PSPACE Bounds for Rank 1 Modal Logic

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- Sufficiently intuitive for use in actual software specification (CCSL, CoCasl)
- Subsumes many modal logics, e.g.
    - Hennessy-Milner logic
    - Graded modal logic, majority
    - Probabilistic modal logic
    - Pauly's coalition logic

Complexity Results in CML

- Finite model construction for CML (LS, FOSSACS 06):
  - complexity results depend on complexity of one-step satisfiability (OSS) — e.g. solvability of linear inequations over rationals (PML) or integers (GML)
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  - $Satisfiability \in NEXPTIME$ if $OSS \in NP$ (GML)
  - $Satisfiability \in EXPTIME$ if $OSS \in P$ (PML) (LS/DP 2005)
- However: K, GML, Coalition Logic (but not PML) are known to be $PSPACE$-complete.

Today:

$$\mathit{Satisfiability} \in \mathit{PSPACE},$$

assuming a tractable complete set of one-step rules

Coalgebra

$T : \mathbf{Set} \to \mathbf{Set}$ functor (e.g. datatype)

Coalgebra (X, ξ) = map $\xi : X \to TX$

ξ: evolution map

$\xi(x)$: structured collection of successor states

Coalgebraic Modal Logic

(Pattinson 04) Predicate lifting for $T : \mathbf{Set} \to \mathbf{Set}$ = nat. transformation

$$\lambda : 2^{-} \to 2^{T^{op}}$$

Λ set of predicate liftings: $\mathcal{L}(\Lambda)$ defined by

$$\phi ::= \bot \mid \phi \wedge \psi \mid \neg\phi \mid [\lambda]\,\phi \quad (\lambda \in \Lambda)$$

Semantics in T-coalgebra (X, ξ):

$$x \models_{(X,\xi)} [\lambda]\,\phi \iff \xi(x) \in \lambda_X [\![\phi]\!]_{(X,\xi)}$$

Examples

- K: $TX = \mathcal{P}(X)$, $\lambda^{\forall}_X(A) = \mathcal{P}(A) \subset \mathcal{P}(X)$, $[\lambda^{\forall}] = \Box$
- Atomic Propositions: $TX = \mathcal{P}V$,
  $\lambda^a_X(A) = \{B \in \mathcal{P}(V) \mid a \in B\}$; $[\lambda^a]\phi = a$
- Graded Modal Logic: $TX =$ Bags $\sum n_i x_i$ over X,
  $\lambda^k_X(A) = \{\sum n_i x_i \mid \sum_{x_i \in A} n_i > k\}$
  $\longrightarrow$ operators $\Diamond_k = [\lambda^k]$
  $\Diamond_k \phi$: ϕ holds in more than k successor states

Example: Probabilistic Modal Logic

$D_\omega X =$ finitely supported probability distributions P over X;
$TX = D_\omega X \times \mathcal{P}(V)$.

T-coalgebras = probabilistic transition systems/
probabilistic type spaces (Heifetz/Mongin 01)
(similarly: reactive and generative probabilistic automata).

Liftings: atomic propositions plus, for $p \in [0,1] \cap \mathbb{Q}$,

$$\lambda^p(A) = \{P \mid PA \geq p\}.$$

→ operators $L_p = [\lambda^p]$ of probabilistic modal logic.

Example: Coalition Logic

(Pauly 2002)

$$TX = \exists\ \overbrace{\Sigma_1 \dots \Sigma_n}^{\text{sets of strategies}}\ .\ \underbrace{\prod \Sigma_i \to X}_{\text{outcome function}}\ ,$$

where $N = \{1, \dots, n\}$ set of agents.

For a coalition $C \subset N$,

$$\lambda^C A = \{f : \prod \Sigma_i \to X \mid \exists \sigma_C .\, \forall \sigma_{N-C} .\, f(\sigma_C, \sigma_{N-C}) \in A\}.$$

$\longrightarrow$ operators $[\lambda^C] = [C]$ of coalition logic

$[C]\phi =$ 'C can force ϕ'.

