R does not prove $\neg\psi$ from ϕ, i.e.

- If $\Gamma_0\sigma \subseteq \neg\psi$
- then check that $\phi \wedge \neg\Gamma_i\sigma$ is satisfiable for some $i = 1, \dots, n$.

Generic Algorithms Via LCCR

- PSPACE over acyclic TBoxes (Schröder/Pattinson LICS 06)
  - e.g. conditional logics with cautious monotony (Schröder/Pattinson/Hausmann ECAI 10)
  - also with nominals (e.g. $\mathcal{ALCHOQ}$) (Myers/Pattinson/Schröder FOSSACS 09)
- EXPTIME for GCIs and with nominals (Schröder/Pattinson/Kupke IJCAI 09)
  - via global caching (Gore/Kupke/Pattinson/Schröder IJCAR 10)
- Completeness and EXPTIME global caching for flat FP logics (Schröder/Venema CONCUR 2010)
  - Alternating-time μ-calculus (Alur et al. 2002)
  - Graded μ-calculus (Kupferman et al. 2002)
- Implemented reasoner CoLoSS
  (Coalgebraic Logic Satisfiability Solver)

Fuzzy Logic

- Truth values $[0,1]$
- Propositional connectives: various semantics, e.g.
  - Zadeh: $a \wedge b = \min(a,b)$, $\neg a = 1-a$, $a \to b = \max(1-a,b)$
  - Gödel: Intuitionistic logic over $[0,1]$ ($\to$ discontinuous)
  - Łukasiewicz: $a \wedge b = \max(a+b-1,0)$, $\neg a = 1-a$
    - Here, one has $\neg a \vee \_$ as a continuous right adjoint of $a \wedge \_$.
    - Zadeh is codable into Łukasiewicz.

Fuzzy $\mathcal{ALC}$

(Straccia 2001, 2005)

Fuzzy concepts $C : X \to [0, 1]$

Fuzzy roles $R : X \times X \to [0, 1]$

$$[\![\exists R.\, C]\!](x) = \bigvee_{y \in X} R(x, y) \wedge C(y)$$

Coalgebraic Fuzzy DL

All the same as before, but

$$[\![L]\!] : [0,1]^{-} \to [0,1]^{T^{op}}$$

and

$$[\![\phi]\!]_C : X \to [0,1]$$

with

$$[\![L\phi]\!]_C = [\![L]\!]([\![\phi]\!]_C) \circ \xi.$$

(Schröder/Pattinson IJCAI 2011)

Coalgebraic Fuzzy DLs: Examples

- Fuzzy $\mathcal{ALC}$: $[\![\exists]\!]_X(A)(B) = \bigvee_{y\in X} B(y) \wedge A(y)$
- Fuzzification of monotone crisp operator L with $[\![L]\!]^c : 2^- \to 2^{T^{op}}$:

$$[\![L]\!]^f_X(A)(t) = \sup\{a \in [0,1] \mid t \in [\![L]\!]^c_X A_a\}$$

  where $A_a = \{x \in X \mid A(x) \geq a\}$.
- Probably (P) (Zadeh 1968):

$$[\![P]\!]_X(A)(\mu) = \sum_{x\in X} A(x)\cdot\mu(x)$$

- Generally (G):

$$[\![G]\!]_X(A)(\mu) = \sup_{a\in[0,1]} a \wedge \mu([\![C]\!]_a)$$

Fuzzy Local Reduction

- Constraints $\sigma \bowtie \kappa$ with $\sigma : V \to \mathcal{F}(\Sigma)$, $\kappa : V \to [0,1]$
- Local constraints $(\phi, \sigma \bowtie \kappa)$ with $\gamma \subseteq (V \to [0,1])$, $\sigma : W \to \mathrm{Prop}(V)$
- Top-level decomposition $\sigma = \sigma^\sharp \sigma^\flat$
- $\mathrm{Th}(\sigma) = \{\kappa : V \to [0,1] \mid \sigma = \kappa \text{ satisfiable}\}$
- Local reduction: $\sigma \bowtie \kappa$ sat. iff $(\mathrm{Th}(\sigma^\flat), \sigma^\sharp \bowtie \kappa)$ sat.
- If the local logic has the polysize-model property for $\leq$, get exponential model property
  - Example: Fuzzy $\mathcal{ALC}$
  - Non-examples: probably, generally.

