Fundamental Constructions for Coalgebras, Corings, and Comodules

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Abstract

We study the various categories of corings, coalgebras, and comodules from a categorical perspective. Emphesis is given to the question which properties of these categories can be seen as instances of general categorical resp. algebraic results. We obtain new results concerning the existence of limits and of factorizations of morphisms.

MSC 2000: Primary 16W30, Secondary 18C05

Keywords: Coalgebras, corings, comodules, limits, factorizations

Introduction

The categories $\mathbf{Coalg}_R$ of R–coalgebras, for a commutative ring R, and $\mathbf{Comod}_A$ of A–comodules for a given R–coalgebra A have attracted attention and quite a number of their categorical properties have been obtained, as well as some others labelled "unexpected" or "curious". In analogy also the categories $\mathbf{Coring}_A$ of A–corings w.r.t. a (not necessarily commutative) R–algebra A and $\mathbf{Comod}_{\mathcal{C}}$ of $\mathcal{C}$–comodules for a given A–coring $\mathcal{C}$ have been investigated (see [8] for a comprehensive overview). We will offer a common approach to these and other results through "Universal Coalgebra" the dual of a generalization of "Universal Algebra" as established by Birkhoff. We first recall the following fundamental definitions.

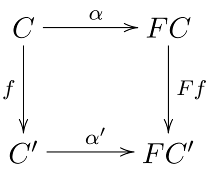
1 Definition Let $F\colon \mathbf{C} \longrightarrow \mathbf{C}$ be an endofunctor of the category $\mathbf{C}$.

An F*–algebra* is a pair (C, α) where $\alpha\colon FC \longrightarrow C$ is a $\mathbf{C}$–morphism. An F*–algebra homomorphism* $f\colon (C, \alpha) \longrightarrow (C', \alpha')$ is a $\mathbf{C}$–morphism $f\colon C \longrightarrow C'$ such that the diagram

$$\begin{array}{ccc} FC & \xrightarrow{\alpha} & C \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle f} \\ FC' & \xrightarrow{\alpha'} & C' \end{array}$$

commutes. F–algebras and their homomorphisms constitute the category $\mathbf{Alg}F$.

An F*–coalgebra* is a pair (C, α) where $\alpha\colon C \longrightarrow FC$ is a $\mathbf{C}$–morphism. An F–*coalgebra homomorphism* $f\colon (C, \alpha) \longrightarrow (C', \alpha')$ is a $\mathbf{C}$–morphism $f\colon C \longrightarrow C'$ such that the diagram



commutes. F–coalgebras and their homomorphisms constitute the category $\mathbf{Coalg}F$.

Note that Birkhoff's Ω–Algebras for a signature $\Omega = (\Omega_n)$ form the category $\mathbf{Alg}H_\Omega$ for the endofunctor H_Ω on the category $\mathbf{Set}$ of sets and mappings (with Σ denoting the coproduct (disjoint union) in $\mathbf{Set}$) defined by

$$H_\Omega(X) = \sum_n \Omega_n \times X^n$$

so motivating the notion of F–algebra.

The categories of A-corings, R–coalgebras as well as their respective module categories are defined as full subcategories of categories $\mathbf{Coalg}F$ of F–coalgebras for suitable functors F as follows

1. The category $\mathbf{Coalg}_R$ of R–coalgebras for a commutative unital ring R is the full subcategory of $\mathbf{Coalg}T_R$ for

$$\begin{array}{rccc} T_R\colon & \mathbf{Mod}_R & \longrightarrow & \mathbf{Mod}_R \\ & M & \longmapsto & (M \otimes_R M) \times R \end{array}$$

spanned by those T_R–coalgebras making the following diagrams commute where $m\colon M \longrightarrow M \otimes M$ and $e\colon M \longrightarrow R$ are the coordinates of $\alpha\colon M \longrightarrow T_R M$ and the double arrows indicate the obvious isomorphisms:

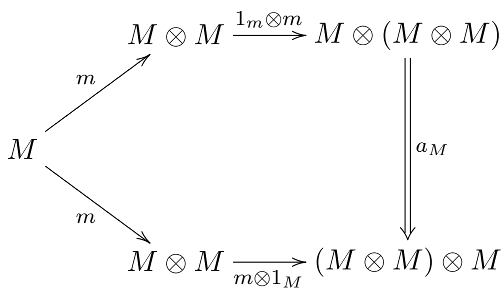


Diagram 1 (co–associative law)

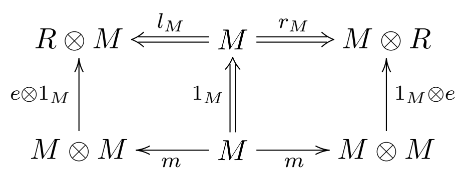


Diagram 2 (co–neutral law)

2. Similarly, given an R–coalgebra A, the category ${}_A\mathbf{Comod}$ of left A–comodules is the full subcategory of $\mathbf{Coalg}_A M$ for

$$\begin{array}{rccc} {}_AM\colon & \mathbf{Mod}_R & \longrightarrow & \mathbf{Mod}_R \\ & M & \longmapsto & A \otimes_R M \end{array}$$

spanned by those ${}_AM$–coalgebras $(M, \alpha\colon M \longrightarrow A \otimes_R M)$ which make the following diagrams commute

