PSPACE Bounds for Rank 1 Modal Logics

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LICS 06

Introduction

- Complexity of 'static' modal logics typically $PSPACE$, e.g.
  - K (KB, $S4$, $\ldots$): witness algorithm for shallow Kripke models
  - Graded modal logic (GML): constraint set algorithm (Tobies 01)
  - Logic of knowledge and probability: shallow model method based on local small model property (Fagin/Halpern 94)
  - Epistemic logic (Vardi 89), coalition logic (Pauly 02): shallow neighbourhood models.
- Generalize method of (Vardi 89) to arbitrary rank-1 logics
- Obtain uniform shallow-model based $PSPACE$ algorithm
- Semantic basis: coalgebraic modal logic

Disclaimer

- $PSPACE$ completeness of probabilistic modal logic follows from earlier results by Fagin/Halpern.

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- To make up, we prove $PSPACE$ completeness of majority logic (Pacuit/Salame KR 2004).

Coalgebra

$T : \mathbf{Set} \to \mathbf{Set}$ functor (e.g. datatype)

Coalgebra (X, ξ) = map $\xi : X \to TX$

ξ: transition map

$\xi(x)$: structured collection of observations/successor states

Coalgebraic Modal Logic

(Pattinson 04) Predicate lifting for $T : \mathbf{Set} \to \mathbf{Set} =$ nat. transformation

$$\lambda : 2^{-} \to 2^{T^{op}}$$

Λ set of predicate liftings:

$$\phi ::= \bot \mid \phi \wedge \psi \mid \neg \phi \mid [\lambda]\,\phi \qquad (\lambda \in \Lambda)$$

Semantics in T-coalgebra (X, ξ):

$$x \models_{(X,\xi)} [\lambda]\,\phi \iff \xi(x) \in \lambda_X [\![\phi]\!]_{(X,\xi)}$$

Examples

- K: $TX = \mathcal{P}(X)$, $\lambda^{\forall}_X(A) = \mathcal{P}(A) \subset \mathcal{P}(X)$, $[\lambda^{\forall}] = \Box$
- Atomic Propositions: $TX = \mathcal{P}V$,
  $\lambda^a_X(A) = \{B \in \mathcal{P}(V) \mid a \in B\}$; $[\lambda^a]\phi = a$
- Probabilistic Modal Logic:
  $D_\omega X$ = finitely supported probability measures P over X;
  $TX = D_\omega X \times \mathcal{P}(V)$;
  $\lambda^p(A) = \{(P, B) \mid PA \geq p\}$; $[\lambda^p] = L_p$
  $L_p\phi$ = 'ϕ holds in the next step with probability $\geq p$'
- Coalition Logic: $[C]\phi$ 'coalition C can force ϕ'.

Example: Majority Logic

$$TX = \text{Bags } \textstyle\sum n_i x_i \text{ over } X$$

$$\lambda^k_X(A) = \{\textstyle\sum n_i x_i \mid \sum_{x_i \in A} n_i > k\}, \quad k \geq 0$$

$$\lambda^W_X(A) = \{\textstyle\sum n_i x_i \mid \sum_{x_i \in A} n_i \geq \sum_{x_i \notin A} n_i\}$$

→ operators $\Diamond_k = [\lambda^k]$ of graded modal logic:

$\Diamond_k \phi =$ 'ϕ holds in more than k successor states',

plus weak majority operator $W = [\lambda^W]$

$W\phi =$ 'ϕ holds in at least half of the successor states'

One-Step Rules

A one-step rule over V is a rule R of the form $\frac{\phi}{\psi}$, where

$$\phi \in \mathsf{Prop}(V) \qquad \text{(Rank 0)}$$

$$\psi \text{ clause over atoms } [\lambda]a,\ a \in V \qquad \text{(Rank 1)}$$

R one-step sound if

$$X \models \phi\tau \implies TX \models \psi\tau$$

for all $\mathcal{P}(X)$-valuations τ.

