The dual rings of an R-coring revisited

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Abstract

It is shown that for every monoidal bi-closed category $\mathbb{C}$ left and right (semi)-dualization by means of the unit object not only defines a pair of adjoint functors, but that these functors are monoidal as functors from $\mathbb{C}^{\mathsf{op}}$, the dual monoidal category of $\mathbb{C}$ into the transposed monoidal category $\mathbb{C}^t$. We, thus, generalize the case of a symmetric monoidal category, where this kind of dualization is a special instance of convolution. We apply this construction to the monoidal category of bimodules over a not necessarily commutative ring R and so obtain various contravariant dual ring functors defined on the category of R-corings. It becomes evident that previous, hitherto apparently unrelated constructions of this kind are all special instances of our construction and, hence, coincide. Finally we show that Sweedler's Dual Coring Theorem is a simple consequence of our approach and that these dual ring constructions are compatible with the processes of (co)freely adjoining (co)units.

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Introduction

The purpose of this analysis of dual rings of R-corings is twofold. First we compare and explain conceptually the known constructions of such rings as given in [16] and [17] (see also [7]). Second we apply the methods of [13] and prove that these dualizations are compatible with the operations of adjoining units and counits to R-rings and R-corings respectively.

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Concerning the first mentioned topic recall that Sweedler in [16] constructs, for a given R-coring $\mathsf{C} = (C, \Delta, \epsilon)$, i.e., a comonoid in the monoidal category ${}_R\mathbb{M}\mathrm{od}_R$ of R-R-bimodules (with tensor product $- \otimes_R -$ over R and $\lambda_C \colon R \otimes_R C \to C$ and $\rho_C \colon C \otimes_R R \to C$ the canonical isomorphisms), left and right dual unital rings

$\mathsf{Sw}_l(\mathsf{C}) := ({}^*C, m_l, u_l)$, with ${}^*C := {}_R\hom(C, R)$, the abelian group of left R-module homomorphisms from C to R, multiplication m_l acting on pairs $(\mu, \nu) \in {}^*C \times {}^*C$ as $m_l(\mu, \nu) = \mu \circ \rho_C \circ (C \otimes_R \nu) \circ \Delta$ and unital element $u_l = \epsilon$.

$\mathsf{Sw}_r(\mathsf{C}) := (C^*, m_r, u_r)$ with $C^* := \hom_R(C, R)$, the abelian group of right R-module homomorphisms from C to R, multiplication m_r acting on pairs $(\mu, \nu) \in C^* \times C^*$ as $m_r(\mu, \nu) = \nu \circ \lambda_C \circ (\mu \otimes_R C) \circ \Delta$ and unital element $u_r = \epsilon$.

These rings are shown to be equipped with ring antihomorphisms[1] $\lambda_{\mathsf{C}} \colon R \to \mathsf{Sw}_l(\mathsf{C})$ and $\rho_{\mathsf{C}} \colon R \to \mathsf{Sw}_r(\mathsf{C})$ respectively, where $\lambda_{\mathsf{C}}(b)(c) = \epsilon(cb)$ and $\rho_{\mathsf{C}}(b)(c) = \epsilon(bc)$ for each $c \in C$ and $b \in R$.

Thus, $(\mathsf{Sw}_l(\mathsf{C}), \lambda_{\mathsf{C}})$ and $(\mathsf{Sw}_r(\mathsf{C}), \rho_{\mathsf{C}})$ in fact are objects in the comma category $R^{\mathsf{op}} \downarrow {}_1\mathbf{Ring}$. The assignments $\mathsf{C} \mapsto (\mathsf{Sw}_l(\mathsf{C}), \lambda_{\mathsf{C}})$ and $\mathsf{C} \mapsto (\mathsf{Sw}_r(\mathsf{C}), \rho_{\mathsf{C}})$ moreover are shown to define (contravariant) functors from ${}_\epsilon\mathbf{Coring} := \mathbf{Comon}({}_R\mathbb{M}\mathrm{od}_R)$ into $R^{\mathsf{op}} \downarrow {}_1\mathbf{Ring}$.

Recall also that Takeuchi in [17] in a more categorical way defines a contravariant functor D_l from ${}_\epsilon\mathbf{Coring}$ into the category $\mathbf{Mon}({}_{R^{\mathsf{op}}}\mathbb{M}\mathrm{od}_{R^{\mathsf{op}}})$ of monoids in the monoidal category ${}_{R^{\mathsf{op}}}\mathbb{M}\mathrm{od}_{R^{\mathsf{op}}}$ of R^{op}-R^{op}-bimodules (with tensor product $- \otimes_{R^{\mathsf{op}}} -$ over R^{op}), where the underlying abelian group of $\mathsf{D}_l(\mathsf{C})$ again is ${}^*C$. Neither Takeuchi nor the more recent survey [7] relate the latter construction to Sweedler's.

As we are going to show in Section 2.2, both constructions are essentially the same and imply Sweedler's Dual Coring Theorem in a simple way. We here will show in addition (see Remark 15 (3)), how to understand conceptually a further dual ring construction given in [16] as well.

All of this is based on the purely categorical result that, given a not necessarily symmetric monoidal bi-closed category $\mathbb{C} = (\mathbf{C}, - \otimes -, I)$ as, e.g., the category ${}_R\mathbb{M}\mathrm{od}_R$ of R-R-bimodules with its standard monoidal structure, the (contravariant) left and right internal hom-functors $[-, I]_l$ and $[-, I]_r$ can be seen as monoidal functors (see Theorem 6). To the best of our knowledge this fact, certainly known in case $\mathbb{C}$ is symmetric monoidal, is not known yet in the non-symmetric case. This shows in particular, that it is misleading to some extent, to consider dualization (of coalgebras) simply as a special instance of the convolution construction: Dualization is a construction in its own right, which coincides with (a special instance) of convolution in the symmetric case.

In order to make our analysis accessible to readers not too familiar with the theory of monoidal categories we recall its basic elements as far as they are needed.

[1] We here use Sweedler's original notation for these maps: Thus, λ_{C} and ρ_{C} here should not be mistaken for the left and right unit constraints λ_C and ρ_C in a monoidal category, i.e., the canonical isomorphisms mentioned above.

1 Basics

1.1 Monoidal categories and functors

Throughout $\mathbb{C} = (\mathbf{C}, - \otimes -, I, \alpha, \lambda, \varrho)$ denotes a monoidal category with α the associativity and λ and ϱ the left and right unit constraints. If $\mathbb{C}$ is even symmetric monoidal, the symmetry will be denoted by $\tau = (C \otimes D \xrightarrow{\tau_{C D}} D \otimes C)_{C,D}$.

Recall that $\mathbb{C}$ is called *monoidal left closed*, provided that, for each $\mathbf{C}$-object C the functor $C \otimes -$ has a right adjoint $[C, -]_l$. If each functor $- \otimes C$ has a right adjoint, denoted by $[C, -]_r$, $\mathbb{C}$ is called *monoidal right closed*. $\mathbb{C}$ is called *monoidal bi-closed*, provided that $\mathbb{C}$ is monoidal left and right closed.

The counits $C \otimes [C, X]_l \to X$ and $[C, X]_r \otimes C \to X$ of these adjunctions will be denoted by ev^l and ev^r respectively. By paramatrized adjunctions (see [11]) one thus has functors $[-, -]_r$ and $[-, -]_l$ $\mathbf{C}^{\mathsf{op}} \times \mathbf{C} \to \mathbf{C}$. In particular, for each X in $\mathbf{C}$ there are the contravariant functors $[-, X]_r$ and $[-, X]_l$ on $\mathbb{C}$.

For $C \xrightarrow{f} D$ in $\mathbf{C}$, the morphism $[D, X]_l \xrightarrow{[f,X]_l} [C, X]_l$ is the unique morphism such that the following diagram commutes

$$\begin{array}{ccc} C \otimes [D, X]_l & \xrightarrow{C \otimes [f,X]_l} & C \otimes [C, X]_l \\ {\scriptstyle f \otimes [D,X]_l}\downarrow & & \downarrow{\scriptstyle ev_l} \\ D \otimes [D, X]_l & \xrightarrow[ev_l]{} & X \end{array}$$

Similarly for $[-, X]_r$.

In each such category there are (see e.g. [10, Eqns. 1.25, 1.26, 1.27])

1. natural isomorphisms $\hom_{\mathbf{C}}(I, [C, D]_l) \simeq \hom_{\mathbf{C}}(C, D) \simeq \hom_{\mathbf{C}}(I, [C, D]_r)$;

2. natural isomorphisms $C \xrightarrow{j_C} [I, C]_l$ and $C \xrightarrow{i_C} [I, C]_r$, corresponding by adjunction to $I \otimes C \xrightarrow{\lambda_C} C$ and $C \otimes I \xrightarrow{\rho_C} C$ respectively;

3. natural isomorphisms

$$[C, [D, A]_r]_r \to [C \otimes D, A]_r \quad \text{and} \quad [C, [D, A]_l]_l \to [C \otimes D, A]_l,$$

whose images under $\hom_{\mathbf{C}}(I, -)$ are, respectively, the isomorphisms

$$\hom_{\mathbf{C}}(C, [D, A]_r) \simeq \hom_{\mathbf{C}}(C \otimes D, A) \quad \text{and} \quad \hom_{\mathbf{C}}(D, [C, A]_l) \simeq \hom_{\mathbf{C}}(C \otimes D, A)$$

expressing the adjunctions for right and left tensoring. These isomorphisms will be noted by $\Pi^r_{C,D}$ and $\Pi^l_{C,D}$ respectively in the special instance $A = I$.

4. natural transformations

$$\Theta^r_{C,D} \colon D \otimes [C, I]_r \to [C, D]_r \quad \text{and} \quad \Theta^l_{C,D} \colon [C, I]_l \otimes D \to [C, D]_l$$

corresponding by adjunction to

$$(D \otimes [C,I]_r) \otimes C \simeq D \otimes ([C,I]_r \otimes C) \xrightarrow{D \otimes ev^r_{C,I}} D \otimes I \xrightarrow{\rho_D} D$$

and, respectively,

$$C \otimes ([C,I]_l \otimes D) \simeq (C \otimes [C,I]_l) \otimes D \xrightarrow{ev^l_{C,I} \otimes D} I \otimes D \xrightarrow{\lambda_D} D.$$

Given a monoidal category $\mathbb{C} = (\mathbf{C}, -\otimes -, I, \alpha, \lambda, \varrho)$, there are the following simple ways of constructing new monoidal categories.