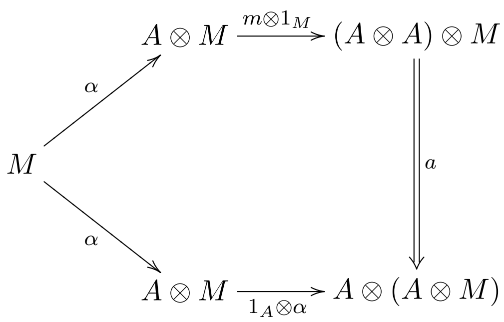


Diagram 3

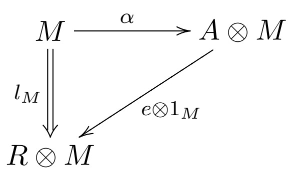


Diagram 4

where $\langle m, e\rangle\colon A \longrightarrow (A \otimes A) \times R$ is the R–coalgebra structure of A.

3. Considering instead of ${}_AM$ the functor

$$\begin{array}{rccc} M_A\colon & \mathbf{Mod}_R & \longrightarrow & \mathbf{Mod}_R \\ & M & \longmapsto & M \otimes_R A \end{array}$$

one would obtain the category $\mathbf{Comod}_A$ of right A–comodules as a subcategory of $\mathbf{Coalg} M_A$.

4. Corings and their comodule categories can be described in the same way. To obtain A–corings one simply replaces the functor $T_R\colon \mathbf{Mod}_R \longrightarrow \mathbf{Mod}_R$ by

$$\begin{array}{rccc} T_A\colon & {}_A\mathbf{Mod}_A & \longrightarrow & {}_A\mathbf{Mod}_A \\ & M & \longmapsto & (M \otimes_A M) \times A \end{array}$$

where now A is an R–algebra and ${}_A\mathbf{Mod}_A$ denotes the category of (A, A)–bimodules with the usual tensor product $- \otimes_A -$.

5. In order to obtain, for given A–corings $\mathcal{C}$ and $\mathcal{D}$, the categories ${}_\mathcal{C}\mathbf{Comod}$, $\mathbf{Comod}_\mathcal{C}$ and ${}_\mathcal{C}\mathbf{Comod}_\mathcal{D}$ of left $\mathcal{C}$–comodules, right $\mathcal{C}$–comodules and $(\mathcal{C}, \mathcal{D})$–cobicomodules, respectively (see [8]), one replaces the functors ${}_AM$ and M_A by

$$\begin{array}{rccc} {}_{\mathcal{C}}M\colon & A\text{–}\mathbf{Mod} & \longrightarrow & A\text{–}\mathbf{Mod} \\ & M & \longmapsto & \mathcal{C}\otimes_A M \end{array}$$

$$\begin{array}{rccc} M_{\mathcal{C}}\colon & \mathbf{Mod}\text{–}A & \longrightarrow & \mathbf{Mod}\text{–}A \\ & M & \longmapsto & M\otimes_A \mathcal{C} \end{array}$$

$$\begin{array}{rccc} \text{or}\quad {}_{\mathcal{C}}M_{\mathcal{D}}\colon & {}_A\mathbf{Mod}_B & \longrightarrow & {}_A\mathbf{Mod}_B \\ & M & \longmapsto & (\mathcal{C}\otimes_A M)\times(M\otimes_A \mathcal{D}) \end{array}$$

with A–**Mod**, **Mod**–A, and ${}_A\mathbf{Mod}_B$ denoting the categories of left, right A–modules and (A, B)–bimodules, respectively.

Clearly, categorical properties of, say, the category $\mathbf{Coalg}_R$ of R–coalgebras then are determined by

- properties of the category $\mathbf{Mod}_R$,
- properties of the functor T_R,
- the way $\mathbf{Coalg}_R$ is embedded in $\mathbf{Coalg}T_R$.

Now there is a well established theory how to study equationally defined subcategories $\mathbf{K}$ of categories $\mathbf{Alg}F$ of functor algebras for functors $F\colon \mathbf{C}\longrightarrow\mathbf{C}$ by means of properties of $\mathbf{C}$, F, and the way $\mathbf{K}$ is embeddded in $\mathbf{Alg}F$ (see [2]). The main results of this theory can be summarized as follows, generalizing basic results of classical universal algebra.

2 Theorem *Let* $\mathbf{C}$ *be a complete and cocomplete regularly cowellpowered category with regular factorizations. Let* $F\colon \mathbf{C}\longrightarrow\mathbf{C}$ *be a functor which preserves regular epimorphisms and which, moreover, is a constructive varietor.*

Then for a full and isomorphism closed subcategory $\mathbf{K}$ *of* $\mathbf{Alg}F$ *the following are equivalent:*

1. $\mathbf{K}$ *is a variety,*
2. $\mathbf{K}$ *is comonadic over* $\mathbf{C}$*,*
3. $\mathbf{K}$ *is closed in* $\mathbf{Alg}F$ *under*
   - *products and subalgebras, and also*
   - *homomorphic images which are carried by retractions in* $\mathbf{C}$*,*

For any such subcategory the following hold:

1. $\mathbf{K}$ *is regularly epireflective in* $\mathbf{Alg}F$*.*
2. $\mathbf{K}$ *is complete and limits in* $\mathbf{K}$ *are created by the underlying functor* $|-|\colon \mathbf{Alg}F\longrightarrow\mathbf{C}$.