One-Step Rules

A one-step rule over V is a rule R of the form $\frac{\phi}{\psi}$, where

$$\phi \in \mathsf{Prop}(V) \qquad \text{(Rank 0)}$$

$$\psi \text{ clause over atoms } [\lambda]a,\ a \in V \qquad \text{(Rank 1)}$$

R one-step sound if

$$X \models \phi\tau \implies TX \models \psi\tau$$

for all $\mathcal{P}(X)$-valuations τ.

One-Step Completeness

Set $\mathcal{R}$ of rules (strictly) one-step complete if

for all $\mathfrak{A} \subset \mathcal{P}(X)$ and

every clause ϕ over atoms $[\lambda]A$, $A \in \mathfrak{A}$,

if $TX \models \phi$, then

ϕ is derivable using

Prop($\mathfrak{A}$)-instances ($\mathfrak{A}$-instances) of $\mathcal{R}$.

Deduction

Deduction system induced by $\mathcal{R}$:

- Propositional reasoning
- Instances of rules in $\mathcal{R}_C$, i.e. $\mathcal{R}$ plus congruence rule

$$\frac{a \leftrightarrow b}{[\lambda]a \leftrightarrow [\lambda]b}$$

Theorem Deduction is sound if $\mathcal{R}$ is one-step sound

Theorem Deduction is complete if $\mathcal{R}$ is one-step complete

Hintikka Sets

- Set Σ of formulae closed $\iff$
  - closed under subformulae and
  - closed under normalized negation $\sim$.
- Hintikka set $H \subset \Sigma$:
  - $\bot \notin H$,
  - $\phi \wedge \psi \in H \iff \phi \in H \wedge \psi \in H$ for $\phi \wedge \psi \in \Sigma$
  - $\neg \phi \in H \iff \phi \notin H$ for $\neg \phi \in \Sigma$.
- $\Sigma(\phi) =$ closure of $\{\phi\}$.

A Shallow Model Theorem

(Method of (Vardi, LICS 89))

Theorem $\mathcal{R}$ strictly one-step complete $\implies$

χ satisfiable iff

$\chi \in H$ for some $\Sigma(\chi)$-Hintikka set H such that

for $\phi/\psi \in \mathcal{R}_C$, σ subst., $\psi\sigma$ clause over $\Sigma(\chi)$,

$$\underbrace{H \models \neg\psi\sigma}_{\mathit{match}} \implies \neg\phi\sigma \text{ satisfiable.}$$

Proof: ‘construct’ a shallow tree model recursively from models for the $\neg\phi\sigma$ (non-constructive existence proof).

Finding Strictly One-Step Complete Sets

Theorem The set of all sound one-step rules is strictly one-step complete.

Corollary CML has the shallow model property.

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Rule resolution: $[\lambda]a \in \psi_1$, $\neg[\lambda]a \in \psi_2$

$$\frac{\phi_1}{\psi_1}, \quad \frac{\phi_2}{\psi_2} \rightsquigarrow \frac{\phi_1 \wedge \phi_2}{(\psi_1 \cup \psi_2) - \{[\lambda]a, \neg[\lambda]a\}}$$

Theorem Resolution closed one-step complete rule sets are strictly one-step complete.

Example: K

$$\frac{a}{\Box a} \qquad \frac{a \wedge b \to c}{\Box a \wedge \Box b \to \Box c}.$$

is one-step complete. Resolution closure:

$$\frac{(\bigwedge_{i=1}^{n} a_i) \to b}{(\bigwedge_{i=1}^{n} \Box a_i) \to \Box b} \; (n \geq 0)$$

Example: Probabilistic Modal Logic

$$\frac{\overbrace{\sum_{i=1}^m 1_{a_i} - \sum_{j=1}^n 1_{b_j} \geq k}^{\text{arithmetic of characteristic functions}}}{\bigwedge_{j=1}^n L_{q_j} b_j \to \bigvee_{i=1}^m L_{p_i} a_i} \quad (k \in \mathbb{Z}),$$

with side condition

$$\begin{aligned} \sum_{i=1}^m p_i - \sum_{j=1}^n q_j &\leq k, \text{ and} \\ \text{if } m = 0 \text{ then } -\sum_{j=1}^n q_j &< k, \end{aligned}$$

is resolution closed (and strictly one-step complete).