- $\mathbb{C}^t = (\mathbf{C}, -\otimes^t -, I, \alpha^t, \lambda^t, \rho^t)$ is a monoidal category with $C \otimes^t D = D \otimes C$, $\alpha^t_{ABC} = \alpha_{CBA}, \rho^t_C = \lambda_C$ and $\lambda^t_C = \rho_C$. $\mathbb{C}^t$ is called the *transpose* of $\mathbb{C}$.
- $\mathbb{C}^{\mathsf{op}} = (\mathbf{C}^{\mathsf{op}}, -\otimes -, I, \alpha^{\mathsf{op}}, \lambda^{\mathsf{op}}, \rho^{\mathsf{op}})$ is a monoidal category with constraints being the inverses of those in $\mathbb{C}$. $\mathbb{C}^{\mathsf{op}}$ is called the *dual* of $\mathbb{C}$.

By $\mathbf{Mon}\mathbb{C}$ and $\mathbf{Comon}\mathbb{C}$ we denote the categories of monoids (M, m, e) in $\mathbb{C}$ and of comonoids (C, Δ, ϵ) in $\mathbb{C}$, respectively. Omitting the units e and counits ϵ respectively we obtain the categories $\mathbf{Sgr}\mathbb{C}$ of semigroups and $\mathbf{Cosgr}\mathbb{C}$ of co-semigroups in $\mathbb{C}$.

1 Fact Concerning these constructions the following facts are obvious.

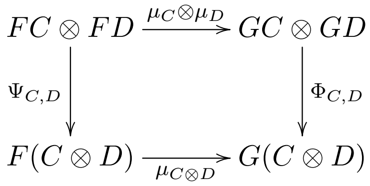
1. If $\mathbb{C}$ is symmetric monoidal, then $\mathbb{C}$ and $\mathbb{C}^t$ are monoidally equivalent (even isomorphic).
2. $(\mathbb{C}^t)^{\mathsf{op}} = (\mathbb{C}^{\mathsf{op}})^t$,
3. $\mathbf{Mon}(\mathbb{C}^t) = \mathbf{Mon}\mathbb{C}$ and $\mathbf{Comon}(\mathbb{C}^t) = \mathbf{Comon}\mathbb{C}$.
4. $\mathbf{Mon}\mathbb{C}^{\mathsf{op}} = (\mathbf{Comon}\mathbb{C})^{\mathsf{op}}$.
5. If $\mathbb{C}$ is monoidal bi-closed, then so is $\mathbb{C}^t$; its internal hom-functors can be chosen as $[C,-]^t_l = [C,-]_r$ and $[C,-]^t_r = [C,-]_l$.

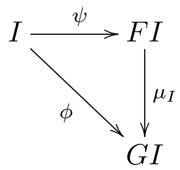
We briefly recall the following definitions and facts wich are fundamental for this note.

2 Definition Let $\mathbb{C} = (\mathbf{C}, -\otimes -, I)$ and $\mathbb{C}' = (\mathbf{C}', -\otimes' -, I')$ be monoidal categories. A *monoidal functor from* $\mathbb{C}$ *to* $\mathbb{C}'$ is a triple (F, Φ, ϕ), where $F\colon \mathbf{C} \to \mathbf{C}'$ is a functor, $\Phi_{C_1,C_2}\colon FC_1 \otimes' FC_2 \to F(C_1 \otimes C_2)$ is a natural transformation and $\phi\colon I' \to FI$ is a $\mathbf{C}$-morphism, subject to certain coherence conditions (see e.g. [15]). A monoidal functor is called *strong monoidal*, if Φ and ϕ are isomorphisms and *strict monoidal*, if Φ and ϕ are identities.

An *opmonoidal functor* from $\mathbb{C}$ to $\mathbb{C}'$ is a monoidal functor from $\mathbb{C}^{\mathsf{op}}$ to $\mathbb{C}'^{\mathsf{op}}$.

Given monoidal functors $\mathbb{F} = (F, \Psi, \psi)$ and $\mathbb{G} = (G, \Phi, \phi)$ from $\mathbb{C}$ to $\mathbb{D}$, a natural transformation $\mu\colon F \Rightarrow G$ is called a *monoidal transformation* $\mathbb{F} \Rightarrow \mathbb{G}$, if the following diagrams commute for all C, D in $\mathbf{C}$.





A *monoidal isomorphism* is a natural isomorphism which is a monoidal transformation.

A *monoidal equivalence* of monoidal categories is given by monoidal functors $\mathbb{F} = (F, \Psi, \psi)$ and $\mathbb{G} = (G, \Phi, \phi)$ and monoidal isomorphisms $\eta\colon \mathsf{Id} \Rightarrow GF$ and $\epsilon\colon FG \Rightarrow \mathsf{Id}$.

3 Remarks ([3]) 1. The composition of monoidal functors is a monoidal functor.

2. By a *monoidal subcategory* of a monoidal category $\mathbb{C} = (\mathbf{C}, - \otimes -, I)$ we mean a full subcategory $\mathbf{A}$ of $\mathbf{C}$, closed under tensor products and containing I. The embedding E of $\mathbf{A}$ into $\mathbf{C}$ then is a monoidal functor with the monoidal structure given by identities.
   Let $\mathbb{F} = (F, \Phi, \phi)\colon \mathbb{C} \to \mathbb{D}$ be a monoidal functor, and let $\mathbb{C}'$ and $\mathbb{D}'$ be monoidal subcategories of $\mathbb{C}$ and $\mathbb{D}$ respectively. By a *restriction* of $\mathbb{F}$ to these subcategories is meant a monoidal functor $\mathbb{F}' = (F', \Phi', \phi')\colon \mathbb{C}' \to \mathbb{D}'$ satisfying (a) $F'C = FC$ for all C in $\mathbf{C}'$, (b) $\Phi'_{A,B} = \Phi_{A,B}$ for all A, B in $\mathbf{C}'$, and (c) $\phi' = \phi$. This is equivalent to saying that $\mathbb{F}\mathbb{E}_{\mathbb{C}'} = \mathbb{E}_{\mathbb{D}'}\mathbb{F}'$.

3. If (G, Ψ, ψ) is a monoidal functor and F is left adjoint to G with unit η, then (F, Φ, ϕ) is opmonoidal, where ϕ corresponds by adjunction to ψ and $\Phi_{C,D}$ corresponds by adjunction to $\Psi_{FC,FD} \circ (\eta_C \otimes \eta_D)$. This defines a bijection between monoidal structures on G and opmonoidal structures on F.

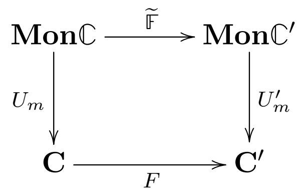
4. If (G, Ψ, ψ) and (F, Φ, ϕ) are monoidal functors where F is left adjoint to G, then this adjunction is called a *monoidal adjunction*, provided that its unit and counit are monoidal transformations. In this situation
   (a) (F, Φ, ϕ) is a strong monoidal and, hence, an opmonoidal functor;
   (b) the natural isomorphism $\hom(F-, -) \simeq \hom(-, G-)$ is a monoidal isomorphism.

5. If (G, Ψ, ψ) is a monoidal and (F, Φ, ϕ) a strong opmonoidal (hence monoidal) functor, where F is left adjoint to G, then (b) above implies that the adjunction is monoidal.

If $\mathbb{C}$ is symmetric monoidal then the internal hom functor $[-,-]\colon \mathbb{C}^{\mathsf{op}} \otimes \mathbb{C} \to \mathbb{C}$ is monoidal (see [5]) and, thus, also each functor $[-, X]\colon \mathbb{C}^{\mathsf{op}} \to \mathbb{C}$ is monoidal; this will not necessarily be the case for a non-symmetric $\mathbb{C}$. For an important fragment of this see however Section 1.2 below.

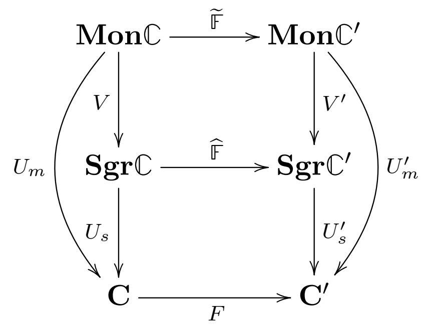
4 Proposition ([3, Chap. 3]) *Let $\mathbb{F} := (F, \Phi, \phi)\colon \mathbb{C} \to \mathbb{C}'$ be a monoidal functor.*

1. $\widetilde{\mathbb{F}}(M, m, e) = (FM, FM \otimes FM \xrightarrow{\Phi_{M,M}} F(M \otimes M) \xrightarrow{Fm} FM, I' \xrightarrow{\phi} FI \xrightarrow{Fe} FM)$ *and* $\widetilde{\mathbb{F}}f = Ff$ *defines an* induced functor $\widetilde{\mathbb{F}}\colon \mathbf{Mon}\mathbb{C} \to \mathbf{Mon}\mathbb{C}'$, *such that the diagram*



commutes (with forgetful functors U_m and U'_m).

2. *In the same way one obtains a functor* $\widehat{\mathbb{F}}\colon \mathbf{Sgr}\mathbb{C} \to \mathbf{Sgr}\mathbb{C}'$, *such that the diagram*



commutes (with forgetful functors V, V', U_s and U'_s).

Given a monoidal adjunction $\mathbb{F} \dashv \mathbb{G}$ with unit $\eta\colon \mathsf{Id} \to GF$ and counit $\epsilon\colon FG \to \mathsf{Id}$, then the induced functors $\tilde{\mathbb{F}}$ and $\tilde{\mathbb{G}}$ form an adjunction whith unit η' and counit ϵ', such that $U_m\eta' = \eta$ and $U'_m\epsilon' = \epsilon$. Similarly for $\hat{\mathbb{F}}$ and $\hat{\mathbb{G}}$.

