Hopf Monoids in Varieties

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Abstract

We show that entropic varieties provide the canonical setting for a generalization of Hopf algebra theory. In particular we show that all naturally occurring functors in this context have the expected adjoints, when generalized to this level. In particular, universal measuring comonoids exist over every entropic variety, as do generalized group algebras, and these carry a canonical Hopf structure. This is done partly by specializing more general results from category theory, and partly by generalizing classical and recent results about Hopf algebras over modules. As a byproduct a list of questions concerning properties of the monoidal structure of an entropic variety is produced, which may be of interest more generally.

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Introduction

The fundamental concepts Hopf algebra theory can be formulated within the framework of symmetric monoidal categories $\mathcal{C}$ (see e.g. [5], [23]). However, if one wants to generalize familiar results about Hopf algebras to this general setting, one needs to make special assumptions on $\mathcal{C}$. In particular, the following properties of $\mathcal{C}$ are of importance.

1. The existence of adjunctions for the naturally occurring forgetful functors in this context requires that
   (a) $\mathcal{C}$ has (extremal epi, mono)- and (epi, extremal mono)-factorizations and the tensor product $f \otimes g$ of extremal epimorphisms f and g in $\mathcal{C}$ is an extremal epimorphism (see [23]).
   (b) $\mathcal{C}$ is locally presentable and the functors $C \otimes -$ preserve directed colimits (see [23]).
2. In order to have a dual monoid functor with a left adjoint or, more generally, internal convolution monoids, one needs $\mathcal{C}$ to be symmetric monoidal closed and locally presentable (see [25]).

   If one wants the internal convolution monoids to be monoids in the usual sense, one needs, in addition, a monoidal forgetful functor $\mathcal{C} \to$ **Set**, equivalently, a faithful functor $\mathcal{C} \to \mathbf{Set}$ and the condition that the tensor product is given by universal bimorphisms in the sense of [8] (see below).

3. If one wants to have the important example of a Hopf algebra structure on group algebras generalized, one needs a forgetful functor $\mathcal{C} \to \mathbf{Set}$ with a left adjoint, forming a monoidal adjunction, as will be shown below.

There is a well known class of monoidal categories meeting all of these conditions, namely the class of one-sorted finitary varieties with a commutative theory $\mathcal{T}$, in other words, the class of all entropic varieties in the sense of [11].

It, thus, seems more than natural to develop Hopf algebra theory over entropic varieties as a generalization of classical Hopf algebra theory over commutative rings. Somewhat surprisingly, hardly anything of this kind seems to have been done so far, though Daveys's paper [11] paved the way for such an endeavor already more than 30 years ago. The only attempts known to this author are the recent papers [27] and [1], which however concentrate on entropic Jónsson-Tarski varieties, that is, they only deal with the generalization of Hopf algebra theory over commutative rings to Hopf algebra theory over commutative semirings[1]. It is not really surprising that this can be done. We hence refrain from discussing concepts and properties which can only be generalized to entropic Jónsson-Tarski varieties, such as tensor-bialgebras and primitive elements; that is left to a sequel to this paper.

The intention of this paper is the following: By explicating that a substantial and non-trivial part of Hopf algebra theory can be generalized to entropic varieties we would like to raise interest in further research on these varieties. This will be done, more specifically, by showing how the attempt to generalize certain important results on Hopf algebras leads, in a natural way, to a couple of non-trivial questions about (entropic) varieties, which we consider interesting, independent of Hopf algebra theory. We here only mention the following ones; for more see Section 4.

1. Characterize (the algebraic theory of) those varieties $\mathcal{V}$ for which every subalgebra of a finitely generated sualgebra is finitely generated. Recall that, for $\mathcal{V} = \mathbf{Mod}_R$, the variety of modules over a commutative ring R, this is the case if R, that is (in some sense) the algebraic theory of $\mathbf{Mod}_R$, is noetherian.

2. Give conditions on an embedding $S \xrightarrow{i} A$ in an entropic variety $\mathcal{V}$ to be *entropically pure*, that is that, $B \otimes S \xrightarrow{B\otimes i} B \otimes A$ is an embedding, for every algebra B. Analyze the relation between such embeddings with pure embeddings in the sense of Universal Algebra (recall that both notions coincide in $\mathbf{Mod}_R$).

3. Give conditions on an algebra B in an entropic variety $\mathcal{V}$ to be *entropically flat*, that is that $B \otimes S \xrightarrow{B\otimes i} B \otimes A$ is an embedding, for every embedding $S \xrightarrow{i} A$ (Recall that this holds for each B in $\mathbf{Mod}_R$, if an only if B is flat in the sense of module theory).

[1]Entropic Jónsson-Tarski varieties are up to concrete equivalence the categories of semi-modules over commutative semirings (see e.g. [18] or [26]).

Finally a word towards the language used in this paper: Since the objects of the monoidal categories under consideration in this note are algebras, we avoid talking about Hopf algebras and bialgebras in varieties (except when these are module categories) and prefer the terms Hopf monoids and bimonoids in order to avoid possible confusion.

The paper is organized as follows:

In Section 1 we collect some known facts about entropic varieties with an emphasis on the monoidal functors of importance and on the introduction of some new properties, algebras or varieties should have in this context.

In Section 2 we first recall the basic concepts of monoid and comonoid (including generalizations of group algebras and universal measuring comonoids). Next we specialize the concept of Hopf monoid in a symmetric monoidal category and the existence of adjoints to the obvious forgetful functors in this context to entropic varieties and discuss group-like elements and (bi)modules over a monoid.

