Coalgebraic Logics in Artificial Intelligence and Reactive Systems

Lutz Schröder

DFKI Bremen and Department of Mathematics and Computer Science, Universität Bremen

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Introduction: Modal Logic

Modal logics are a central logical tool in computer science and AI:

- Applications e.g. to
  - reactive systems
  - knowledge representation
  - multi-agent systems
- Good semantic properties, e.g. invariance under bisimulation
- May be tailored to offer the right expressive means for a given domain
- Often have good computational properties (unlike, e.g., FOL/HOL)

Introduction: Coalgebra

- Large variety of domain-specific Logics of different syntax, semantics, and complexity
- Coalgebra acts as a unifying semantic theory of modal logic and supports
  - generic complete deduction systems
  - generic decidability results
  - generic algorithms and complexity bounds
  - generic implementations
  - systematic logic design

Modal Logic (Version 1)

Sichere Kognitive Systeme

Modal logic extends (classical) propositional logic with additional operators, e.g.

$$\phi ::= \bot \mid p \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \Box\phi \qquad (p \in P)$$

with $\Box\phi$ read e.g. 'necessarily ϕ'
(dually: $\Diamond\phi :\equiv \neg\Box\neg\phi$ 'possibly ϕ')

Modal logic is a logic of relational structures:

- Models $((X,R),V)$, where
  - (X,R) is a Kripke frame, i.e. $R \subseteq X \times X$
  - V is a valuation $P \to \mathcal{P}(X)$.
- Satisfaction is per state $x \in X$

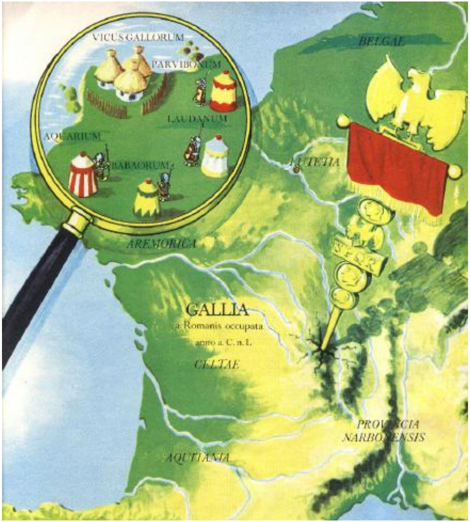
$$x \models \Box\phi \text{ iff } \forall y.\, xRy \Rightarrow y \models \phi.$$

$\Box$ satisfies normality:

$$\Box\top \qquad \Box(a \wedge b) \leftrightarrow (\Box a \wedge \Box b)$$

Everything is Relational

- Temporal logics
  - $\Box\phi =: G\phi$ = 'Generally/Forever ϕ'
- Epistemic logics
  - $\Box\phi$ = 'I know that ϕ'
- Logics of Belief
- Standard Deontic Logic
  - $\Box\phi =: O\phi$ = 'It is obligatory that ϕ'
- Description logic
  - $\mathrm{Mother} = \mathrm{Woman} \wedge \exists \mathrm{hasChild}.\top$


VICUS GALLORUM
BELGAE
LAUDANUM
AQUARIUM
BABAORUM
AREMORICA
GALLIA
a Romanis occupata
CELTAE
AQUITANIA

Everything?

Some Non-Normal Modal Logics

- Graded modal logic: $\Box_k \phi$ = 'with at most k exceptions ϕ'
  - Does have relational semantics, but better: multigraphs $\bullet \xrightarrow{n} \bullet$
- Probabilistic modal logic: $L_p \phi$ = 'with probability $\geq p$, ϕ'
- Agent logics $E_a \phi$ = 'agent a brings it about that ϕ'

$$\neg E_a \top$$

- Deontic logics for dilemmas:

$$O \neg \text{leave} \land O \text{leave} \not\to O \text{suicide}$$

- Conditional logic: $a \Rightarrow b$ 'if a, then normally b'

$$(\text{monday} \Rightarrow \text{bus}) \not\to (\text{monday} \land \text{strike} \Rightarrow \text{bus})$$

Modal Logic, Version 2

Neighbourhood semantics:

- Models (X, R, V) where
  - (X, R) neighbourhood frame, i.e.