1.2 Dualization in monoidal closed categories

Let $\mathbb{C}$ be a monoidal bi-closed category. We call the functor $[-, I]_l$, introduced in Section 1.1 above, the *left dualization functor of* $\mathbb{C}$. Analogously there is the *right dualization functor* $[-, I]_r$.

Note that, from a categorical perspective, we rather should call these functors *semi-dualization functors* following [14, Def. 4.6]. The linear dual of an R-module M is a dual in the categorical sense only, if M is reflexive. But since our focus is on modules and bimodules, we prefer to omit the prefix 'semi' in this note.

5 Proposition *For every monoidal bi-closed category $\mathbb{C}$ the left and the right dualization functors form a dual adjunction, i.e.,* $\mathbf{C} \xrightarrow{[-,I]_r} \mathbf{C}^{\mathsf{op}}$ *is left adjoint to* $\mathbf{C}^{\mathsf{op}} \xrightarrow{[-,I]_l} \mathbf{C}$.

Proof: Compose $\hom_{\mathbf{C}}(I, \Pi^r_{C,D})$ and the inverse of $\hom_{\mathbf{C}}(I, \Pi^l_{C,D})$ (see Section 1.1) in order to obtain the requested natural isomorphisms

$$\hom_{\mathbf{C}}(C, [D, I]_r) \simeq \hom_{\mathbf{C}}(C \otimes D, I) \simeq \hom_{\mathbf{C}}(D, [C, I]_l).$$

$\square$

6 Theorem *For every monoidal left closed category $\mathbb{C}$ the functor $[-, I]_l \colon \mathbb{C}^{\mathsf{op}} \to \mathbb{C}^t$ is monoidal.*

Proof: Suppressing the constraints we denote, for $\mathbf{C}$-objects C, D, by $\bar{\Phi}_{C,D}$ the $\mathbf{C}$-morphism

$$(C \otimes D) \otimes [D, I]_l \otimes [C, I]_l \xrightarrow{\mathsf{id} \otimes ev^l_{D,I} \otimes \mathsf{id}} C \otimes I \otimes [C, I]_l \xrightarrow{\rho_C \otimes \mathsf{id}} C \otimes [C, I]_l \xrightarrow{ev^l_{C,I}} I$$

This family of morphisms obviously is natural in C and D.

Denoting by $\Phi_{C,D} \colon [D, I]_l \otimes [C, I]_l \to [C \otimes D, I]_l$ the morphism corresponding to $\bar{\Phi}_{C,D}$ by adjunction, i.e., the (unique) morphism making the following diagram commute

(1)
$$\begin{array}{ccc}
(C \otimes D) \otimes [C \otimes D, I]_l & \xrightarrow{ev^l_{C \otimes D, I}} & I \\
\uparrow{\scriptstyle \mathsf{id} \otimes \Phi_{C,D}} & & \uparrow{\scriptstyle ev^l_{CI}} \\
 & & C \otimes [C, I]_l \\
 & & \uparrow{\scriptstyle \rho_C \otimes \mathsf{id}} \\
(C \otimes D) \otimes [D, I]_l \otimes [C, I]_l & \xrightarrow[\mathsf{id} \otimes ev^l_{D,I} \otimes \mathsf{id}]{} & C \otimes I \otimes [C, I]_l
\end{array}$$

one thus obtains a natural transformation $\Phi_{C,D} \colon [C, I]_l \otimes^t [D, I]_l \to [C \otimes D, I]_l$.

Denoting by $\phi := j_I \colon I \to [I, I]_l$ the isomorphism corresponding by adjunction to $I \otimes I \xrightarrow{\rho_I = \lambda_I} I$ (see Section 1.1), $([-, I]_l, \Phi, \phi)$ is a monoidal functor.

To prove coherence commutativity of the following diagrams is to be shown.

(2)
$$\begin{array}{ccccc}
I \otimes [C, I] & \xrightarrow{\phi \otimes [C,I]} & [I, I] \otimes [C, I] & \xrightarrow{\Phi_{C,I}} & [C \otimes I, I] \\
 & {\scriptstyle \lambda_{[C,I]}} \searrow & & \swarrow {\scriptstyle [\rho_C^{-1}, I]} & \\
 & & [C, I] & &
\end{array}$$

(3)
$$\begin{array}{ccccc}
[C, I] \otimes I & \xrightarrow{[C,I] \otimes \phi} & [C, I] \otimes [I, I] & \xrightarrow{\Phi_{I,C}} & [I \otimes C, I] \\
 & {\scriptstyle \rho_{[C,I]}} \searrow & & \swarrow {\scriptstyle [\lambda_C^{-1}, I]} & \\
 & & [C, I] & &
\end{array}$$

$$
\begin{array}{ccccc}
[C,I]\otimes([B,I]\otimes[A,I]) & \xrightarrow{[C,I]\otimes\Phi_{A,B}} & [C,I]\otimes[A\otimes B,I] & \xrightarrow{\Phi_{A\otimes B,C}} & [(A\otimes B)\otimes C),I] \\
\downarrow{\scriptstyle \alpha_{[C,I],[B,I],[A,I]}} & & & & \downarrow{\scriptstyle [\alpha^{-1}_{A,B,C},I]} \\
([C,I]\otimes[B,I])\otimes[A,I] & \xrightarrow{\Phi_{B,C}\otimes[A,I]} & [B\otimes C,I]\otimes[A,I] & \xrightarrow{\Phi_{A,B\otimes C}} & [A\otimes(B\otimes C),I]
\end{array} \tag{4}
$$

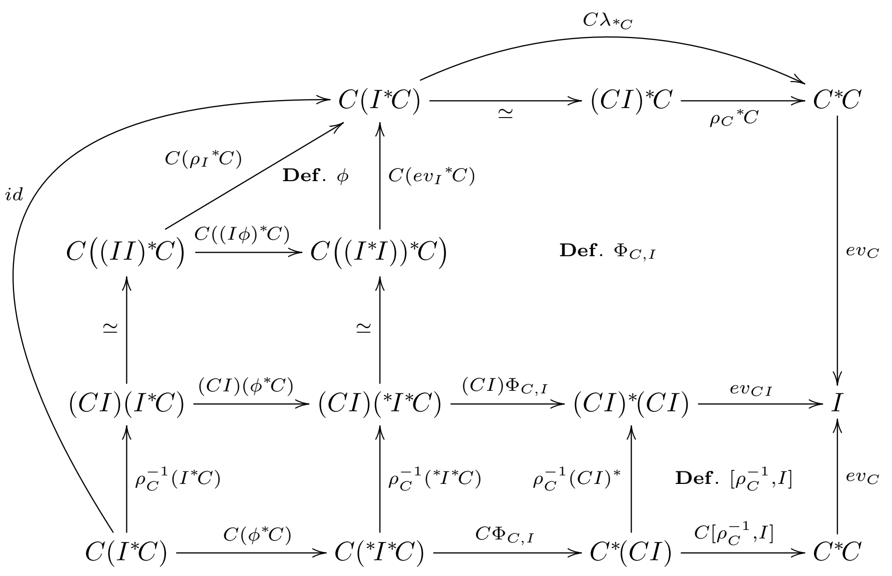
For typographical reasons we abbreviate our notation by omitting in the diagrams below the symbol $\otimes$, i.e., we simply write CD instead of $C \otimes D$, and we write ${}^*C$ instead of $[C,I]_l$ and ev_C instead of $ev^l_{C,I}$. We also denote all identities simply by id. Unspecified associativities we denote by $\simeq$. In particular, Diagram 1 can be, without suppressing the constraints, abbreviated as

$$
\begin{array}{ccccc}
(CD)^*(CD) & & \xrightarrow{ev_{CD}} & & I \\
& & & & \uparrow{\scriptstyle ev_C} \\
\uparrow{\scriptstyle id\Phi_{C,D}} & & & & C^*C \\
& & & & \uparrow{\scriptstyle \rho_C id} \\
(CD)({}^*D{}^*C) & \xrightarrow[\simeq]{} & (C(D^*D))^*C & \xrightarrow[(id ev_D)id]{} & (CI)^*C.
\end{array} \tag{5}
$$

The following diagram commutes: The cells with curved arrows do so by coherence; where previous definitions are used, this is indicated; the remaining cells commute trivially. Now,

$$ev_C \circ C \otimes \lambda_{{}^*C} = ev_C \circ C \otimes ([\rho_C^{-1}, I] \circ \Phi_{C,I} \circ (\phi \otimes {}^*C))$$

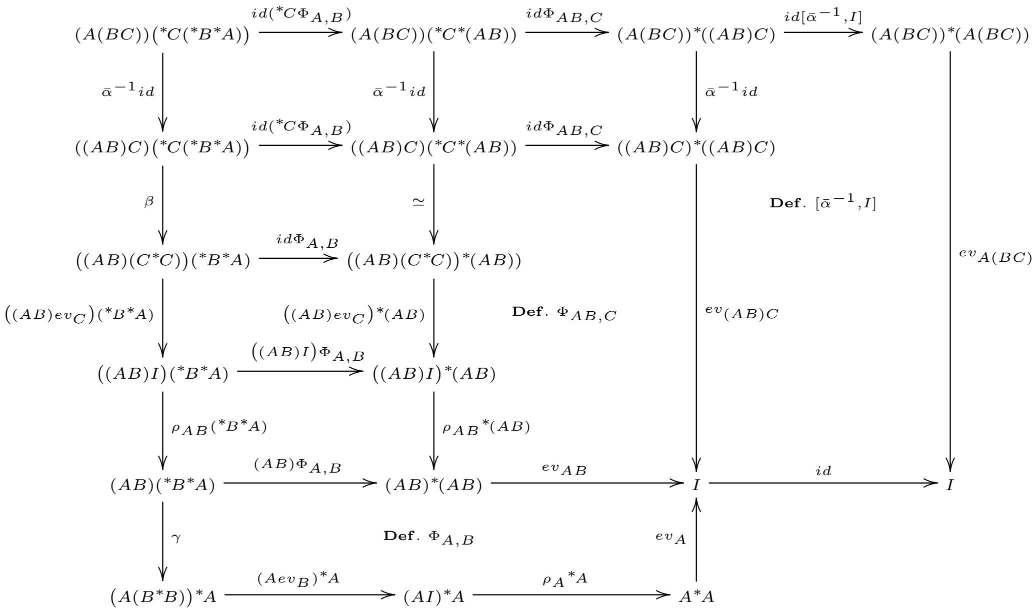
is equivalent to commutativity of Diagram (2).