Congruence rule: **(C)** $\dfrac{a \leftrightarrow b}{[\lambda]a \leftrightarrow [\lambda]b}$

One-Step Completeness

Set $\mathcal{R}$ of rules (strictly) one-step complete if

for all $\mathfrak{A} \subset \mathcal{P}(X)$ and

every clause ϕ over atoms $[\lambda]A$, $A \in \mathfrak{A}$,

if $TX \models \phi$, then

ϕ is derivable using

Prop($\mathfrak{A}$)-instances ($\mathfrak{A}$-instances) of $\mathcal{R}$.

Hintikka Sets

- Set Σ of formulae closed $\iff$
    - closed under subformulae and
    - closed under normalized negation $\sim$.
- Hintikka set $H \subset \Sigma$:
    - $\bot \notin H$,
    - $\phi \wedge \psi \in H \iff \phi \in H \wedge \psi \in H$ for $\phi \wedge \psi \in \Sigma$
    - $\neg\phi \in H \iff \phi \notin H$ for $\neg\phi \in \Sigma$.
- $\Sigma(\phi) =$ closure of $\{\phi\}$.

A Shallow Model Theorem

(Method of (Vardi, LICS 89))

Theorem $\mathcal{R}$ strictly one-step complete $\implies$

χ satisfiable iff

$\chi \in H$ for some $\Sigma(\chi)$-Hintikka set H such that

for $\phi/\psi \in \mathcal{R} \cup \{C\}$, σ subst., $\psi\sigma$ clause over $\Sigma(\chi)$,

$$\underbrace{H \models \neg\psi\sigma}_{match} \implies \neg\phi\sigma \text{ satisfiable.}$$

Proof: 'construct' a shallow tree model recursively from models for the $\neg\phi\sigma$ (non-constructive existence proof).

Finding Strictly One-Step Complete Sets

Theorem The set of all sound one-step rules is strictly one-step complete.

Corollary CML has the shallow model property.

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Rule resolution: $[\lambda]a \in \psi_1$, $\neg[\lambda]a \in \psi_2$

$$\frac{\phi_1}{\psi_1}, \quad \frac{\phi_2}{\psi_2} \rightsquigarrow \frac{\phi_1 \wedge \phi_2}{(\psi_1 \cup \psi_2) - \{[\lambda]a, \neg[\lambda]a\}}$$

Theorem Resolution closed & one-step complete $\implies$ strictly one-step complete.

Example: K

One-step complete rule set:

$$\frac{a}{\Box a} \qquad \frac{a \wedge b \to c}{\Box a \wedge \Box b \to \Box c}.$$

Resolution closure:

$$\frac{(\bigwedge_{i=1}^{n} a_i) \to b}{(\bigwedge_{i=1}^{n} \Box a_i) \to \Box b}\ (n \geq 0)$$

Example: Majority Logic

Resolution closed one-step complete rule set:

$$\frac{\overbrace{\sum a_i + \sum_1^v c_r + m \leq \sum b_j + \sum_1^w d_s}^{\text{arithmetic of characteristic functions}}}{\bigwedge \Diamond_{k_i} a_i \wedge \bigwedge W c_r \to \bigvee \Diamond_{l_j} b_j \vee \bigvee W d_s} \quad (m \in \mathbb{Z})$$

with side condition

$$\begin{aligned} \sum (k_i + 1) - \sum l_j + w - 1 - \max(m, 0) \geq 0 \\ v - w + 2m \geq 0. \end{aligned}$$

Obtained from known one-step complete set of 7 axioms.