3. $\mathbf{K}$ *is cocomplete.*

4. $\mathbf{K}$ *has image–factorizations and, more general, regular factorizations of cones which are created by* $|-|$.

In the above theorem completeness of $\mathbf{Alg}F$ is trivial and only needs completeness of $\mathbf{C}$. The properties 1. and 3. are consequences of 4. and property 4. is a consequence of the assumptions on F and $\mathbf{C}$ w.r.t regular factorizations.

We want to explore to what extent properties of the categories $\mathbf{Coalg}_R$, $\mathbf{Coring}_A$ and their respective categories of comodules can be obtained from the above theorem by simple dualization, i.e., without further proof.

The dualization principle at work here is the following simple observation

3 Fact *For any functor* $F\colon \mathbf{C} \longrightarrow \mathbf{C}$ *one has*

$$\mathbf{Coalg}F = (\mathbf{Alg}^{\mathsf{op}})^{\mathsf{op}}$$

where $F^{\mathsf{op}}\colon \mathbf{C}^{\mathsf{op}} \longrightarrow \mathbf{C}^{\mathsf{op}}$ *is the functor dual to* F *(i.e., acting on* $\mathbf{C}^{\mathsf{op}}$*, the dual of* $\mathbf{C}$*, as* F*).*

Thus, in order to obtain properties for the categories mentioned above, we need to know whether the functors T_R and M_A and their appropriate modifications satisfy the duals of the assumptions in Theorem 2. In the process of doing so we will also explain notions used above, which might not be familiar to the non–categorical reader as, e.g., (constructive) varietor; we will in fact then immediately explain the dualized version.

A first example of how this works shall end this introduction. The following is well known and easy to prove:

4 Fact *For endofunctors* F *on any category* $\mathbf{C}$*, the underlying functor* $\mathbf{Alg}F \longrightarrow \mathbf{C}$ *creates limits and, moreover, colimits of those types which are preserved by* F*.*

The duality principle thus yields

5 Proposition *The categories* $\mathbf{Coalg}T_R$ *and* $\mathbf{Coalg}M_A$ *have all colimits, provided* $\mathbf{C}$ *is cocomplete. These are obtained by forming the respective colimit in* $\mathbf{C}$ *and supplying this with the unique coalgebra structure which makes the colimit injections coalgebra homomorphisms.*

Another simple consequence of (the dual of) Fact 4 is the following result, a much more involved proof of which is given, e.g., in [9, 1.5.29].

6 Proposition *If* R *is a field then for each object* $\mathbb{C}$ *in* $\mathbf{Coalg}T_R$ *and* $\mathbf{Coalg}M_A$ *respectively, the following hold:*

1. *The subcoalgebras of* $\mathbb{C}$ *form a complete lattice* $\mathrm{sub}\mathbb{C}$.

2. *In* $\mathrm{sub}\mathbb{C}$ *one has* $\bigwedge = \bigcap$.

Proof: The functors T_R and M_A preserve subspaces and intersections of subspace of a given vector space (see [12, p. 568]). Since these intersections are limits the result follows from Fact 4. $\square$

1 The categories $\mathbf{Coalg}T_I$ and $\mathbf{Coalg}M_A$

In the following let $(\mathbf{C}, \otimes, I)$ be a monoidal category, where

- $\mathbf{C}$ is a variety in the sense of Birkhoff, hence, in particular, a locally finitely presentable category, which is regularly cowellpowered and has a faithful underlying functor $|-|\colon \mathbf{C} \longrightarrow \mathbf{Set}$, which creates limits and directed colimits,
- the tensor product is given by universal bimorphisms in the sense of [4]; in particular, each functor $C \otimes -\colon \mathbf{C} \longrightarrow \mathbf{C}$ is a left adjoint.

Examples of such categories are

1. the category $\mathbf{Mod}_R$ of R modules for a commutative unital ring R, and
2. the category ${}_A\mathbf{Mod}_A$ of (A, A)–bimodules for a unital (not necessarily commutative)ring A.

Given a category as above we consider the functors

$$\begin{array}{llll} T^2\colon & \mathbf{C} & \longrightarrow & \mathbf{C} \\ & C & \longmapsto & C \otimes C \end{array} \qquad \begin{array}{llll} T_I\colon & \mathbf{C} & \longrightarrow & \mathbf{C} \\ & C & \longmapsto & (C \otimes C) \times I \end{array}$$

$$\begin{array}{llll} {}_AM\colon & \mathbf{C} & \longrightarrow & \mathbf{C} \\ & C & \longmapsto & A \otimes C \end{array} \qquad \begin{array}{llll} M_A\colon & \mathbf{C} & \longrightarrow & \mathbf{C} \\ & C & \longmapsto & C \otimes A \end{array}$$

where, in the last two cases, $(A, A \xrightarrow{\langle m,e\rangle} T_IA)$ is a T_I–coalgebra. Their respective categories $\mathbf{Coalg}T_I$, $\mathbf{Coalg}M_A$, and $\mathbf{Coalg}_AM$ then are concrete over $\mathbf{C}$ by means of obvious underlying functors $|-|$.