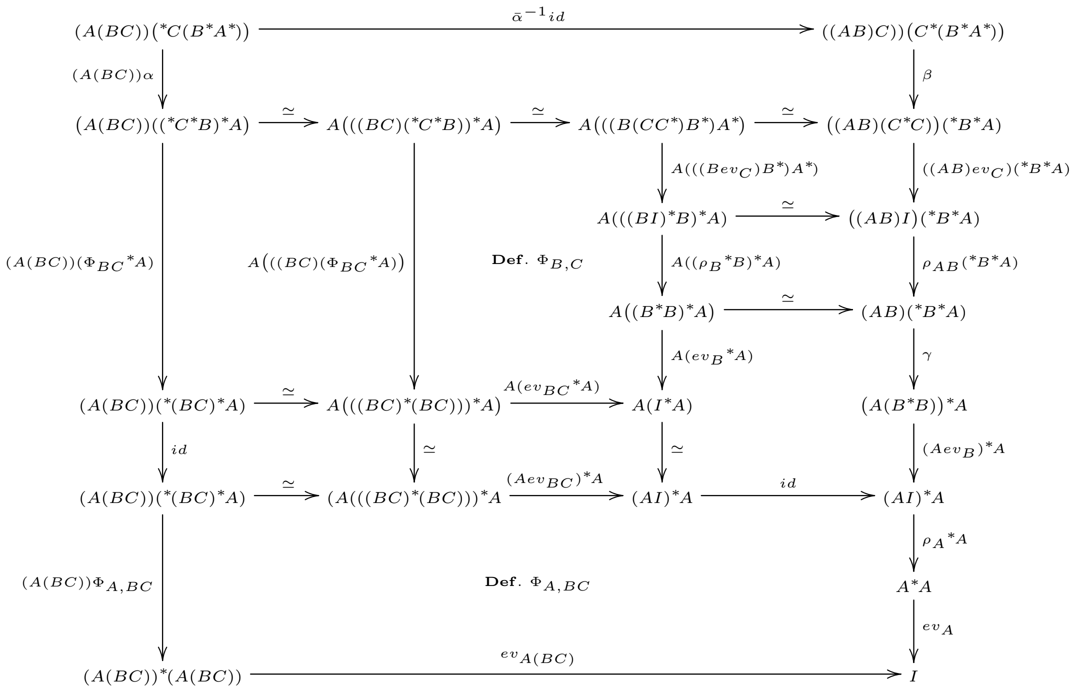


Concerning Diagram 4 we first observe that the following diagrams commute, where we put $\alpha = \alpha_{^*C,^*B,^*A}$ and $\bar{\alpha} = \alpha_{A,B,C}$. Again the cells referring to previously defined data are marked respectively; associativities are only labelled by $\simeq$. The top cell of the second diagram commutes by coherence. One thus gets

$$ev_{A(BC)} \circ (A(BC)) \otimes (\Phi_{A,BC} \circ \Phi_{BC}{}^*A \circ \alpha) = ev_{A(BC)} \circ (A(BC)) \otimes ([\bar{\alpha}^{-1}, I] \circ \Phi_{AB,C} \circ {}^*C\Phi_{A,B})$$

which is equivalent to commutativity of Diagram 4.





$\square$

Since $\mathbb{C}$ is left closed if and only if $\mathbb{C}^t$ is right closed (with $[-,I]_l$ in $\mathbb{C}$ being the same as $[-,I]_r$ in $\mathbb{C}^t$ — see above) one immediately gets

7 Corollary *For every monoidal right closed category $\mathbb{C}$ the functor $[-,I]_r\colon \mathbb{C}^{\mathsf{op}} \to \mathbb{C}^t$ is monoidal.*

The monoidal structure on $[-,I]_r$ is given by the isomorphism $\psi := i_I\colon I \to [I,I]_r$ corresponding by adjunction to $I \otimes I \xrightarrow{\lambda_I=\rho_I} I$ (see again Section 1.1) and the natural transformation Ψ whose components are those morphisms making the following diagram commute.

$$\begin{array}{ccc}
[D\otimes C,I]_r \otimes (D\otimes C) & \xrightarrow{ev^r_{D\otimes C,I}} & I \\
\uparrow{\scriptstyle \Psi_{D,C}\otimes \mathsf{id}} & & \uparrow{\scriptstyle ev^r_{CI}} \\
 & & [C,I]_r \otimes C \\
 & & \uparrow{\scriptstyle \mathsf{id}\otimes\lambda_C} \\
[C,I]_r \otimes [D,I]_r \otimes (D\otimes C) & \xrightarrow[\mathsf{id}\otimes ev^r_{D,I}\otimes\mathsf{id}]{} & [C,I]_r \otimes I \otimes C
\end{array}$$

8 Remarks 1. The opmonoidal structure (Ψ,ψ) on $[-,I]_r$ just defined corresponds to the monoidal structure (Φ,ϕ) on $[-,I]_l$ in the sense of Remark 3 (3).

2. The natural transformation Ψ is related to the natural transformations $\Theta^r_{C,D}$ and Π^r of Section 1.1 as follows:

$$\Psi_{C,D} = [D,I]_r \otimes [C,I]_r \xrightarrow{\Theta^r_{C,[D,I]_r}} [C,[D,I]_r]_r \xrightarrow{\Pi^r_{C,D}} [C\otimes D,I]_r$$

Similarly for Φ.

1.3 Bimodules

Throughout this section R denotes a not necessarily commutative unital ring and ${}_R\mathbf{Mod}_R$ the category of R-R-bimodules. We note that this category is abelian.

By ${}_R\mathbf{Mod}_R \xrightarrow{|-|} \mathbf{Ab}$ we denote the underlying functor from ${}_R\mathbf{Mod}_R$ into the category of abelian groups.

1.3.1 The monoidal category of R-R-bimodules

The category ${}_R\mathbf{Mod}_R$ carries a monoidal structure given by the tensor product $-\otimes_R-$ over R and the bimodule R as unit object. The resulting monoidal category ${}_R\mathbb{M}\mathrm{od}_R$ is a non-symmetric monoidal category if R fails to be commutative, and it is monoidal bi-closed (see e.g. [6]) as follows

- $[M,-]_r$, the right adjoint of $-\otimes_R M$, is given by $[M,N]_r = \hom_R(M,N)$, the R-R-bimodule of right R-linear maps;

- $[M,-]_l$, the right adjoint of $M\otimes_R -$, is given by $[M,N]_l = {}_R\hom(M,N)$, the R-R-bimodule of left R-linear maps.

In particular one has the maps $R \xrightarrow{j_R} [R,R]_l = {}_R\hom(R,R)$ and $R \xrightarrow{i_R} [R,R]_r = \hom_R(R,R)$ (see Section 1.1). In the sequel we will refer to these isomorphisms, which are the maps sending the unit $1\in R$ to the identity id_R, as ϕ_R and ψ_R respectively.

For bimodules M and N the canonical surjection $|M|\otimes_{\mathbb{Z}}|N| \to |M\otimes_R N|$ will be denoted by ${}_Rcan_{M,N}$. The family of these surjections forms a natural transformation. Denoting by $\mathbb{Z} \xrightarrow{\chi_R} R$ the unique unital ring homomorphism, one gets a monoidal functor

$$(|-|, {}_Rcan, \chi_R)\colon {}_R\mathbb{M}\mathrm{od}_R \to \mathbb{A}\mathrm{b} \tag{6}$$

from the monoidal category of bimodules into the monoidal category of abelian groups, the latter equipped with its standard tensor product.

With τ the symmetry of the tensor product of abelian groups, one also gets a monoidal functor

$$(|-|, {}_Rcan\circ\tau, \chi_R)\colon {}_R\mathbb{M}\mathrm{od}_R{}^t \to \mathbb{A}\mathrm{b} \tag{7}$$

1.3.2 A monoidal isomorphism

With R^{op} the opposite ring of R one has a canonical isomorphism ${}_R\mathbf{Mod}_R \to {}_{R^{\mathsf{op}}}\mathbf{Mod}_{R^{\mathsf{op}}}$, switching left and right R-actions, which will be denoted by $\overline{(-)}$.

Note that there are natural isomorphisms $\overline{N}\otimes_{R^{\mathsf{op}}}\overline{M} \xrightarrow{\sigma_{M,N}} \overline{M\otimes_R N}$ acting as $y\otimes_{R^{\mathsf{op}}} x \mapsto x\otimes_R y$ (see e.g. [9, p. 132]). Since[2] $\overline{R} = R^{\mathsf{op}}$,

$$(\overline{(-)}, \sigma, \mathsf{id}_{R^{\mathsf{op}}})\colon {}_R\mathbb{M}\mathrm{od}_R{}^t \to {}_{R^{\mathsf{op}}}\mathbb{M}\mathrm{od}_{R^{\mathsf{op}}} \tag{8}$$

is a (strong) monoidal functor (in fact a monoidal isomorphism) ${}_R\mathbb{M}\mathrm{od}_R{}^t \simeq {}_{R^{\mathsf{op}}}\mathbb{M}\mathrm{od}_{R^{\mathsf{op}}}$ and, thus, lifts by Propostition 4 to functorial isomorphisms

$$\mathbf{Mon}({}_R\mathbb{M}\mathrm{od}_R) = \mathbf{Mon}({}_R\mathbb{M}\mathrm{od}_R{}^t) \simeq \mathbf{Mon}({}_{R^{\mathsf{op}}}\mathbb{M}\mathrm{od}_{R^{\mathsf{op}}})$$

$$\mathbf{Comon}({}_R\mathbb{M}\mathrm{od}_R) = \mathbf{Comon}({}_R\mathbb{M}\mathrm{od}_R{}^t) \simeq \mathbf{Comon}({}_{R^{\mathsf{op}}}\mathbb{M}\mathrm{od}_{R^{\mathsf{op}}}).$$

1.3.3 Dualization functors for ${}_R\mathbf{Mod}_R$

Recall from Section 1.2 the right and the left dualization functor on ${}_R\mathbf{Mod}_R$. These are the contravariant functors $(-)^* := [-,R]_r$ and ${}^*(-) := [-,R]_l$, which — on the level of sets — act as $\hom_R(-,R)$ and ${}_R\hom(-,R)$ respectively (see Section 1.3.1).