$$R \subseteq X \times \mathcal{P}(X)$$

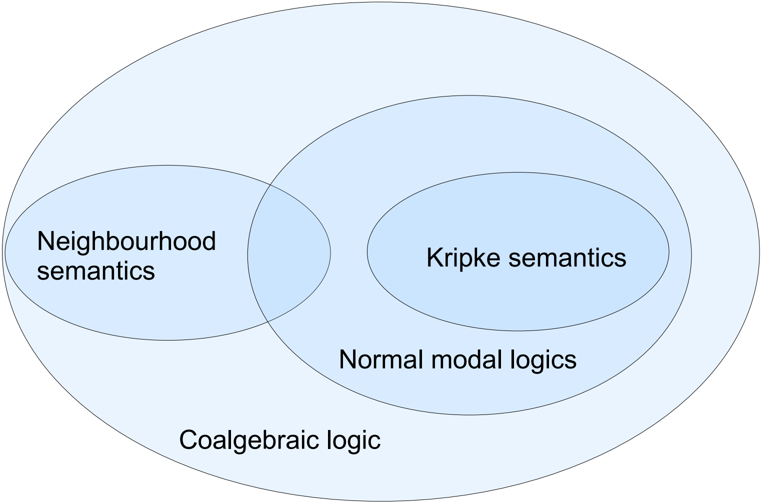
- $x \models \Box\phi$ iff $xR[\![\phi]\!]$ where $[\![\phi]\!] = \{y \in X \mid y \models \phi\}$.

This does cover nearly everything, but

- is unintuitive
- does not capture the intended semantics
  - and in fact gives up nearly all semantic structure
- is often unsuitable for metatheory and efficient reasoning.

$\rightarrow$ Look for a general framework that retains semantic structure

The Modal Semantics Landscape



A Reformulation of Kripke Semantics

- Frames are $\mathcal{P}$-coalgebras

$$\xi : X \to \underbrace{\mathcal{P}}_{\text{Functor}}(X)$$

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$$\xi(x) \in \{A \in \mathcal{P}(X) \mid A \subseteq [\![\phi]\!]\}$$

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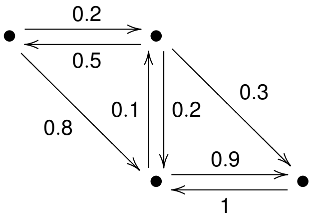
- $x \models \Box\phi$ iff

$$\xi(x) \in \{A \in \mathcal{P}(X) \mid A \subseteq [\![\phi]\!]\} =: \underbrace{[\![\Box]\!]_X}_{\text{Predicate Lifting}}([\![\phi]\!])$$

Example: Quantitative Uncertainty

Uncertain transitions

Probabilistic frames (Markov chains)



$$\xi : X \to D(X)$$

($D(X)$ = discrete probability distributions on X)

- Modal operators L_p 'in the next step, it holds with probability $\geq p$ that'
- $x \models L_p\phi$ iff

$$\xi(x) \in \{P \mid P[\![\phi]\!] \geq p\} =: [\![L_p]\!][\![\phi]\!].$$

- Variants: belief, possibility, linear inequalities, expectations, ...
- Applications:
  - Quantitative uncertainty (Halpern)
  - Rational agency (Heifetz/Mongin)
  - Black box bisimulation testing (Larsen/Skou)

Modal Logic, Version 3: Coalgebraic Modal Logic

- General modal similarity types (collections of modal operators)
- Abstract over the type of systems:
  - Set functor (parametrised datatype) $T : \mathbf{Set} \to \mathbf{Set}$
  - Systems = T-coalgebras

$$\xi : X \to TX$$

- Abstract over the interpretation of modal operators L:
  - Predicate liftings $[\![L]\!]_X : \mathcal{P}(X) \to \mathcal{P}(TX)$, natural in X
  - $x \models L\phi$ iff

$$\xi(x) \in [\![L]\!]_X([\![\phi]\!])$$

Nearly Everything is Coalgebraic

Logic	Systems	Syntax	Functor
Normal modal logics	Kripke frames	$\Box\phi$	powerset $\mathcal{P}(X)$
Probabilistic modal logics	Markov chains	$L_p\phi$ $\sum a_i P(\phi_i) \geq b$	distributions $D(X)$
Graded modal logics	Multigraphs	$\geq nR.\phi$ $\sum a_i \#(\phi_i) \geq b$	multisets $\mathcal{B}(X) = X \to \mathbb{N}_\infty$
Conditional logics	Conditional frames	$\phi \Rightarrow \psi$	selection functions $\mathcal{P}(X) \to \mathcal{P}(X)$
Classical modal logics	Neighbourhood frames	$\Box\phi$	neighbourhoods $\mathcal{P}(\mathcal{P}(X))$
Coalition logic	Game frames	$[C]\phi$	Games $\exists(S_i).(\prod S_i \to X)$

(Schröder/Pattinson/Cirstea/Kurz/Venema et al. 2004–2008)

So Can One Do Something With It?