T_I–coalgebras $(C, C \xrightarrow{\langle m,e\rangle} (C \otimes C) \times I)$ which make the diagrams 1 and 2 commute (with R replaced by I) form the category $\mathbf{Comon}$ of *comonoids* in $(C, \otimes, I)$. ${}_AM$–coalgebras $(C, C \xrightarrow{\alpha} A \otimes C)$ which make the diagrams 3 and 4 commute, form the category ${}_A\mathbf{Coact}$ of *left A–coactions* in $(C, \otimes, I)$; *right A–coactions* are defined by means of M_A in the obvious way.

Combining the various options we obtain

- $\mathbf{Coalg}_R$, the category of R–coalgebras for a commutative unital ring R is the category of comonoids in $(C, \otimes, I) = (\mathbf{Mod}_R, - \otimes_R -, R)$,
- $\mathbf{Comod}_A$, the category of right A–modules for an R–algebra A (R a commutative ring) is the category of right A–coactions in $(\mathbf{Mod}_R, - \otimes_R -, R)$,
- ${}_A\mathbf{Comod}$, the category of left A–comodules for an R–algebra A is the category of left A–coactions in $(\mathbf{Mod}_R, - \otimes_R -, R)$,
- $\mathbf{Coring}_A$, the category of A–corings for a unital (not necessarily commutative) R–algebra A is the category of comonoids in $(C, \otimes, I) = ({}_A\mathbf{Mod}_A, - \otimes_A -, A)$.

The categories of comodules for corings do not fit into this picture literally, since their respective functors M act on a category like A–**Mod**, which is not monoidal. The abstract setting here is that of a monoidal category $(\mathbf{C}, \otimes_{\mathbf{C}}, I)$ a faithful functor $|-|\colon \mathbf{C} \to \mathbf{C}'$ and a bifunctor $\mathbf{C}\times\mathbf{C}' \xrightarrow{-\otimes-} \mathbf{C}'$ with $|C_1 \otimes_{\mathbf{C}} C_2| = |C_1| \otimes |C_2|$ and natural isomorphisms $|I| \otimes - \simeq \mathsf{id}'_{\mathbf{C}}$, and $(|C_1| \otimes |C_2|) \otimes C' \simeq |C_1| \otimes (|C_2| \otimes C')$ subject to the condition that, for each C in $\mathbf{C}$, $|C| \otimes -$ is a left adjoint. The paradigmatic example is $|-|\colon {}_A\mathbf{Mod}_A \to A$–**Mod**. The arguments and results to follow nevertheless here apply as well.

We start stating the following consequence of our basic assumptions.

7 Lemma *M_A preserves all colimits and finite products, since it is a left adjoint.*

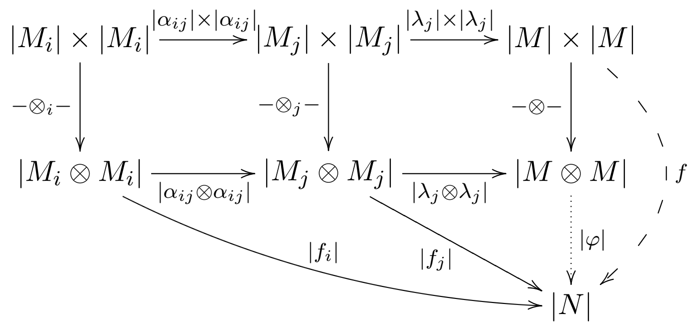
8 Proposition *The functor*

$$T^2\colon \begin{array}{ccc} \mathbf{C} & \longrightarrow & \mathbf{C} \\ M & \longmapsto & M \otimes M \end{array}$$

preserves directed colimits.

For the sake of the non–categorical reader we mention that directed colimits often are called "direct limits", a notion we consider misleading. Though the statement is probably well known we give an elementary proof of it (For a more categorical argument based on monoidal closedness see [11]) because of its importance in the present context.

Proof: Let $M_i \xrightarrow{\alpha_{ij}} M_j$ be a directed diagram in $\mathbf{C}$ and $M_j \xrightarrow{\lambda_j} M$ its colimit. Since the underlying functor $|-|\colon \mathbf{C} \longrightarrow \mathbf{Set}$ preserves directed colimits (note that this also is a consequence of Fact 4 since polynomial functors H_Ω on **Set** preserve directed colimits) and so does $X \longmapsto X \times X$ on **Set**, the top row of the following commutative diagram is a (directed) colimit:



The lower triangle in this diagram refers to any compatible family of $\mathbf{C}$–morphisms $M_j \otimes M_j \longrightarrow N$, which obviously determines a family $|f_j| \circ - \otimes_j -$, compatible with the top row diagram, thus inducing a unique map f making the outer right hand "square" commute.

From the construction of directed colimits in **Set** one deduces easily that f is a bimorphism in the sense of [4], hence induces a unique $\varphi\colon M \otimes M \longrightarrow N$ in $\mathbf{C}$ such that the right hand "triangle" commutes as well. Now the required equation

$$\varphi \circ (\lambda_j \otimes \lambda_j) = f_j$$

follows from the universal property of $- \otimes_j -$. The required uniqueness of φ is an immediate consequence of uniqueness of f w.r.t. commutativity of the right hand triangle. □

9 Corollary *T_I preserves directed colimits.*

Proof: This follows from Proposition 8 since any functor $C \times -$ preserves directed colimits in **Set** and thus in any variety. □

Since the category **C** is locally finitely presentable by assumption the following is a consequence of Corollary 7 and Lemma 9 (see [2, 3.25] or [6]) in connection with [1, 20.56].