Proposition 5 now specializes to

9 Proposition *Considering the left and right R-dualization functors as functors $({}_R\mathbf{Mod}_R)^{\mathsf{op}} \xrightarrow{{}^*(-)} {}_R\mathbf{Mod}_R \xrightarrow{(-)^*} ({}_R\mathbf{Mod}_R)^{\mathsf{op}}$, the functor $(-)^*$ is left adjoint to ${}^*(-)$. The units and counits of this adjunction are the maps $M \xrightarrow{\eta_M} {}^*(M^*)$ and $M \xrightarrow{\epsilon_M} ({}^*M)^*$ with $x\mapsto (y\mapsto y(x))$.*

[2]Here R and R^{op} are considered as an R-R-bimodule and an R^{op}-R^{op}-bimodule respectively.

By Section 1.2 we obtain

10 Proposition *1. The dualization functors are monoidal functors as follows*

(a) $\mathbb{D}_l := ({}^*(-), \Phi, \phi_R)\colon {}_R\mathbb{M}\mathrm{od}_R{}^{\mathsf{op}} \to {}_R\mathbb{M}\mathrm{od}_R{}^t$

(b) $\mathbb{D}_r := ((-)^*, \Psi, \psi_R)\colon {}_R\mathbb{M}\mathrm{od}_R{}^{\mathsf{op}} \to {}_R\mathbb{M}\mathrm{od}_R{}^t$

2. Composing these with the strong monoidal functor of 1.3.2 one obtains monoidal functors

(a) $\overline{\mathbb{D}_l} := (\overline{{}^*(-)}, \overline{\Phi} \circ \sigma, \overline{\phi_R})\colon {}_R\mathbb{M}\mathrm{od}_R{}^{\mathsf{op}} \to {}_{R^{\mathsf{op}}}\mathbb{M}\mathrm{od}_{R^{\mathsf{op}}}$

(b) $\overline{\mathbb{D}_r} := (\overline{(-)^*}, \overline{\Psi} \circ \sigma, \overline{\psi_R})\colon {}_R\mathbb{M}\mathrm{od}_R{}^{\mathsf{op}} \to {}_{R^{\mathsf{op}}}\mathbb{M}\mathrm{od}_{R^{\mathsf{op}}}$

11 Remark The action of the maps

$$ {}^*M \otimes_R {}^*N \xrightarrow{\Phi_{N,M}} {}^*(N \otimes_R M) \quad \text{and} \quad M^* \otimes_R N^* \xrightarrow{\Psi_{N,M}} (N \otimes_R M)^* $$

can, in view of the commutative diagrams of Section 1.2, be described by the following equations, for all $\mu \in {}^*M$ (resp. M^*), $\nu \in {}^*N$ (resp. N^*) and $x \in M, y \in N$,

$$ \big(\Psi_{N,M}(\mu \otimes_R \nu)\big)(y \otimes x) = \mu(\nu(y)x) \tag{9} $$

$$ \big(\Phi_{N,M}(\mu \otimes_R \nu)\big)(y \otimes x) = \nu(y\mu(x)) \tag{10} $$

Equivalently,

$$ \Psi_{N,M}(\mu \otimes_R \nu) = \mu \circ \lambda_M \circ (\nu \otimes_R M) \tag{11} $$

$$ \Phi_{N,M}(\mu \otimes_R \nu) = \nu \circ \rho_N \circ (N \otimes_R \mu) \tag{12} $$

Note that Sweedler's natural transformation μ of [16, Lemma 3.4 (b)], if specialized to $X = Y = V = R$ and with the replacements $R = M$ and $S = N$ coincides with Φ, while the natural transformation η of [16, Lemma 3.4 (a)] with the corresponding specializations and replacements coincides with Ψ.

We finally remark that the hom-functor of ${}_R\mathbf{Mod}_R$ can be lifted to a functor ${}^*(-)^*\colon {}_R\mathbf{Mod}_R^{\mathsf{op}} \to \mathbf{Ab}$. This functor can be supplied with a monoidal structure as follows: Denote by $\Xi_{M,N}\colon {}^*M^* \otimes_{\mathbb{Z}} {}^*N^* \to {}^*(M \otimes_R N)^*$ the group homomorphism with $\mu \otimes_{\mathbb{Z}} \nu \mapsto \rho_R \circ (\mu \otimes_R \nu)$, by $\mathbb{Z} \xrightarrow{\xi} {}^*R^*$ the group homomorphism with $1 \mapsto \mathsf{id}_R$, and by $\Xi^t_{M,N}$ the composition ${}^*M^* \otimes_{\mathbb{Z}} {}^*N^* \xrightarrow{\tau_{{}^*M^*,{}^*N^*}} {}^*N^* \otimes_{\mathbb{Z}} {}^*M^* \xrightarrow{\Xi_{N,M}} {}^*(N \otimes_R M)^* = {}^*(M \otimes^t_R N)^*$. Note that this construction can be obtained by a purely categorical argument as well, using the fact that ${}_R\mathbf{Mod}_R$ is enriched over $\mathbf{Ab}$ (see [5]). Then the following holds.

12 Proposition *1. $\mathbb{D} := ({}^*(-)^*, \Xi, \xi)\colon {}_R\mathbb{M}\mathrm{od}_R{}^{\mathsf{op}} \to \mathbb{A}\mathrm{b}$ is a monoidal functor.*

2. $\mathbb{D}^t := ({}^*(-)^*, \Xi^t, \xi)\colon ({}_R\mathbb{M}\mathrm{od}_R{}^t)^{\mathsf{op}} \to \mathbb{A}\mathrm{b}$ is a monoidal functor.

It is well known that every adjunction induces a largest equivalence. In more detail: Let $\mathbf{A} \xrightarrow{G} \mathbf{B} \xrightarrow{F} \mathbf{A}$ form an adjunction with unit $\eta : \mathsf{Id}_{\mathbf{B}} \Rightarrow GF$ and counit $\epsilon \colon FG \Rightarrow \mathsf{Id}_{\mathbf{A}}$, then the restrictions of G and F to the full subcategories $Fix\eta$ and $Fix\epsilon$ spanned by those objects of $\mathbf{B}$ and $\mathbf{A}$, whose components of η and ϵ respectively are isomorphisms, provide an equivalence. And $Fix\eta$ and $Fix\epsilon$ obviously are the largest subcategories of $\mathbf{B}$ and $\mathbf{A}$ respectively, which are equivalent under F and G. Applied to the adjunction of Proposition 9 above we thus obtain a duality (= dual equivalence) between the full subcategories spanned by those bimodules M which are reflexive as left and right R-modules respectively.

To characterize these is not possible in general. However, one can describe a duality between somewhat smaller subcategories, induced by the dual adjunction $(-)^* \dashv {}^*(-)$, which then even is a *monoidal* duality as follows, where we denote by $\mathbf{FGP}_R$ and ${}_R\mathbf{FGP}$ the full subcategories of ${}_R\mathbf{Mod}_R$, spanned by all bimodules which are finitely generated projective as right and as left R-modules respectively (see also [16] and [17]).

13 Proposition 1. *The categories $\mathbf{FGP}_R$ and ${}_R\mathbf{FGP}$ are closed in ${}_R\mathbf{Mod}_R$ under tensor products. In particular, these categories form monoidal subcategories $\mathbb{FGP}_R$ and ${}_R\mathbb{FGP}$ of ${}_R\mathbb{M}\mathrm{od}_R$.*

2. *The dualization functors can be restricted as follows:*

$${}^*(-) \colon {}_R\mathbf{FGP}^{\mathsf{op}} \to \mathbf{FGP}_R \text{ and } (-)^* \colon \mathbf{FGP}_R \to {}_R\mathbf{FGP}^{\mathsf{op}}$$

and these restrictions provide a duality.

3. *The restrictions of the monoidal functors $\mathbb{D}_l = ({}^*(-), \Phi, \phi_R)$ and $\mathbb{D}_r = ((-)^*, \Psi, \psi_R)$, considered as monoidal functors*[3]

$${}_R\mathbb{FGP}^{\mathsf{op}} \xrightarrow{\mathbb{D}'_l} \mathbb{FGP}^t_R \text{ and } \mathbb{FGP}^t_R \xrightarrow{\mathbb{D}'_r} {}_R\mathbb{FGP}^{\mathsf{op}}$$

provide a monoidal duality.

4. *Considering the dualization functors in the form of Proposition 10 (2) this amounts to a monoidal duality ${}_R\mathbb{FGP}^{\mathsf{op}} \simeq {}_{R^{\mathsf{op}}}\mathbb{FGP}$, given by $\overline{\mathbb{D}_l}'$ and $\overline{\mathbb{D}_r}'$.*

Proof: P and Q are R-R-bimodules from $\mathbb{FGP}_R$, if and only if there exist retractions $R^n \xrightarrow{p} P$ and $R^m \xrightarrow{q} Q$ in $\mathbf{Mod}_R$. Then $R^{nm} \simeq R^n \otimes_R R^m \xrightarrow{p \otimes_R q} P \otimes_R Q$ is a retraction in $\mathbf{Mod}_R$, such that $P \otimes_R Q$ is finitely generated projective in $\mathbf{Mod}_R$. This proves 1., since the tensor product in ${}_R\mathbf{Mod}_R$ has this module as its underlying right R-module.

To prove 2., it suffices to observe that (a) ${}^*R \simeq R$, (b) ${}^*(-)$, being a right adjoint by Proposition 9, preserves products and, thus, finite coproducts as well, since these are biproducts, and (c) ${}^*(-)$ preserves retracts by functoriality.