Decidability in PSPACE

- Close rules under removal of duplicate literals in conclusions
  - Avoids big rules matching small clauses
  - Possible blowup of the rule set
- Require tractability of the rule set
  - Represent rules by codes, up to equivalence of premises
  - Side conditions, clauses of premise, validity of codes in NP
  - Polynomial bound on codes of matching rules
- Traverse shallow model in $APTIME = PSPACE$

Example: Majority Logic

- Closure under reduction:

$$\frac{m \leq \sum r_i a_i + \sum s_j b_j}{\bigvee sgn(r_i)\Diamond_{k_i} a_i \vee \bigvee sgn(s_j) W b_j} \; (r_i, s_j \in \mathbb{Z})$$

(with correspondingly modified side condition)

- Tractability:
  - Equivalence of premises & satisfaction of side condition: linear inequations.
  - Thus: polynomially bounded solution (standard linear programming)

Next: $PSPACE$, semantically

- Semantic criterion:
  - Strong one-step small model property & tractable one-step model checking $\implies$ $PSPACE$
  - Better bound on branching in shallow models
  - Off-the-shelf application to logics of uncertainty (Halpern/Pucella)

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  - Off-the-shelf application to logics of uncertainty (Halpern/Pucella)
- Merits of the above ‘syntactic’ criterion:
  - Potentially handles exponential branching
  - Algorithm computes shallow proof that witnesses
    - ▷ a weak subformula property
    - ▷ encapsulation of cuts in the rule set
  - E.g., obtain complete axiomatization for W alone!

Conclusion

- Coalgebraic modal logic has the shallow model property
- Obtain $PSPACE$ algorithm for satisfiability, given a tractable 'saturated' axiomatization
- Recover known results:
  K, GML, PML, Coalition Logic are in $PSPACE$
- New (easier?) algorithm for GML
- New tight bound: Majority logic is in $PSPACE$

Future Work

- Semantic $PSPACE$ criterion (see above)
- Compositionality
- Semantics-free approach:
    - construct functor for given rank 1 logic
    - obtain fmp, $PSPACE$ bound, proof theoretic properties. . .
- Coalgebraic CTL
- How do we tackle rank n?

Coalition Logic

(Pauly 2002)

$$TX = \exists \overbrace{\Sigma_1 \dots \Sigma_n}^{\text{sets of strategies}} . \underbrace{\prod \Sigma_i \to X}_{\text{outcome function}},$$

where $N = \{1, \dots, n\}$ set of agents.

For a coalition $C \subset N$,

$$\lambda^C A = \{f : \prod \Sigma_i \to X \mid \exists \sigma_C . \forall \sigma_{N-C} . f(\sigma_C, \sigma_{N-C}) \in A\}.$$

$\to$ operators $[\lambda^C] = [C]$ of coalition logic

$[C]\phi =$ ‘C can force ϕ’.

PSPACE-Tractability

Represent rules (modulo propositional equivalence of premises) by codes

Definition $\mathcal{R}$ is PSPACE-tractable if

- rules matching a reduced clause ρ have codes of size polynomially bounded by $|\rho|$
- It can be decided in NP whether
  - a given code represents a rule in $\mathcal{R}$
  - a given rule matches a given reduced clause
  - a given clause belongs to the CNF of the premise of a given rule

Matching for Majority Logic

- Rule codes: $((r_i),(s_j),(k_i),m)$
- Equivalence of premises & Satisfaction of side condition: linear inequations.
- Thus: polynomially bounded solution (standard linear programming)
- CNF of premise:

$$m \leq \sum_{i \in I} r_i c_i \equiv \bigwedge_{r(J)<k} \Big(\bigwedge_{j \in J} c_j \to \bigvee_{j \notin J} c_j \Big),$$

Deduction

Deduction system induced by $\mathcal{R}$:

- Propositional reasoning
- Instances of rules in $\mathcal{R} \cup \{C\}$, i.e. $\mathcal{R}$ plus congruence rule

$$\frac{a \leftrightarrow b}{[\lambda]a \leftrightarrow [\lambda]b}$$

Theorem Deduction is sound if $\mathcal{R}$ is one-step sound

Theorem Deduction is complete if $\mathcal{R}$ is one-step complete