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Sichere Kognitive Systeme

Yes, e.g.

- Finite models / weak completeness (Schröder FOSSACS 06)
- Cut elimination, interpolation
  (Pattinson/Schröder CMCS 08, TABLEAUX 09)
- Decidability
- Tight upper complexity bounds PSPACE/EXPTIME/NEXPTIME
  (Schröder 06, Schröder/Pattinson LICS 06, KR 08, KI 08)
- New results for specific logics, e.g.
  - PSPACE upper bounds for
    - Elgesem's logic of agency
    - Majority logic (priority race with Demri/Lugiez IJCAR 06)
    - Presburger modal logic over reflexive frames
  - New PSPACE algorithms for graded and probabilistic modal logics
  - New coNP upper bound for $CK + \{MP, CEM\}$

Heterogeneous Modal Logics

Real-life modelling tasks use multiple modalities:

„The risk of traffic congestion is normally higher on weekdays.“

Coalgebra supports

- Modular combination of logics – ‘Pick and choose’ – in parallel to
  - algorithms
  - implementations

Coalgebraic Hybrid Logic

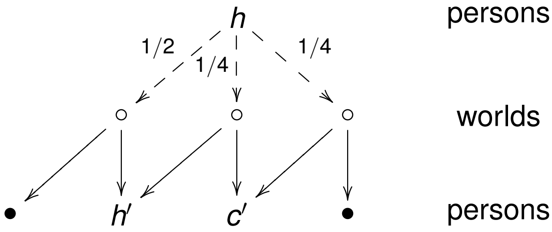
- Extended generic framework with nominals $i, j, \ldots$ and satisfaction operators $@_i \phi$ 'state i satisfies ϕ'
  - Internalisation of ABoxes
- E.g.

$$\models L_p i \wedge L_q j \rightarrow L_q i \vee L_1(\neg(i \wedge j))$$

  in hybrid probabilistic logic.
- Generic results:
  - Finite model property, weakly complete Hilbert calculus
  - Terminating internalised sequent calculus
  - Tight upper bound PSPACE via sequent calculus and semantically (Myers/Pattinson/Schröder FOSSACS 09)
  - Tight upper bound EXPTIME for TBox reasoning (Schröder/Pattinson/Kupke IJCAI 09)

The Uncertain Progeny of Henry VIII

h = Henry VIII
h' = Henry Carey
c' = Catherine Carey



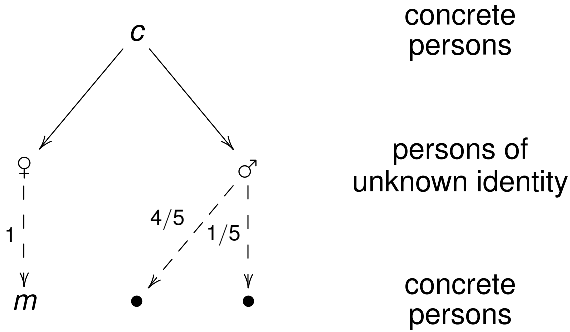
Combined TBox and ABox e.g.

$$\text{king} \to L_{4/5}\exists \text{child.\,illegitimate}$$

$$@_h L_{3/4}\exists \text{child.}\, h'$$

The Uncertain Progeny of Catherine of Aragon

c = Catherine of Aragon
m = Queen Mary



$$\text{queen} \rightarrow \forall \text{child}. \neg L_1 \text{illegitimate}$$

$$@_c \exists \text{child}. L_1 \text{queen}$$

The One-Step Logic

'Flat' logic without nesting of modal operators

Syntax: e.g. one-step pairs (η, ψ) over V:

η propositional over V

ψ conjunctive clause over atoms La, $a \in V$

Semantics: one-step models (X, τ, t) with

- X set, τ hybrid $\mathcal{P}(X)$-valuation
- $t \in TX$.

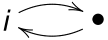
$$(X, \tau, t) \models (\eta, \psi) \text{ iff } [\![\eta]\!]\tau = X \text{ and } t \in [\![\psi]\!]\tau$$

Lemma Every satisfiable one-step pair has an exponential-sized one-step model.

Proof: Identify states whose propositional theories coincide.

Shallow Hybrid Models

- Shallow hybrid models are forests: one root per named state
- Trees are imperfect: $i \wedge \Diamond\Diamond i$



Theorem Every satisfiable hybrid formula ϕ is satisfiable in an exponentially branching shallow hybrid model.