For 3. we first of all need to show that the restricted adjunction is a monoidal one, which (see Remark 3) is the case if and only if $\Psi_{M,N}$ is an isomorphism, provided

[3] Note that Ψ and ψ are invertible and, thus, provide the respective morphisms in ${}_R\mathbf{FGP}^{\mathsf{op}}$ by considering their inverses.

that M and N are finitely generated projective as right R-modules. By Remark 8 we have $\Psi_{N,M} = \Pi^r_{N,M} \circ \Theta_{N,M^*}$, where $\Pi^r_{N,M}$ is an isomorphism; now Θ_{N,M^*} is an isomorphism, provided that M or N is finitely generated projective (see e.g. [2, 20. Ex. 12]). Consequently, the restriction of the dual adjunction to ${}_R\mathbf{FGP}$ is monoidal and then, by 2., a monoidal duality as requested. $\square$

2 R-rings and R-corings

2.1 The categories of R-rings and R-corings

Recall that the category of monoids in $\mathbb{Ab}$ is ${}_1\mathbf{Ring}$, the category of unital rings.

The *category of unital R-rings* is defined as ${}_1\mathbf{Ring}_R := \mathbf{Mon}({}_R\mathbb{Mod}_R)$, the category of monoids in ${}_R\mathbb{Mod}_R$(see [7, 3.24]) or as the comma category $R \downarrow {}_1\mathbf{Ring}$ of unital rings under R (see [4]). In fact, these categories are easily seen to be isomorphic (see below). We will, in the sequel, use the first mentioned definition since it allows to make use of the theory of monoidal categories. (In particular we can define the category $\mathbf{Ring}_R$ of not nessarily unital R-rings as the category of semigroups in ${}_R\mathbb{Mod}_R$.)

However, when doing so one has to be careful, since one needs to distinguish R-rings and R^{op}-rings. As we will see below this distinction cannot be made in a simple categorical way: The categories ${}_1\mathbf{Ring}_R$ and ${}_1\mathbf{Ring}_{R^{\mathsf{op}}}$ are isomorphic! Since $\mathbf{Mon}({}_R\mathbb{Mod}_R) = \mathbf{Mon}({}_R\mathbb{Mod}_R{}^t)$, they even may — or may not — be concretely isomorphic over ${}_1\mathbf{Ring}$, depending of the chosen forgetful functors (see below). Thus, if one wants to distinguish them categorically, one only can do that on the level of concrete categories.

The category of comonoids in ${}_R\mathbb{Mod}_R$ is called the *category of counital R-corings* (see [7]) and will be denoted by ${}_\epsilon\mathbf{Coring}_R$; $\mathbf{Coring}_R$ then is the category of co-semigroups in ${}_R\mathbb{Mod}_R$.

The underlying ring of an R-ring

The monoidal functor of Eqn. (6) induces a functor $I_{\otimes_R} \colon \mathbf{Mon}({}_R\mathbb{Mod}_R) \to {}_1\mathbf{Ring}$ (see Proposition 4), which obviously is faithful[4].

$I_{\otimes_R}$ maps a monoid (M, m, e) to $(M, M \otimes_{\mathbb{Z}} M \xrightarrow{{}_R can} M \otimes_R M \xrightarrow{m} M, \mathbb{Z} \xrightarrow{\chi_R} R \xrightarrow{e} M)$, called the *underlying ring of* (M, m, e); in particular $I_{\otimes_R}(R) = R$.

Analogously there is a functor $I_{\otimes^t_R} \colon \mathbf{Mon}({}_R\mathbb{Mod}_R{}^t) = \mathbf{Mon}({}_R\mathbb{Mod}_R) \to {}_1\mathbf{Ring}$, which is faithful as well.

$I_{\otimes^t_R}$ maps a monoid (M, m, e) to $(M, M \otimes_{\mathbb{Z}} M \xrightarrow{{}_R can \circ \tau} M \otimes_R M \xrightarrow{m} M, \mathbb{Z} \xrightarrow{\chi_R} R \xrightarrow{e} M)$. Hence $I_{\otimes^t_R}(M, m, e) = (I_{\otimes_R}(M, m, e))^{\mathsf{op}}$. In particular $I_{\otimes^t_R}(R) = R^{\mathsf{op}}$.

Some categorical isomorphisms

Let (M, m, e) be an R-ring. Since, as for every monoid in a monoidal category, the unit e is a monoid homomorphism, the morphism $I_{\otimes_R} e = e$ is a ring homomorphism

[4]We use the somewhat clumsy notation $I_{\otimes_R}$ in order to stress the fact that this functor depends on the monoidal structure.

from R into the underlying ring of (M,m,e). In other words, $(I_{\otimes_R}(M,m,e),e)$ is an object of $R \downarrow {}_1\mathbf{Ring}$. This defines a functor $\Phi_R \colon \mathbf{Mon}({}_R\mathbb{Mod}_R) \to R \downarrow {}_1\mathbf{Ring}$ if one puts $\Phi_R f = f$ for each morphim f of monoids in ${}_R\mathbb{Mod}_R$.

Conversely, if (M,p,u) is a monoid in $\mathbb{Ab}$, i.e., a unital ring and $e \colon R \to (M,p,u)$ a morphism in ${}_1\mathbf{Ring}$, M becomes an R-R–bimodule in the obvious way. e then is a morphism in ${}_R\mathbf{Mod}_R$ and p induces a (R-left and -right linear) map $m \colon M \otimes_R M \to M$ with $m \circ {}_Rcan = p$. It is well known (and easy to see) that the assignment $((M,p,u),e) \mapsto (M,m,e)$ defines a functor $\Psi_R \colon R \downarrow {}_1\mathbf{Ring} \to \mathbf{Mon}({}_R\mathbb{Mod}_R)$, if one puts $\Psi_R f = f$, and this is the inverse of Φ_R.

Denoting by V_R the familiar forgetful functor from $R \downarrow {}_1\mathbf{Ring}$ to ${}_1\mathbf{Ring}$ the following diagram commutes.

$$\begin{array}{ccc} R \downarrow {}_1\mathbf{Ring} & \underset{\Phi_R}{\overset{\Psi_R}{\rightleftarrows}} & \mathbf{Mon}({}_R\mathbb{Mod}_R) \\ & {\scriptstyle V_R}\searrow \quad \swarrow{\scriptstyle I_{\otimes_R}} & \\ & {}_1\mathbf{Ring} & \end{array}$$

Thus, the categories $R \downarrow {}_1\mathbf{Ring}$ and $\mathbf{Mon}({}_R\mathbb{Mod}_R)$ are isomorphic as concrete categories over ${}_1\mathbf{Ring}$.

Since the categories $\mathbf{Mon}({}_R\mathbb{Mod}_R)$ and $\mathbf{Mon}({}_R\mathbb{Mod}_R{}^t)$ coincide, the monoidal isomorphism of Eqn. (8) induces a functor $\Omega \colon {}_1\mathbf{Ring}_R \to {}_1\mathbf{Ring}_{R^{\mathsf{op}}}$, and this is an isomorphism; it makes the following diagram commute.

$$\begin{array}{ccc} \mathbf{Mon}({}_R\mathbb{Mod}_R) & \xrightarrow{\quad\Omega\quad} & \mathbf{Mon}({}_{R^{\mathsf{op}}}\mathbb{Mod}_{R^{\mathsf{op}}}) \\ & {\scriptstyle I_{\otimes^t_R}}\searrow \quad \swarrow{\scriptstyle I_{\otimes_{R^{\mathsf{op}}}}} & \\ & {}_1\mathbf{Ring} & \end{array} \tag{13}$$

The isomorphism Ω corresponds to the isomorphism $\Sigma \colon R \downarrow {}_1\mathbf{Ring} \to R^{\mathsf{op}} \downarrow {}_1\mathbf{Ring}$, which maps a ring homomorphism $R \xrightarrow{f} S$ to $R^{\mathsf{op}} \xrightarrow{f} S^{\mathsf{op}}$. In particular, the following diagram commutes.

$$\begin{array}{ccc} \mathbf{Mon}({}_R\mathbb{Mod}_R) & \xrightarrow{\Phi_R} & R \downarrow {}_1\mathbf{Ring} \\ {\scriptstyle\Omega}\downarrow & & \downarrow{\scriptstyle\Sigma} \\ \mathbf{Mon}({}_{R^{\mathsf{op}}}\mathbb{Mod}_{R^{\mathsf{op}}}) & \xrightarrow[\Phi_{R^{\mathsf{op}}}]{} & R^{\mathsf{op}} \downarrow {}_1\mathbf{Ring} \end{array}$$

Similarly there is a functor $\Omega' \colon \mathbf{Sgr}({}_R\mathbb{Mod}_R) \to \mathbf{Sgr}({}_{R^{\mathsf{op}}}\mathbb{Mod}_{R^{\mathsf{op}}})$.

Summarizing this we state: There are the following isomorphisms of concrete categories over ${}_1\mathbf{Ring}$, characterizing the concrete categories of R-rings and R^{op}-rings respectively.