Open Interval Satisfiability

Demand that operators are Lipschitz, i.e. that for every $t \in TX$,

$$[\![L]\!]_X(\_)(t) : [0,1]^X \to [0,1]$$

is k_L-Lipschitz.

- Everything is Lipschitz.
- Łukasiewicz is Lipschitz.
- For Lipschitz logics, theories $\mathrm{Th}(\sigma)$ are compact.

Theorem (Local ε-reduction) Let $\mathcal{L}$ be Lipschitz and have the local finite $<$-model property, and let $\sigma : V \to \mathcal{F}(\Lambda)$ be a substitution, V finite. Then there exists k s.t. $\sigma < \kappa$ is satisfiable iff

$$(U_\varepsilon(\mathrm{Th}(\sigma^\flat)), \sigma^\sharp < \kappa - k\varepsilon)$$

is satisfiable for some $\varepsilon > 0$.

Complexity Bounds

Obtain upper bound EXPSPACE if functor and operators are FO over the reals:

- Translate into exponential-size FO formula over the reals
  - Use local ε-reduction
  - Polynomial-size local $<$-model property allows describing (X, τ) by polynomially many variables
  - Leave ε existentially quantified
  - Note

$$\tau^T(x) = \lambda v \in V.\, \tau(v)(x) \in U_\varepsilon(\mathrm{Th}(\sigma^\flat)) \text{ iff } \sigma^\flat \in U_\varepsilon(\tau^T(x)) \text{ sat.}$$

$\rightarrow$ polynomial-breadth linear-depth recursion.

P does have the local polysize $<$-model property (Hájek 2005), similarly for *G*
(uses continuity argument, does not work for $\leq$)

Conclusions

- Coalgebra provides a uniform framework for modal and hybrid logics
  - Graded operators (knowledge representation, redundancy)
  - Probabilistic operators (quantitative uncertainty, reactive systems)
  - Conditional operators (nonmonotonic reasoning)
  - Alternating-time logics, game logic, logics of agency (multi-agent systems)
- Wide range of generic decision procedures and complexity bounds
- Modular (Schröder/Pattinson ICALP 2007)
- Frequently new bounds and calculi for instance logics, in particular in presence of
  - nominals
  - fixed points
  - fuzzy truth degrees

Ongoing and Future Work

- Manydimensional coalgebraic logics
- Algorithms in coalgebraic FOL
- Vision: generic, efficient modular reasoning tools
  - Ongoing optimization of CoLoSS (PhD thesis Hausmann)

Thanks for your attention!

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CDL with Nominals

- Nominals $i, j, \dots$ are atomic concepts to be interpreted as singletons
- Internalize ABoxes via satisfaction operators

$$@_i C = \text{`}i \text{ satisfies } C\text{'}$$

Deduction over GCIs

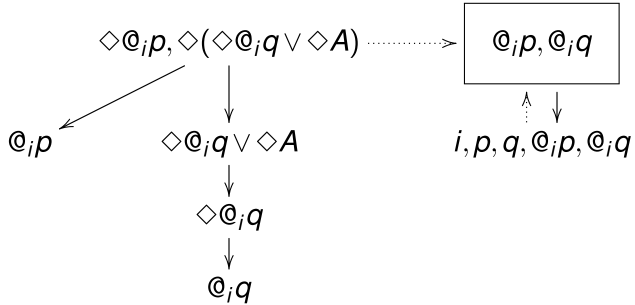
- General concept inclusions $C \sqsubseteq D$
- Tableaux diverge without blocking:
  for gci $\top \sqsubseteq \exists R.\, A$,

$$\cfrac{\top, \exists R.\, A}{\cfrac{A, \exists R.\, A}{\cfrac{A, \exists R.\, A}{\dots}}}$$

- Tedious analysis even for $\mathcal{ALC}$ (Donini/Massacci 99)

A Global Caching Algorithm

Collect @-formulas along a winning strategy:



- Decidability in EXPTIME
- Room for heuristic optimization
- Novel algorithm even for the relational case

(Goré/Kupke/Pattinson/Schröder IJCAR 10)