10 Proposition *The underlying functors* $\mathbf{Coalg}T_I \to \mathbf{C}$ *and* $\mathbf{Coalg}M_A \to \mathbf{C}$ *for any comonoid* A *have right adjoints, and thus are comonadic.*

11 Remark Recall that an endofunctor F on **C** is called *covarietor* provided that $\mathbf{Coalg}F$ has cofree coalgebras, i.e., the underlying functor $\mathbf{Coalg}F \longrightarrow \mathbf{C}$ has a right adjoint. For covarietors in our context it holds that, for each **C**–object X, the couniversal linear map $\epsilon\colon X_\sharp \longrightarrow X$ from the cofree F–coalgebra $X_\sharp$ over X to X is a surjection: since X has a coalgebra structure, e.g., the zero–map, the identity 1_X factors as $1_X = \epsilon y$.

It is of importance to know (see Theorem 2) whether F is even a *constructive covarietor*, i.e., whether the cofree objects in our categories can be constructed by means of a standard construction which we sketch as follows (see [2] for details). For an object X in the base category **X**, assumed to be complete, one constructs a chain

$$1 \xleftarrow{!} X_{1\sharp} \xleftarrow{x_{21}} X_{2\sharp} \xleftarrow{x_{32}} X_{3\sharp} \longleftarrow \ldots \longleftarrow X_{i\sharp} \longleftarrow \ldots$$

by transfinite induction and then, for each $f\colon Y \longrightarrow X$ in **X**, a cone $(f_{i\sharp}\colon Y \longrightarrow X_{i\sharp})_i$ — "computation of coterms" — for this chain. If this chain stops at some ordinal κ, $X_{\kappa\sharp}$ carries an F–coalgebra structure which makes it cofree over X.

12 Proposition *Let* $F\colon \mathbf{C} \longrightarrow \mathbf{C}$ *be a functor which preserves regular epimorphisms (surjections). If* F *is a covarietor it is even a constructive covarietor.*

Proof: In order to show that the canonical chain

$$1 \xleftarrow{!} X_{1\sharp} \longleftarrow X_{2\sharp} \longleftarrow \ldots X_{i\sharp} \longleftarrow$$

stops for each module X, observe first, that the connecting homomorphisms of this chain are all surjections (we use notation as in [2])

i) $X_{1\sharp} \xrightarrow{!} 1$ is surjective,

ii) x_{ij} surjective $\Rightarrow X \times Fx_{ij}$ surjective (since F preserves surjections),

iii) the limit–projections at limit steps are surjective by the construction of directed limits in **Set** (and hence in **C**), since the connecting morphisms of the chain of which the limit is taken are surjective by ii).

Since **C** is regularly co–wellpowered it suffices for the canonical chain to stop to find a cone $Y_X \xrightarrow{f_i} X_{i\sharp}$ in **C**, compatible with that chain, consisting of surjective homomorphisms only. Construct such a cone from a surjective $f\colon Y_X \longrightarrow X$ with a surjective F–coalgebra action $\alpha\colon Y_X \longrightarrow FY_X$ as "computation of coterms" $f_{i\sharp}$:

i) $f_{0\sharp} = !\colon Y_X \longrightarrow 1$ is surjective.

ii) $f_{i\sharp}$ surjective $\Rightarrow f_{i+1\sharp} = \langle f, Ff_{i\sharp} \circ \alpha\rangle \colon Y_X \longrightarrow X_{i+1\sharp} = X \times FX_i^\sharp$ surjective.

iii) $f_{j\sharp}$, for a limit ordinal j, is surjective again by the way directed limits are constructed, since the $f_{i\sharp}, i < j$, all are surjective.

Thus we only need to find, to each module X, an F–coalgebra (Y_X, α) with α surjective and a surjective homomorphism $f\colon Y_X \longrightarrow X$. Since F is a covarietor there exists a cofree coalgebra (Y_X, β) for each module X. By Lambek's Lemma there is an isomorphism (in **C**) $Y_X \xrightarrow{\varphi} X \times FY_X$; thus, the composition α of φ with the projection onto FY_X is an F–coalgebra (Y_X, α) with α surjective. Finally the couniversal homomorphisms $\epsilon\colon Y_X \longrightarrow X$ is surjective (see Remark 11). □

The following now is an immediate consequence.

