PSPACE Bounds for Rank 1 Modal Logics

Lutz Schröder and Dirk Pattinson
Introduction

• Complexity of ‘static’ modal logics typically $PSPACE$, e.g.
  - $K (KB, S4, \ldots )$: witness algorithm for shallow Kripke models
  - Graded modal logic (GML): constraint set algorithm (Tobies 01)
  - Logic of knowledge and probability: shallow model method based on local small model property (Fagin/Halpern 94)
  - Epistemic logic (Vardi 89), coalition logic (Pauly 02): shallow neighbourhood models.

• Generalize method of (Vardi 89) to arbitrary rank-1 logics
• Obtain uniform shallow-model based $PSPACE$ algorithm
• Semantic basis: coalgebraic modal logic
Disclaimer

- $\textit{PSPACE}$ completeness of probabilistic modal logic follows from earlier results by Fagin/Halpern.
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- To make up, we prove $PSPACE$ completeness of majority logic (Pacuit/Salame KR 2004).
Coalgebra

\( T : \text{Set} \rightarrow \text{Set} \) functor (e.g. datatype)

Coalgebra \((X, \xi) = \text{map } \xi : X \rightarrow TX\)

\(\xi\): transition map

\(\xi(x)\): structured collection of observations/successor states
Coalgebraic Modal Logic

(Pattinson 04) **Predicate lifting** for $T : \text{Set} \rightarrow \text{Set} = \text{nat. transformation}

$$\lambda : 2^T \rightarrow 2^{T^\text{op}}$$

$\Lambda$ set of predicate liftings:

$$\phi ::= \bot \mid \phi \land \psi \mid \neg \phi \mid [\lambda] \phi \ (\lambda \in \Lambda)$$

Semantics in $T$-coalgebra $(X, \xi)$:

$$x \models_{(X, \xi)} [\lambda] \phi \iff \xi(x) \in \lambda X [\phi]_{(X, \xi)}$$
Examples

- **K**: $TX = \mathcal{P}(X)$, $\lambda_X^\forall(A) = \mathcal{P}(A) \subseteq \mathcal{P}(X)$, $[\lambda^\forall] = \Box$

- Atomic Propositions: $TX = \mathcal{P}V$, $\lambda^a_X(A) = \{ B \in \mathcal{P}(V) \mid a \in B \}$; $[\lambda^a] \phi = a$

- Probabilistic Modal Logic:
  $D_\omega X = \text{finitely supported probability measures } P \text{ over } X$; $TX = D_\omega X \times \mathcal{P}(V)$; $\lambda^p(A) = \{(P,B) \mid PA \geq p\}$; $[\lambda^p] = L_p$; $L_p \phi = \text{‘} \phi \text{ holds in the next step with probability } \geq p \text{’}$

- Coalition Logic: $[C] \phi \text{ ‘coalition } C \text{ can force } \phi \text{’}$. 
Example: Majority Logic

\[ TX = \text{Bags } \sum n_i x_i \text{ over } X \]

\[ \lambda^k_X(A) = \{ \sum n_i x_i \mid \sum_{x_i \in A} n_i > k \}, \quad k \geq 0 \]

\[ \lambda^W_X(A) = \{ \sum n_i x_i \mid \sum_{x_i \in A} n_i \geq \sum_{x_i \notin A} n_i \} \]

→ operators \[ \Diamond_k = [\lambda^k] \] of graded modal logic:

\[ \Diamond_k \phi = \text{‘} \phi \text{ holds in more than } k \text{ successor states’}, \]

plus weak majority operator \[ W = [\lambda^W] \]

\[ W \phi = \text{‘} \phi \text{ holds in at least half of the successor states’} \]
One-Step Rules

A one-step rule over $V$ is a rule $R$ of the form $\frac{\phi}{\psi}$, where

$\phi \in \text{Prop}(V)$ \hspace{1cm} (Rank 0)

$\psi$ clause over atoms $[\lambda]a$, $a \in V$ \hspace{1cm} (Rank 1)

$R$ one-step sound if

$$X \models \phi_\tau \implies TX \models \psi_\tau$$

for all $\mathcal{P}(X)$-valuations $\tau$.