PROOF (sketch):

- Closure Σ of ϕ under subformulas, $\neg$, $@$
- Expand ϕ to $@$-theory $K \subseteq \Sigma$, put $K_i = \{\phi \mid @_i\phi \in K\}$ (complete ABox)
- Eliminate $@$ from the K_i by substitution
- Lemma: K-satisfiable formulas are satisfiable in shallow K-fragments, i.e. partial coalgebras with hypothetical states K_i
- Satisfy the K_i by gluing of K-fragments.

Fragments

- Models $((X, \xi), V)$ with

$$\xi : X \to TX \text{ partial}$$

  and V hybrid valuation
- $K_i \in X$ for all i
- ξ undefined precisely on the states K_i
- $V(i) = \{K_i\}$
- fragment satisfaction: for $\rho \in \Sigma$,

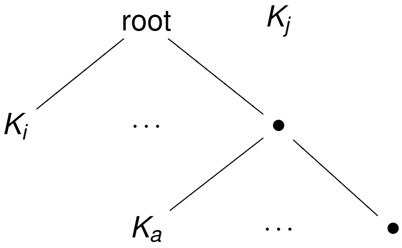
$$K_i \models \rho \iff \rho \in K_i$$

  usual clauses otherwise.

> **Lemma** Every @-free K-satisfiable $\psi \in \mathrm{Prop}(\Sigma)$ has an exponentially branching shallow tree-shaped K-fragment model.

Proof: Recursion over ψ, using exponential-sized one-step models.

A Fragment

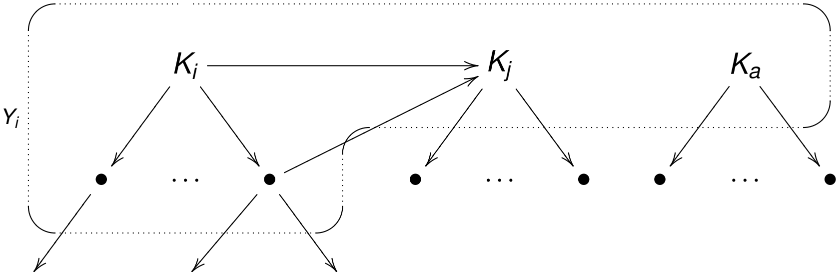


Gluing of Fragments

Sichere Kognitive Systeme

- Sub(Σ) = subformulas under modal operators in Σ
- Turn the K_i into (satisfiable) one-step pairs (η, ψ_i) over Sub(Σ)
  - propositional part η = propositional theory of Sub(Σ) + $i \models (K_i \cap \pm\text{Sub}(\Sigma))$
- Have exponential-sized one-step models (Y_i, τ_i, t_i) for (η, ψ_i) with $K_j \in Y_i$
- $\text{Th}_{\tau_i}(y)$ satisfiable for all $i, y \in Y_i$
- Thus have K-fragment models of the $\text{Th}_{\tau_i}(y)$; may assume root = y
- Form the union of these, identifying the K_i
  - contains the Y_i
- Fill in $\xi(K_i) = t_i$
- Prove by induction over formulas that $K_i \models K_i$.

A Shallow Model



PSPACE Via Shallow Hybrid Models

Definition The strict (lax) one-step satisfiability problem is to check satisfiability of one-step pairs (η, ψ), with input size $|\psi|$ $(|(\eta, \psi)|)$.

Theorem *If strict one-step satisfiability is in PSPACE, then hybrid satisfiability is in PSPACE.*

Proof: Traverse shallow models. In particular:

Algorithm: Check fragmentary K-satisfiability of $\psi \in \mathrm{Prop}(\Sigma)$

1. Decompose $\psi \equiv \psi_0\sigma$ with $\psi_0 \in \Lambda(V) \cup \neg\Lambda(V)$
2. Recursively compute the propositional theory η of σ as the disjunction of all conjunctive clauses χ such that $\chi\sigma$ is fragment K-satisfiable.
3. Check that $(\eta + (i \models (K_i \cap \mathrm{Sub}(\Sigma))), \psi_0)$ is one-step satisfiable

Theorem: If the one-step logic has the polysize model property, then strict one-step satisfiability is in PSPACE iff lax one-step satisfiability is in PSPACE.

Covers most applications, e.g.

- Conditional hybrid logics
- Hybrid coalition logic
- Probabilistic hybrid logic

except . . .

Graded Hybrid Logic

- Strict one-step satisfiability for graded/Presburger modalities is in PSPACE by (Papadimitriou 1981):
  - Integer linear inequalities have componentwise polysize solutions
  - Guess component of solution, add up multiplicities, forget component
- Thus, the combination of
  - Graded modalities/qualified number restrictions
  - Nominals / satisfaction operators
  - Mutiple roles / hierarchies

  is in PSPACE
- This is (more powerful than) the DL $\mathcal{ALCHOQ}$ with empty TBox.