13 Theorem *The functors T_I and M_A, for any A, are constructive covarietors.*

2 Closure Properties

It is of interest to know to what extent the categories $\mathbf{Coalg}_R$ and $\mathbf{Comod}_A$ are closed in their respective categories of functor coalgebras w.r.t. certain categorical constructions. We need the following notions:

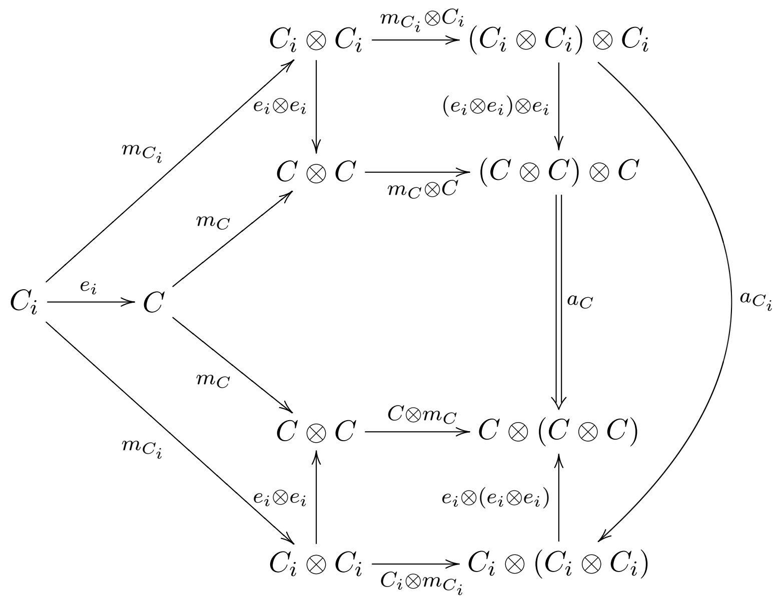
14 Definition A pair $(C, (f_i)_I)$ (where I might be a class) with a family of homomorphisms f_i with common codomain C is called *episink* provided the family $(f_i)_I$ is right cancellable, i.e., $(\forall i \in I\ \forall\ r, s\colon C \longrightarrow D\ \ rf_i = sf_i) \Rightarrow r = s$.

A subcoalgebra (C, α) of (C', α') is called *split* provided the embedding $s\colon C \longrightarrow C'$ splits in **C**, i.e., there is some $r \in \mathrm{Hom}_{\mathbf{C}}(C', C)$ with $rs = \mathsf{id}_C$.

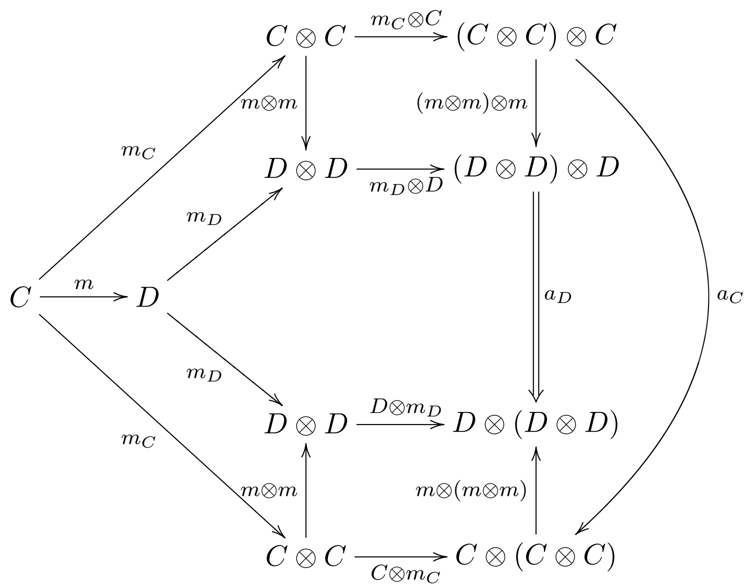
15 Proposition *The category $\mathbf{Coalg}_I$ is closed in $\mathbf{Coalg}T_I$ under episinks and split subcoalgebras, i.e.,*

1. *for every episink $\big((C, \alpha), (f_i)\big)$ in $\mathbf{Coalg}T_I$ the coalgebra (C, α) is an I–coalgebra provided the domain of each f_i is one, and*
2. *every split T_I–subcoalgebra of an I–coalgebra is an I–coalgebra.*

Proof: Ad 1: Let $e_i\colon (C_i, \alpha_i) \to (C, \alpha_C)$ be a nonempty episink (the case $I = \emptyset$ is trivial).
Let every (C_i, α_i) be co-associative. Consider the following diagram: Here the left hand cells commute since the e_i are homomorphisms; the right hand cell commutes by naturality of a; the top and bottom cells commute by (bi)functoriality of $- \otimes -$; the outer frame commutes since the (C_i, α_i) are co-associative. Thus, the inner cell commutes as required, since the e_i are jointly cancellable.



Ad 2: Let $m\colon (C, \alpha_C) \to (D, \alpha_D)$ be a monomorphism splitting in **C** with (D, α_D) co-associative. In the following diagram



the left hand cells commute since m is a homomorphism; the right hand cell commutes by naturality of a; the top and bottom cells commute by (bi)functoriality of $- \otimes -$; the inner cell commutes since the (D, α_D) is co-associative. Thus, the outer cell commutes as required, since with m the morphism $m \otimes (m \otimes m)$ is a (split) monomorphism and therefore cancellable.

In both cases (C, α_C) satisfies the co–neutral law by similar arguments. □

In the same way one proves

16 Proposition *The category* $\mathbf{Comod}_A$ *is closed in* $\mathbf{Coalg}M_A$ *under episinks and split subcoalgebras.*

17 Remark 1. Proposition 16 clearly equally holds for left coactions.

2. Proposition 15 also holds w.r.t. cocommutative coalgebras by means of a similar argument.

3. Note that closure under episinks in particular means closure under all kinds of colimits and under homomorphic images.