Congruence rule: (C) $\frac{a \leftrightarrow b}{[\lambda]a \leftrightarrow [\lambda]b}$
One-Step Completeness

Set $\mathcal{R}$ of rules (strictly) one-step complete if for all $\mathfrak{A} \subset \mathcal{P}(X)$ and every clause $\phi$ over atoms $[\lambda]A$, $A \in \mathfrak{A}$, if $TX \models \phi$, then

$\phi$ is derivable using $\text{Prop}(\mathfrak{A})$-instances ($\mathfrak{A}$-instances) of $\mathcal{R}$. 
Hintikka Sets

• Set $\Sigma$ of formulae closed $\iff$
  - closed under subformulae and
  - closed under normalized negation $\sim$.

• Hintikka set $H \subset \Sigma$:
  - $\bot \notin H$,
  - $\phi \land \psi \in H \iff \phi \in H \land \psi \in H$ for $\phi \land \psi \in \Sigma$
  - $\neg \phi \in H \iff \phi \notin H$ for $\neg \phi \in \Sigma$.

• $\Sigma(\phi) =$ closure of $\{\phi\}$. 
A Shallow Model Theorem

(Method of (Vardi, LICS 89))

**Theorem** \( \mathcal{R} \) strictly one-step complete \( \Rightarrow \)

\( \chi \) satisfiable iff

\[ \chi \in H \text{ for some } \Sigma(\chi) \text{-Hintikka set } H \text{ such that } \]

\[ \text{for } \phi/\psi \in \mathcal{R} \cup \{C\}, \sigma \text{ subst., } \psi\sigma \text{ clause over } \Sigma(\chi), \]

\[ H \models \neg\psi\sigma \quad \Rightarrow \quad \neg\phi\sigma \text{ satisfiable.} \]

**Proof:** ‘construct’ a shallow tree model recursively from models for the \( \neg\phi\sigma \) (non-constructive existence proof).
Finding Strictly One-Step Complete Sets

**Theorem** The set of all sound one-step rules is strictly one-step complete.

**Corollary** CML has the shallow model property.
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**Rule resolution:** \([\lambda]a \in \psi_1, \neg[\lambda]a \in \psi_2\)

\[
\frac{\phi_1}{\psi_1}, \quad \frac{\phi_2}{\psi_2} \sim \frac{\phi_1 \land \phi_2}{(\psi_1 \cup \psi_2) - \{[\lambda]a, \neg[\lambda]a\}}
\]

**Theorem** Resolution closed & one-step complete \(\implies\) strictly one-step complete.
Example: \( K \)

One-step complete rule set:

\[
\frac{a}{\Box a} \quad \frac{a \land b \rightarrow c}{\Box a \land \Box b \rightarrow \Box c}.
\]

Resolution closure:

\[
\frac{\bigwedge_{i=1}^{n} a_i \rightarrow b}{\bigwedge_{i=1}^{n} \Box a_i \rightarrow \Box b (n \geq 0)}
\]
Example: Majority Logic

Resolution closed one-step complete rule set:

\[
\sum a_i + \sum_1^v c_r + m \leq \sum b_j + \sum_1^w d_s \quad (m \in \mathbb{Z})
\]

\[
\land \Diamond k_i a_i \land \land W c_r \rightarrow \lor \Diamond l_j b_j \lor \lor W d_s
\]

with side condition

\[
\sum (k_i + 1) - \sum l_j + w - 1 - \max(m, 0) \geq 0
\]
\[
v - w + 2m \geq 0.
\]

Obtained from known one-step complete set of 7 axioms.
Decidability in PSPACE

• Close rules under removal of duplicate literals in conclusions
  ○ Avoids big rules matching small clauses
  ○ Possible blowup of the rule set

• Require tractability of the rule set
  ○ Represent rules by codes, up to equivalence of premises
  ○ Side conditions, clauses of premise, validity of codes in $NP$
  ○ Polynomial bound on codes of matching rules

• Traverse shallow model in $APTIME = PSPACE$
Example: Majority Logic

- **Closure under reduction:**

\[
m \leq \sum r_i a_i + \sum s_j b_j \\
\bigvee \text{sgn}(r_i) \bigtriangleup k_i a_i \lor \bigvee \text{sgn}(s_j) Wb_j \quad (r_i, s_j \in \mathbb{Z})
\]

(with correspondingly modified side condition)