1. $(R \downarrow {}_1\mathbf{Ring}, V_R) \xrightarrow{\Psi_R} (\mathbf{Mon}({}_R\mathbb{Mod}_R), I_{\otimes_R})$
2. $(R^{\mathsf{op}} \downarrow {}_1\mathbf{Ring}, V_{R^{\mathsf{op}}}) \xrightarrow{\Psi_{R^{\mathsf{op}}}} (\mathbf{Mon}({}_{R^{\mathsf{op}}}\mathbb{Mod}_{R^{\mathsf{op}}}), I_{\otimes_{R^{\mathsf{op}}}}) \xrightarrow{\Omega^{-1}} (\mathbf{Mon}({}_R\mathbb{Mod}_R{}^t), I_{\otimes^t_R})$ $= (\mathbf{Mon}({}_R\mathbb{Mod}_R), I_{\otimes^t_R})$

2.2 Dual rings of a coring

2.2.1 Dual ring functors

Using the equation ${}_\epsilon\mathbf{Coring}_R^{\mathsf{op}} = (\mathbf{Comon}({}_R\mathbb{M}\mathrm{od}_R))^{\mathsf{op}} = \mathbf{Mon}(({}_R\mathbb{M}\mathrm{od}_R)^{\mathsf{op}})$, the left and right monoidal dualization functors $\mathbb{D}_l$ and $\mathbb{D}_r$ induce, by Proposition 4, functors

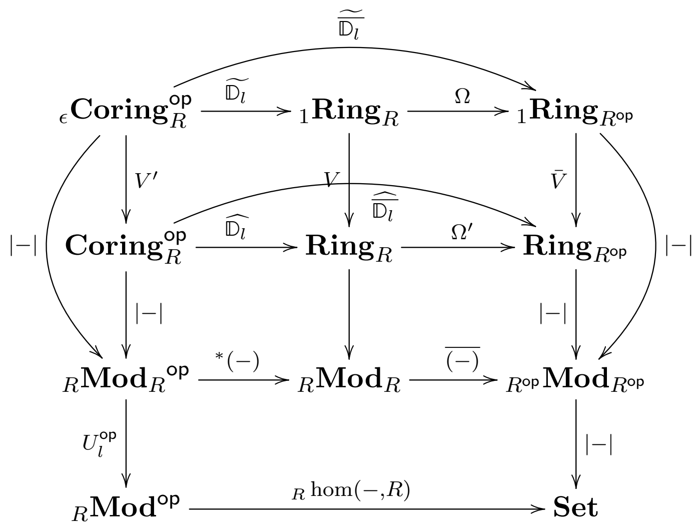
$$\widetilde{\mathbb{D}_l}, \widetilde{\mathbb{D}_r} \colon {}_\epsilon\mathbf{Coring}_R^{\mathsf{op}} \to {}_1\mathbf{Ring}_R \quad \text{and} \quad \widehat{\mathbb{D}_l}, \widehat{\mathbb{D}_r} \colon \mathbf{Coring}_R^{\mathsf{op}} \to \mathbf{Ring}_R$$

while the left and right monoidal dualization functors $\overline{\mathbb{D}_l}$ and $\overline{\mathbb{D}_r}$ induce functors

$$\widetilde{\overline{\mathbb{D}_l}}, \widetilde{\overline{\mathbb{D}_r}} \colon {}_\epsilon\mathbf{Coring}_R^{\mathsf{op}} \to {}_1\mathbf{Ring}_{R^{\mathsf{op}}} \quad \text{and} \quad \widehat{\overline{\mathbb{D}_l}}, \widehat{\overline{\mathbb{D}_r}} \colon \mathbf{Coring}_R^{\mathsf{op}} \to \mathbf{Ring}_{R^{\mathsf{op}}}$$

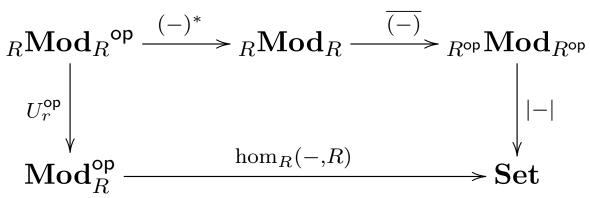
such that (for the left dualization functors) the following diagram commutes, where $|-|$ denotes the various forgetful functors.

(14)



In the corresponding diagram for the right dualization functors the lower cell is

(15)



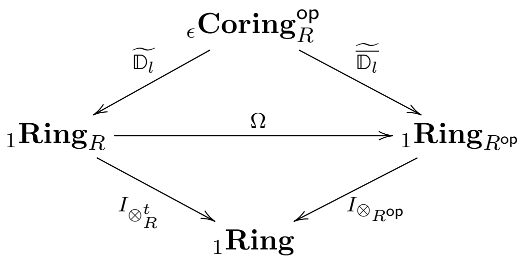
14 Remarks 1. For the sake of clarity we stress the fact already mentioned above, that it can be seen as a matter of taste whether the dual ring of an R-coring is an R-ring or an R^{op}-ring! Both options are available and they correspond to each other by the functor Ω. Both, the more natural (using "natural" tensor products only) and the more conceptual (consider the dual ring functor as a functor induced by a monoidal functor) point of view, suggest however to say: *The dual ring of an R-coring is an R^{op}-ring.*

2. Clearly the monoidal functors $\mathbb{D}$ and $\mathbb{D}^t$ from Proposition 12 induce, again by Proposition 4, functors $\widetilde{\mathbb{D}}, \widetilde{\mathbb{D}^t}\colon {}_{\epsilon}\mathbf{Coring}_R^{\mathsf{op}} \to {}_1\mathbf{Ring}$ (recall that $\mathbf{Comon}({}_R\mathbb{M}\mathrm{od}^t_R) = \mathbf{Comon}({}_R\mathbb{M}\mathrm{od}_R) = {}_{\epsilon}\mathbf{Coring}_R$ by Fact 1).

2.2.2 The ring structure of dual rings

We next describe the various dual rings constructed above as objects of ${}_1\mathbf{Ring} \downarrow R$, ${}_1\mathbf{Ring} \downarrow R^{\mathsf{op}}$ and ${}_1\mathbf{Ring}$ respectively, i.e., in the form $(I(X,m,e),e)$ where I is the respective forgetful functor and $I(X,m,e)$ is a unital ring (considered as a monoid in $\mathbb{A}\mathrm{b}$) whose unit e (considered as a group homomorphism $R \to X$) then in fact is a unital ring homomorphism $I(R) \to I(X,m,e)$. This enables us in particular to compare our constructions with those of [16]:

Given any R-coring (C,Δ,ϵ), we note first that, by Section 2.1, the underlying rings of the R^{op}-ring $\widetilde{\widetilde{\mathbb{D}_l}}(C,\Delta,\epsilon)$ and $\widetilde{\widetilde{\mathbb{D}_r}}(C,\Delta,\epsilon)$ are the opposite rings of the underlying rings of the R-rings $\widetilde{\mathbb{D}_l}(C,\Delta,\epsilon)$ and $\widetilde{\mathbb{D}_r}(C,\Delta,\epsilon)$ respectively. The situation is illustrated by the following commutative diagram.



It thus suffices to consider the latter ones.

Since ${}^*\Delta$ (and Δ^* and ${}^*\Delta^*$ as well) acts by pre-composition with Δ we get, by definition of Φ, Ψ and Ξ respectively, the following descriptions (where $(\mu,\nu) \in {}^*C \times {}^*C$ (resp. $C^* \times C^*$)).

1. $\widetilde{\mathbb{D}_l}(C,\Delta,\epsilon)$ is the unital R-ring $({}^*C, m_l, e_l)$ with

$$m_l = {}^*C \otimes^t_R {}^*C \xrightarrow{\Phi_{C,C}} {}^*(C \otimes_R C) \xrightarrow{{}^*\Delta} {}^*C \quad \text{and} \quad e_l = R \xrightarrow{\phi_R} {}^*R \xrightarrow{{}^*\epsilon} {}^*C.$$

   (a) Its underlying unital ring thus is

$$I_{\otimes_R}(\widetilde{\mathbb{D}_l}(C,\Delta,\epsilon)) = ({}_R\mathrm{hom}(C,R), m_l \circ {}_R can, e_l \circ \chi_R)$$

   with multiplication

$$\begin{aligned} \mu \cdot \nu = m_l(\mu \otimes_R \nu) &= \Phi_{C,C}(\mu \otimes_R \nu) \circ \Delta \\ &= C \xrightarrow{\Delta} C \otimes_R C \xrightarrow{C \otimes \mu} C \otimes_R R \xrightarrow{\rho_C} C \xrightarrow{\nu} R \end{aligned}$$

   and unital element $u_l = e_l \circ \chi_R(1) = {}^*\epsilon \circ \phi_R \circ \chi_R(1) = {}^*\epsilon(\mathsf{id}_R) = \epsilon$.

   (b) The map e_l acts as $e_l(r) = \epsilon(cr)$, for $c \in C$ and $r \in R$.

2. $\widetilde{\mathbb{D}_r}(C,\Delta,\epsilon)$ is the unital R-ring (C^*, m_r, e_r) with

$$m_r = C^* \otimes^t_R C^* \xrightarrow{\Psi_{C,C}} (C \otimes_R C)^* \xrightarrow{\Delta^*} C^* \quad \text{and} \quad e_r = R \xrightarrow{\psi_R} R^* \xrightarrow{\epsilon^*} C^*.$$

   (a) Its underlying unital ring thus is

$$I_{\otimes_R}(\widetilde{\mathbb{D}_r}(C,\Delta,\epsilon)) = (\hom_R(C,R), m_r \circ {}_R can, e_r \circ \chi_R)$$

   with multiplication

$$\begin{aligned} \mu \cdot \nu = m_r(\mu \otimes_R \nu) &= \Psi_{C,C}(\mu \otimes_R \nu) \circ \Delta \\ &= C \xrightarrow{\Delta} C \otimes_R C \xrightarrow{\nu \otimes C} C \otimes_R R \xrightarrow{\lambda_C} C \xrightarrow{\mu} R \end{aligned}$$

   and unital element $u_r = e_r \circ \chi_R(1) = \epsilon^* \circ \psi_R \circ \chi_R(1) = \epsilon^*(\mathsf{id}_R) = \epsilon$.

   (b) The map e_r acts as $e_r(r)(c) = \epsilon(rc)$, for $c \in C$ and $r \in R$.

3. $\widetilde{\mathbb{D}}(C,\Delta,\epsilon)$ is the unital ring $({}^*C^*, m, e)$ with

$$m = {}^*C^* \otimes_{\mathbb{Z}} {}^*C^* \xrightarrow{\Xi_{C,C}} {}^*(C \otimes_R C)^* \xrightarrow{{}^*\Delta^*} {}^*C^* \quad \text{and} \quad e = \mathbb{Z} \xrightarrow{\xi} {}^*R^* \xrightarrow{{}^*\epsilon^*} {}^*C^*.$$

   Its multiplication, thus, is given by

$$\begin{aligned} \mu \cdot \nu = m(\mu \otimes_{\mathbb{Z}} \nu) &= \Xi_{C,C}(\mu \otimes_{\mathbb{Z}} \nu) \circ \Delta \\ &= C \xrightarrow{\Delta} C \otimes_R C \xrightarrow{\mu \otimes \nu} R \otimes_R R \xrightarrow{\rho_R} R \end{aligned}$$

   while its unital element is ϵ.