3 Factorizations

The image–factorization of a homomorphism is an important tool in algebra, which is intimately related to the notion of subalgebra. In (**Set**–based) algebra an image is always given by a regular epimorphism (surjection), a subalgebra by a monomorphism (injection); hence in coalgebra one might expect the dual of this situation. Being aware of the fact that what was stated above for (**Set**–based) algebra heavily depends on the fact that every **Set**–functor preserves surjections (assuming AC) the dual case would require preservation of monomorphism s which, for example is not given in general for the functors T_I and M_A. To ensure this property we therefore will have to take provision by additional assumptions. We therefore start by examining the embeddings of subcoalgebras and projections to homomorphic images in $\mathbf{Coalg}T_I$ and $\mathbf{Coalg}M_A$.

18 Proposition *Let F denote any of the functors T_I and M_A. Then the following holds:*

1. *The epimorphisms in* $\mathbf{Coalg}F$ *are precisely the surjective homomorphisms.*

2. *Every injective homomorphism in* $\mathbf{Coalg}F$ *is a strong monomorphism.*

The same holds in any full coreflective subcategory of $\mathbf{Coalg}F$*.*

Proof: 1. is clear since the underlying functor $\mathbf{Coalg}F \longrightarrow \mathbf{C}$ is a faithful left adjoint (and this argument also applies to the case of a coreflective subcategory).

2. We need to show that every commutative diagram in a full coreflective subcategory of $\mathbf{Coalg}F$

$$\begin{array}{ccc} (A,\alpha) & \xrightarrow{\;e\;} & (B,\beta) \\ {\scriptstyle f}\downarrow & \overset{d}{\swarrow} & \downarrow{\scriptstyle g} \\ (C,\gamma) & \xrightarrow[m]{} & (D,\delta) \end{array}$$

with e an epimorphism and m injective admits a unique diagonal d. By 1. and the fact that injective linear maps are strong monomorphisms in $\mathbf{C}$, $d\colon B \longrightarrow C$ exists uniquely as a linear map, which is easily seen to be a $\mathbf{Coalg}F$–morphism. □

19 Remark It can be shown that the converse of statement 2. above does not hold.

20 Remark In general it is not possible to define an image of a homomorphism in our categories. Consider the following simple example in $\mathbf{Coalg}T_{\mathbb{Z}}$:

Let $\varphi\colon (\mathbb{Z}_4, \mathsf{id}_{\mathbb{Z}_2}) \longrightarrow (\mathbb{Z}_8, 0)$ be defined by $\varphi([x]_4) = [2x]_8$. Consider the $T_{\mathbb{Z}}$–coalgebras $(\mathbb{Z}_4, \alpha)$ and $(\mathbb{Z}_8, \beta)$ where $\alpha([x]) = [3x]$ and $\beta([x]) = [2x]$. Then $\varphi\colon \mathbb{Z}_4 \longrightarrow \mathbb{Z}_8$ with $\varphi(\mathbb{Z}_4, \alpha) \longrightarrow (\mathbb{Z}_8, \beta)$ is a coalgebra homomorphism.
Now $Im\varphi = \{[0]_8,\ [2]_8,\ [4]_8,\ [6]_8\}$ and with $\varphi\colon \mathbb{Z}_4 \xrightarrow{e} Im\varphi \overset{m}{\hookrightarrow} \mathbb{Z}_4$ we have $e \otimes e([x]_4) = [2x]_8,\ m \otimes m([x]_8) = [2x]_8$.

The only coalgebra actions on $Im\varphi$ are 0, id and "multiplication by 2 or 3". One easily checks that the first three options would not make e a homomorphism, while for the last option m fails to be one.

Things are much better behaved in case F is an endofunctor on $\mathbf{Mod}_R$ where the ring R is regular; to see this we first need the following lemma.

21 Lemma *Every cocone $(M, (M_i \xrightarrow{f_i} M)_I)$ in $\mathbf{Mod}_R$, where I even might be a class, can be factored as $f_i = M_i \xrightarrow{e_i} U \xrightarrow{m} M$ where $(U, (e_i))$ is an episink and m is injective. Moreover, these factorizations are (essentially) unique.*

Proof: If $I = \emptyset$ the statement means: there exists a submodule $U \subset M$ so that the empty cocone on U is an episink: take $U = 0$!

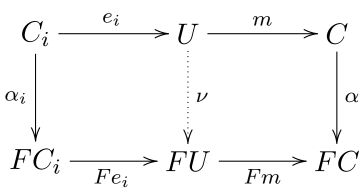
For $I \neq \emptyset$ let m be the embedding of the submodule U of M generated by $\cup f_i[M_i]$ (note that — even though I might be a class, there is only a set of submodules $f_i[M_i]$), i.e., $U = \sum f_i[M_i]$, the sum taken over a suitable subset of I, and $e_i\colon M_i \longrightarrow U$ the corestriction of f_i, for each $i \in I$. □

22 Proposition *Let F be an endofunctor on $\mathbf{Mod}_R$ which preserves monomorphisms (injective linear map). Then the statement of the lemma also holds in $\mathbf{Coalg}F$. Moreover, the factorization*

$$f_i = (C_i,\ \alpha_i) \xrightarrow{e_i} (U, \nu) \xrightarrow{m} (C, \alpha)$$

in $\mathbf{Coalg}F$ is obtained by supplying $U = \sum f_i[C_i]$ (as a submodule of C) with the unique F–coalgebra structure making m and all e_i coalgebra homomorphisms.