- **Tractability:**
  - Equivalence of premises & satisfaction of side condition: linear inequations.
  - Thus: polynomially bounded solution (standard linear programming)
Next: $PSPACE$, semantically

- Semantic criterion:
  - Strong one-step small model property & tractable one-step model checking $\Rightarrow PSPACE$
  - Better bound on branching in shallow models
  - Off-the-shelf application to logics of uncertainty (Halpern/Pucella)
Next: \textit{PSPACE}, semantically

\begin{itemize}
\item Semantic criterion:
  \begin{itemize}
  \item Strong \textit{one-step small model property} \& \\
  \textit{tractable one-step model checking} $\implies \textit{PSPACE}$
  \item Better bound on branching in shallow models
  \item Off-the-shelf application to logics of uncertainty (Halpern/Pucella)
  \end{itemize}
\item Merits of the above ‘syntactic’ criterion:
  \begin{itemize}
  \item Potentially handles exponential branching
  \item Algorithm computes shallow proof that witnesses
    \begin{itemize}
    \item a weak \textit{subformula property}
    \item encapsulation of \textit{cuts} in the rule set
    \end{itemize}
  \item E.g., obtain complete axiomatization for $W$ alone!
  \end{itemize}
\end{itemize}
Conclusion

• Coalgebraic modal logic has the shallow model property

• Obtain \textit{PSPACE} algorithm for satisfiability, given a tractable ‘saturated’ axiomatization

• Recover known results: \( K \), GML, PML, Coalition Logic are in \textit{PSPACE}

• New (easier?) algorithm for GML

• \textbf{New tight bound:} Majority logic is in \textit{PSPACE}
Future Work

• Semantic $PSPACE$ criterion (see above)

• Compositionality

• Semantics-free approach:
  ○ construct functor for given rank 1 logic
  ○ obtain fmp, $PSPACE$ bound, proof theoretic properties. . .

• Coalgebraic CTL

• How do we tackle rank $n$?
Coalition Logic

(Pauly 2002)

\[ TX = \exists \left\{ \Sigma_1 \ldots \Sigma_n \right\} \cdot \prod \Sigma_i \rightarrow X, \]

where \( N = \{1, \ldots, n\} \) set of agents.

For a coalition \( C \subset N \),

\[ \lambda^C A = \{ f : \prod \Sigma_i \rightarrow X \mid \exists \sigma_C. \forall \sigma_{N-C}. f(\sigma_C, \sigma_{N-C}) \in A \}. \]

\( \lambda^C \) operators \([\lambda^C] = [C]\) of coalition logic

\([C] \phi = \text{‘}C\text{ can force } \phi\text{’}.\)
**PSPACE-Tractability**

Represent rules (modulo propositional equivalence of premises) by codes

**Definition** \( \mathcal{R} \) is PSPACE-tractable if

- rules matching a reduced clause \( \rho \) have codes of size polynomially bounded by \( |\rho| \)
- It can be decided in NP whether
  - a given code represents a rule in \( \mathcal{R} \)
  - a given rule matches a given reduced clause
  - a given clause belongs to the CNF of the premise of a given rule

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L. Schröder and D. Pattinson: PSPACE Bounds for Rank 1 Modal Logics; LICS 06
Matching for Majority Logic

- Rule codes: \(((r_i), (s_j), (k_i), m)\)
- Equivalence of premises & Satisfaction of side condition: linear inequations.
- Thus: polynomially bounded solution (standard linear programming)
- CNF of premise:

\[
m \leq \sum_{i \in I} r_i c_i \equiv \bigwedge_{r(J)<k} \left( \bigwedge_{j \in J} c_j \rightarrow \bigvee_{j \notin J} c_j \right),
\]
Deduction

Deduction system induced by $\mathcal{R}$:

- Propositional reasoning
- Instances of rules in $\mathcal{R} \cup \{C\}$, i.e. $\mathcal{R}$ plus congruence rule

$$a \leftrightarrow b$$

$$\frac{[\lambda]a \leftrightarrow [\lambda]b}{[\lambda]a \leftrightarrow [\lambda]b}$$

Theorem Deduction is sound if $\mathcal{R}$ is one-step sound

Theorem Deduction is complete if $\mathcal{R}$ is one-step complete