4. $\widetilde{\mathbb{D}^t}(C,\Delta,\epsilon) = (\widetilde{\mathbb{D}}(C,\Delta,\epsilon))^{\mathsf{op}}$

15 Remark 1. When comparing our constructions with Sweedler's (see Introduction) one gets by inspection:

   (a) $\mathsf{Sw}_l(C,\Delta,\epsilon) = \big(I_{\otimes_R}(\widetilde{\mathbb{D}_l}(C,\Delta,\epsilon))\big)^{\mathsf{op}} = I_{\otimes^t_R}(\widetilde{\mathbb{D}_l}(C,\Delta,\epsilon)) = I_{\otimes_{R^{\mathsf{op}}}}(\widetilde{\widetilde{\mathbb{D}_l}}(C,\Delta,\epsilon))$.

   (b) $\lambda_{\mathsf{C}} = e_l$

   Consequently, up to the isomorphism $\Phi_{R^{\mathsf{op}}}$ of Section 2, Sweedler's construction is nothing but the functor $\widetilde{\widetilde{\mathbb{D}_l}}$, which clearly coincides with Takeuchi's functor D_l.

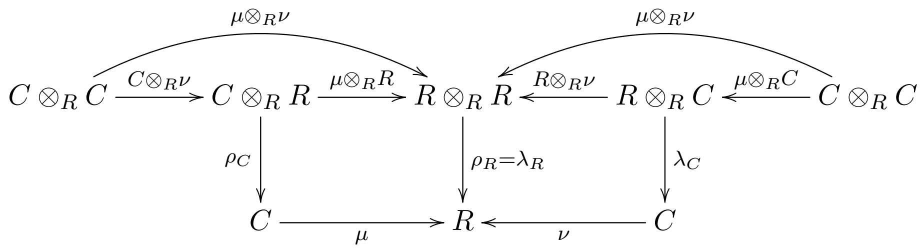
2. Similarly, Sweedler's construction $(\mathsf{Sw}_r(C,\Delta,\epsilon), \rho_C)$ coincides with $\widetilde{\widetilde{\mathbb{D}_r}}$, up to the isomorphism $\Phi_{R^{\mathsf{op}}}$.

3. In Example 3.6 [16] Sweedler provides a multiplication on $C^* = \hom_R(C,R)$, which — since, what there is called η is our $\Psi_{C,C}$ (see Remark 11) — is the map m_r of 2. above. The ring he describes here is $I_{\otimes_R}(\widetilde{\mathbb{D}_l}(C,\Delta,\epsilon))$. The difference, thus, can be described as follows: while in his original construction he applies the underlying functor $I_{\otimes^t_R}$ to the monoidal construction $\widetilde{\mathbb{D}_l}(C,\Delta,\epsilon))$, here he applies the underlying functor $I_{\otimes_R}$; and $I_{\otimes^t_R}(M,m,e)$ equals $(I_{\otimes_R}(M,m,e))^{\mathsf{op}}$ (see above).

4. The following result should be expected and is easy to check: Take an R-coring (C, Δ, ϵ) and form the dual R^{op}-rings $\widetilde{\mathbb{D}_l}(C, \Delta, \epsilon)$ and $\widetilde{\mathbb{D}_r}(C, \Delta, \epsilon)$. Now consider (C, Δ, ϵ) as an R^{op}-coring and form the respective dual R-rings, using the monoidal isomorphism from Section 1.3.2, which allows for seeing an R-(co)ring as an R^{op}-(co)ring. Then the left dual R-ring of (C, Δ, ϵ) coincides with the right dual R^{op}-ring $\widetilde{\mathbb{D}_r}(C, \Delta, \epsilon)$, considered as an R-ring.

5. We leave it to the reader to state the respective statements concerning not necessarily unital dual rings of not necessarily counital R-corings.

16 Remark If R is a commutative ring and $\mathbf{Mod}_R$ is considered instead of ${}_R\mathbf{Mod}_R$, all of these construction coincide, as is immediate from the commutativity of



which is a consequence of the fact that, for R commutative, the monoidal structure on $\mathbf{Mod}_R$ is symmetric.

Moreover, for each R-coalgebra C this construction not only gives a monoid in $\mathbb{A}\mathbb{b}$ but even a monoid in $\mathbb{M}\mathbb{o}\mathbb{d}_R$, i.e., an R-algebra, the so-called *dual algebra of the coalgebra* C. This in fact is a special instance of the familiar *convolution algebra* construction. The latter is best described as follows (see e.g [5]):

1. For each symmetric monoidal closed category $\mathbb{C}$ its internal hom-functor is a monoidal functor $[-,-]\colon \mathbb{C}^{\mathsf{op}} \otimes \mathbb{C} \to \mathbb{C}$.

2. Noting that $\mathbf{Mon}(\mathbb{C}^{\mathsf{op}} \otimes \mathbb{C}) = \mathbf{Mon}\mathbb{C}^{\mathsf{op}} \times \mathbf{Mon}\mathbb{C} = (\mathbf{Comon}\mathbb{C})^{\mathsf{op}} \times \mathbf{Mon}\mathbb{C}$ one thus obtains a functor $\widetilde{[-,-]}\colon (\mathbf{Comon}\mathbb{C})^{\mathsf{op}} \times \mathbf{Mon}\mathbb{C} \to \mathbf{Mon}\mathbb{C}$ which, in case $\mathbb{C} = \mathbb{M}\mathbb{o}\mathbb{d}_R$, is the convolution algebra functor.

Proposition 13 then takes the form of the generalization of the familiar duality between the categories $\mathbf{Coalg}_k$ of k-algebras and $\mathbf{Alg}_k$ of k-algebras for a field k (see e.g. [8]) to arbitrary commutative rings: the categories ${}_{fgp}\mathbf{Coalg}_R$ and ${}_{fgp}\mathbf{Alg}_R$ of R-coalgebras and R-algebras respectively, which are finitely generated projective as R-modules, are dually equivalent.

2.3 Dual corings of R-rings

By Proposition 13 one can restrict the functors $\mathbb{D}_l$ and $\mathbb{D}_r$ to obtain a monoidal equivalence, given by the restrictions $\overline{\mathbb{D}'}_l\colon {}_R\mathbb{FGP}^{\mathsf{op}} \to {}_{R^{\mathsf{op}}}\mathbb{FGP}$ and $\overline{\mathbb{D}'}_r\colon {}_{R^{\mathsf{op}}}\mathbb{FGP} \to {}_{R^{\mathsf{op}}}\mathbb{FGP}^{\mathsf{op}}$. By Proposition 4 the induced functors

$$\widetilde{\overline{\mathbb{D}'}}_l\colon \mathbf{Mon}({}_R\mathbb{FGP}^{\mathsf{op}}) \to \mathbf{Mon}({}_{R^{\mathsf{op}}}\mathbb{FGP}) \quad \text{and} \quad \widetilde{\overline{\mathbb{D}'}}_r\colon \mathbf{Mon}({}_{R^{\mathsf{op}}}\mathbb{FGP}) \to \mathbf{Mon}({}_R\mathbb{FGP}^{\mathsf{op}}).$$

form an adjunction, where the natural isomorphisms $\mathsf{Id} \Rightarrow \widetilde{\widetilde{\mathbb{D}'}}_l\widetilde{\widetilde{\mathbb{D}'}}_r$ and $\widetilde{\widetilde{\mathbb{D}'}}_r\widetilde{\widetilde{\mathbb{D}'}}_l \Rightarrow \mathsf{Id}$ coincide, in ${}_R\mathbf{Mod}_R$, with the units and counits of the adjunction $(-)^* \dashv {}^*(-)$. Since the latter are isomorphisms, this adjunction in fact is a duality.

In other words, one not only can assign to each (counital) R-coring C with underlying module C in ${}_R\mathbf{FGP}$ a dual (unital) R^{op}-ring (whose underlying module lies in ${}_{R^{\mathsf{op}}}\mathbf{FGP}$), but conversely, one can assign to each such coring a dual R-ring, and this defines a dual equivalence between the subcategory $\mathbf{Comon}({}_R\mathbb{FGP})$ of ${}_\epsilon\mathbf{Coring}_R$ and the subcategory $\mathbf{Mon}({}_{R^{\mathsf{op}}}\mathbb{FGP})$ of ${}_1\mathbf{Ring}_{R^{\mathsf{op}}}$. Similarly, there is a duality $(\mathbf{Comon}(\mathbb{FGP}_R))^{\mathsf{op}} \simeq \mathbf{Mon}(\mathbb{FGP}_{R^{\mathsf{op}}})$. This is the content of the *Dual Coring Theorem* of [16] (see also [17]).

Note that these dualities also exist in the non(co)unital case.

3 Unitarization and counitarization

Returning to Diagram (14) we recall that, by Theorem 4 and Remark 7 of [13], the functors V and $\bar{V}$ have left adjoints A and $\bar{A}$ respectively, the *unitarization functors*; V', the dual of the *counitarization functor*, has a left adjoint as well. Since ${}^*(-)$ is right adjoint to $(-)^*$, ${}^*(-)$ preserves (binary) products and $(-)^*$ preserves (binary) coproducts. Since ${}_R\mathbf{Mod}_R$ has biproducts and both dualization functors are additive, they preserve binary products and binary coproducts. Thus, the hypothesis of Theorem 9 of [13] are satisfied by both dualization functors and we get the following compatibility results for the operations of (co)unitarization and (left and right) dualization.

17 Proposition *Let R be a unital ring. Then the following hold for every (not necessarily counital) R-coring (C, Δ):*

1. *The unitarization of the left dual $\widehat{\mathbb{D}_l}(C, \Delta)$ coincides with the left dual $\widetilde{\mathbb{D}_l}A'(C, \Delta)$ of the counitarization of (C, Δ).*

2. *The unitarization of the right dual $\widehat{\mathbb{D}_r}(C, \Delta)$ coincides with the right dual $\widetilde{\mathbb{D}_r}A'(C, \Delta)$ of the counitarization of (C, Δ).*

Analogous statements clearly hold with respect to the dual ring functors $\widetilde{\widetilde{\mathbb{D}_l}}$ and $\widetilde{\widetilde{\mathbb{D}_r}}$ and with respect to the functors $\widetilde{\mathbb{D}}$ and $\widetilde{\mathbb{D}^t}$. We leave the precise formulation to the reader.

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