Proof: The required coalgebra structure on U is obtained as the "diagonal" of the following diagram, which exists by the previous lemma, since Fm is injective by assumption.



□

In order to be able to apply this proposition (see also the introductory remark of this section) we will from now on assume that R is regular — equivalently: $M \otimes i$ is injective for each R–module M and each injective R–linear map i.

23 Theorem *Let R be a regular ring. Then any of the categories* $\mathbf{Coalg}T_R$ *and* $\mathbf{Coalg}M_A$ *has*

1. *image–factorizations of (families) of homomorphisms,*
2. *kernels, which are subcoalgebras,*
3. *all limits.*

Proof: 1. is clear from the previous proposition.

For 2. we use a standard categorical argument: let $f : \mathbb{C} \longrightarrow \mathbb{C}'$ be a coalgebra homomorphism; consider the sink $(C_i, (f_i)_I)$ of all homomorphisms

$\mathbb{C}_i \xrightarrow{f_i} \mathbb{C}$ with $f \circ f_i = 0$.

Then this sink has a factorization

$\mathbb{C}_i \xrightarrow{e_i} \mathbb{K} \xrightarrow{m} \mathbb{C}$

according to Proposition 22 and $\mathbb{K} \xrightarrow{m} \mathbb{C}$ is easily seen to be a kernel of f.

3. follows from 2. since a comonadic category with equalizers is complete ([7]).

□

With image–factorizations at hand we can generalize Proposition 6 as follows:

24 Proposition *Let R be a regular ring. Then for each object $\mathbb{C}$ in any of the categories* $\mathbf{Coalg}T_R$ *and* $\mathbf{Coalg}M_A$ *the subobjects of $\mathbb{C}$ form a complete lattice.*

Proof: Let $\mathbb{C}_i = (C_i, \alpha_i)$ be a family of subobjects of $\mathbb{C} = (C, \alpha)$. By Proposition 22 $U = \sum_I C_i$ carries a unique coalgebra structure β such that all embeddings $C_i \hookrightarrow U$ and the embedding $U \hookrightarrow C$ are coalgebra homomorphisms. (U, β) then clearly is the supremum of all $\mathbb{C}_i$ in the set of subcoalgebras of $\mathbb{C}$. □

To what extent these results can be carried over to R–coalgebras and A–comodules will be shown in the next section.

4 The Covarieties of Coalgebras, Corings and Comodules

Covarieties are those full coequational subcategories of categories of the form $\mathbf{Coalg}F$ (F an endofunctor on a complete category) which have cofree coalgebras. They are precisely the comonadic categories of $\mathbf{C}$ (see [2] for details).

By the results of the previous sections we obtain by simple dualization of Theorem 2 our main result as follows.

25 Theorem *Let R be a regular ring. Then the categories $\mathbf{Coalg}_R$ and $\mathbf{Comod}_A$ are covarieties, in particular they are complete and have cofree objects. $\mathbf{Coalg}_R$ and $\mathbf{Comod}_A$ are (regular mono–)coreflective in $\mathbf{Coalg}T_R$ and $\mathbf{Coalg}M_A$ respectively. The regular monomorphisms in these categories are precisely the injective morphisms, the epimorphisms are precisely the surjective ones and every homomorphism has an image–factorization, which is created by the underlying functor.*

26 Remark It might be noted explicitely that for $\mathbf{K} = \mathbf{Coalg}_R$ and $\mathbf{K} = \mathbf{Comod}_A$ respectively

1. the coreflection of a functor–coalgebra into $\mathbf{K}$ is obtained by forming the factorization of the cocone of all morphisms

$$(\mathbb{K}_f \xrightarrow{f} \mathbb{C})_{\mathbb{K}\in \mathsf{ob}\mathbf{K},f}$$

   according to Propositon 22. If $\mathbb{K}_f \xrightarrow{e_f} \mathbb{C}^* \xrightarrow{m} \mathbb{C}$ is this factorization, then $\mathbb{C}^* \xrightarrow{m} \mathbb{C}$ is the coreflection of $\mathbb{C}$ into $\mathbf{K}$.

2. the limits in $\mathbf{K}$ are the coreflections of the limits formed in the respective category of functor–coalgebras.

27 Remark The following are obvious consequences:

1. Again, the results obtained hold for all modifications of T_R and M_A, respectively, as well as for the category ${}_c\mathbf{Coalg}_R$ of cocommutative coalgebras; concerning A–corings and their module categories it is clearly the ring A (not R) which is required to be regular.

2. $\mathbf{Comod}_A$ is comonadic over A–$\mathbf{Mod}$ for any A (not necessarily regular). This is well known (see, e.g., [8, 3.13, 18.13, 18.28]) and, in fact, only the dual of a simple generalization of [10] or [11, 1.5] to the setting described at the end of section 1.

3. For regular rings we obtain the known fact that $\mathbf{Coalg}_R$ and ${}_c\mathbf{Coalg}_R$ respectively is comonadic over $\mathbf{Mod}_R$.

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