Conclusions

- Coalgebra provides a uniform framework for modal and hybrid logics
  - Graded operators (knowledge representation, redundancy)
  - Probabilistic operators (quantitative uncertainty, reactive systems)
  - Conditional operators (nonmonotonic reasoning)
  - Coalition logic, logics of agency (multi-agent systems)
- Composition of coalgebraic logics helps in modelling various types of uncertainty
  - E.g. statistical vs. subjective probabilities
- Semantics-based upper bound PSPACE for coalgebraic hybrid logic
  - New bound for $\mathcal{ALCHOQ}$

Ongoing and Future Work

- Named Models for coalgebraic hybrid logic
  - Pure completeness, local binder $\downarrow x.\,\phi$
- First-order coalgebraic logic
  - Correspondence language for coalgebraic modal and hybrid logic
  - Generalised Rosen/van Benthem theorem
- Manydimensional coalgebraic logic (jointly with Carsten Lutz)
  - e.g. random Kripke models

$$\mu \in D(X \to \mathcal{P}(X))$$

  - Modal description logics
- Vision: generic, modular reasoning tools
  - Prototype CoLoSS (`http://www.informatik.uni-bremen.de/cofi/CoLoSS/`)

Reasoning with Global Assumptions

- Assume TBox Γ
- Γ-models: every state satisfies Γ.
- Goal: decide satisfiability of formulae in Γ-models
  - Known to be EXPTIME complete for $\mathcal{ALCHOQ}$ (Tobies 00)
- Problem: Tableaux do not terminate (without loop checks): for $\Gamma = \{\Diamond p\}$

$$\cfrac{\cfrac{\cfrac{\top, \Diamond p}{p, \Diamond p}}{p, \Diamond p}}{\dots}$$

  - Complicated analysis even for $\mathcal{ALC}$ (Donini/Massacci 99)

EXPTIME Upper Bound for CHL with TBox

Sichere Kognitive Systeme

Generic solution:

- Assume locally cutfree axiomatisation of modal part
  - known for many examples: graded, probabilistic, coalition logic, ...
  - Used in earlier work on generic PSPACE bounds
- Tableau graph contains an edge for every demand arising from possible rule applications
- Construct coalgebraic models on tableau graphs
- Reduce existence of tableau graph to existence of winning strategies in tableau game
- Under EXPTIME tractability (polynomial bound on codes of matching rules), game boards are exponential-sized
  - Decide existence of winning strategy in exponential time

[Schröder/Pattinson/Kupke IJCAI 09]

One-Step Rules

A one-step rule over V is a rule R of the form $\frac{\phi}{\psi}$, where

ϕ propositional over V (Rank 0)

ψ clause over atoms La, $a \in V$ (Rank 1)

Require that these are sound and cut-free complete for the one-step logic:

- No nesting of modal operators
- Semantics over models (X, τ, t) where
  - τ $\mathcal{P}(X)$-valuation
  - $t \in TX$

Example: One-Step Cutfree Complete Rules

Sichere Kognitive Systeme

K ($\Box$ with Kripke semantics):

$$\frac{\bigwedge_{i=1}^{n} a_i \to b}{\bigwedge_{i=1}^{n} \Box a_i \to \Box b} \quad (n \geq 0)$$

Graded modal logic:

$$\frac{\sum_{i=1}^{n} r_i a_i \geq 0}{\bigwedge_{r_i<0} \Diamond_{k_i} a_i \to \bigvee_{r_i>0} \Diamond_{k_i} a_i} \quad (\sum_{r_i<0} |r_i|(k_i+1) \geq 1 + \sum_{r_i>0} r_i k_i)$$

where

$$\sum_{i=1}^{n} r_i a_i \geq 0 \equiv \bigwedge_{\substack{J \subseteq \{1,\ldots,n\} \\ \sum_{j \in J} r_j < 0}} \Big(\bigwedge_{j \in J} a_j \to \bigvee_{j \notin J} a_j \Big)$$

Main Lemma

Tableau graph for formula ϕ over TBox Γ and complete ABox K

- Nodes: Sets H of subformulas of ϕ, Γ s.t.
  - $H \supseteq \Gamma$
  - if $i \in H$, then $\phi \in H \iff @_i\phi \in K$
- Whenever H falsifies the conclusion of a rule instance, there exists an edge to G falsifying one of the premises.

Theorem Under one-step cutfree completeness, every tableau graph supports a coalgebraic model of ϕ.

Corollary ϕ is satisfiable over Γ, K iff there exists a tableau graph for ϕ.