Reduction Closed Rule Sets

Want to match only reduced $\Sigma(\chi)$-clauses, i.e. clauses consisting of pairwise distinct literals. (Otherwise get infinitely many matching rules.)

Definition $\mathcal{R}$ reduction closed if every rule instance can be replaced by a rule instance with reduced conclusion, i.e.

$$\frac{\phi}{\ldots \vee \epsilon[\lambda]a \vee \epsilon[\lambda]b} \in \mathcal{R} \implies \exists \frac{\phi'}{\ldots \vee \epsilon[\lambda]a} \in \mathcal{R}.\ \phi[a/b] \to \phi'.$$

Example: K

$$\frac{(\bigwedge_{i=1}^{n} a_i) \rightarrow b}{(\bigwedge_{i=1}^{n} \Box a_i) \rightarrow \Box b}$$

is reduction closed.

Example: Probabilistic Modal Logic

Closure under reduction:

$$\frac{\sum_{i=1}^{n} r_i 1_{a_i} \geq k}{\bigvee_{1 \leq i \leq n} sgn(r_i) L_{p_i} a_i}$$

$(n \geq 1,\ r_1, \dots, r_n \in \mathbb{Z} - \{0\},\ k \in \mathbb{Z})$, with side condition

$$\begin{aligned} \textstyle\sum_{i=1}^{n} r_i p_i \leq k, \ \ \text{and} \\ \text{if } \forall i.\, r_i < 0, \text{ then} \textstyle\sum_{i=1}^{n} r_i p_i < k. \end{aligned}$$

The PSPACE Algorithm

1. Guess a $\Sigma(\chi)$-Hintikka set H.
2. Recursively and nondeterministically check

$$\left.\begin{array}{l}\phi/\psi \in \mathcal{R}_C \\ H \models \neg\psi\sigma\end{array}\right\} \implies \neg\phi\sigma \text{ satisfiable.}$$

   for $\psi\sigma$ reduced.

This is in APTIME=PSPACE if the matching between reduced clauses over H and (codes of) rules is PSPACE-tractable.

PSPACE-Tractability

Represent rules (modulo propositional equivalence of premises) by codes

Definition $\mathcal{R}$ is PSPACE-tractable if

- rules matching a reduced clause ρ have codes of size polynomially bounded by $|\rho|$
- It can be decided in NP whether
    - a given code represents a rule in $\mathcal{R}$
    - a given rule matches a given reduced clause
    - a given clause belongs to the CNF of the premise of a given rule

Matching for PML

- Rule codes: $(r_1, \ldots, r_n, p_1, \ldots, p_n, k)$
- Equivalence of premises & Satisfaction of side condition: linear inequations.
- Thus: polynomially bounded solution (standard linear programming)
- CNF of premise:

$$\sum_{i \in I} r_i \phi_i \geq k \equiv \bigwedge_{r(J) < k} \Big(\bigwedge_{j \in J} \phi_j \to \bigvee_{j \notin J} \phi_j \Big),$$

Thus: PML is in PSPACE

Conclusion

- Coalgebraic modal logic has the shallow model property
- Obtain *PSPACE* algorithm for satisfiability, given a tractable ‘saturated’ axiomatization
- Recover known results:
  K, GML, Coalition Logic are in PSPACE
- New (easier?) algorithm for GML
- New tight bound: PML is in PSPACE

Future Work

- Semantics-free criterion for PSPACE decidability in combination with MCS model
- Compositional method: multisorted modal logic
- How do we tackle rank n?