In Section 3 we discuss the question to what extent Sweedler's finite dual coalgebra construction can be generalized to more general varieties than categories of modules, generalizing some recent results of [25]. It is here that the questions concerning entropical purity arise.

The final Section 4 contains a list of problems concerning the monoidal structure in an entropic variety, which arose in this work (extending the respective list above), and which we believe are of a more general interest for universal algebraists.

Finally, for the convenience of the reader not too familiar with the theory of monoidal categories, there is added an appendix containing the basics on monoidal functors, which are indispensable in this note.

1 Preliminaries

1.1 Terminology

By a variety $\mathcal{V}$ we mean a finitary one-sorted variety, considered as a concrete category over **Set**, the category of sets. Throughout we make use of the following notational conventions: The free $\mathcal{V}$-algebra over an n-element set will be denoted by Fn. The set of $\mathcal{V}$-homomorphisms $A \to B$ is denoted by $\mathcal{V}(A, B)$. Given an element x of a $\mathcal{V}$-algebra A, the $\mathcal{V}$-homomorphism $F1 \to A$ with $1 \mapsto x$ will be denoted by x as well. For a homomorphism $A \xrightarrow{f} B$ and an algebra C in an entropic variety we usually denote the homomorphism $A \otimes C \xrightarrow{f \otimes id_C} B \otimes C$ simply by $f \otimes C$.

Recall that every variety is a locally finitely presentable category. It has, in particular, (extremal episink, mono)- and (episink, extremal monosource)-factorizations in the sense of [3].

Definition 1 A variety $\mathcal{V}$ will be called

1. *noetherian*, provided that every subalgebra of a finitely generated $\mathcal{V}$-algebra is finitely generated as well.

2. *set-like*, if finite products of finitely generated $\mathcal{V}$-algebras are finitely generated[2].

Examples 1

1. Every semi-additive variety is set-like, since it has biproducts. In particular the varieties $\mathbf{Mod}_R$ of R-modules over commutative rings R and $\mathbf{SMod}_S$ of semi-modules over commutative semirings S are set-like.

2. $\mathbf{Mod}_R$ is noetherian if and only if the ring R is noetherian.

3. $\mathbf{SMod}_S$ is noetherian if and only if the semiring S is noetherian in the sense of [12].

4. Every variety, in which the free algebra over a finite set is finite, is a set-like and noetherian variety. In particular the varieties **Set** of sets, $\mathbf{Set}_*$ of pointed sets, and **SLat** of semilattices are set-like and noetherian.

1.2 Monomorphisms in varieties

While the various classes of epimorphisms (or *epis* for short) in a variety $\mathcal{V}$ are well understood in the sense that one has for a homomorphism $A \stackrel{e}{\to} B$ in $\mathcal{V}$ the following equivalences

$$e \text{ is a regular epi} \iff e \text{ is an extremal epi} \iff e \text{ is surjective}$$

and every surjective homomorphism is an epimorphism, but not conversely as shown by the embedding $m\colon \mathbb{Z} \hookrightarrow \mathbb{Q}$ in the variety of rings; this is an epimorphism and a monomorphism, but not an extremal epimorphism, for if it were, it would be an isomorphism[3].

The situation for monomorphisms (for short *monos*) m is as follows: m is a monomorphism if and only if m is injective; moreover, there are the following implications, where none of them is reversible in general.

$$m \text{ is a regular mono} \implies m \text{ is an extremal mono} \implies m \text{ is a mono}$$

Lemma 1 *For a variety $\mathcal{V}$ the following are equivalent*

1. *Every injective homomorphism is an extremal monomorphism.*

2. *Every epimorphism is surjective.*

[2]For a more conceptual description of this condition, using the language of categorical algebra see [26].

[3]In any category a morphism is an isomorphism if and only if it is an epimorphism and an extremal monomorphism.

1.3 Entropic varieties

It is well known (see [10] or [11]), that every variety whose theory is commutative, is a symmetric monoidal closed category; following [11] we call any such symmetric monoidal closed category $\mathcal{V}$ an *entropic variety.* The tensor product of $\mathcal{V}$ is given by universal bimorphisms. In more detail, for algebras A and B in $\mathcal{V}$ their tensor product $A \otimes B$ is characterized by the fact, that there is a bimorphism $A \times B \xrightarrow{-\otimes-} A \otimes B$ over which each bimorphism $A \times B \to C$ factors uniquely as $f = g \circ (- \otimes -)$ with a homomorphism $g\colon A \otimes B \to C$. $A \otimes B$ is generated by $\{a \otimes b \mid (a, b) \in A \times B\}$. The unit object I for this monoidal structure is the free algebra $F1$. The canonical isomorphism $F1 \otimes A \xrightarrow{can_A} A$ is given by the bimorphism $(t, a) \mapsto t^A(a)$ where t is a unary term and t^A its interpretation in A. The symmetry is given by the $\mathcal{V}$-homomorphisms $\sigma_{A,B}\colon A \otimes B \to B \otimes A$ with $a \otimes b \mapsto b \otimes a$. The internal hom-functor $[-,-]\colon \mathcal{V}^{\mathsf{op}} \times \mathcal{V} \to \mathcal{V}$ of $\mathcal{V}$ is given by the $\mathcal{V}$-algebras $[A, B]$ of all $\mathcal{V}$-homomorphisms from A to B, considered as subalgebras of B^A (see e.g. [11]). As usual, the counits of the adjunctions $A \otimes - \dashv [A, -]$, that is, the couniversal maps $A \otimes [A, B] \to B$, are called *evaluations* and denoted by *ev*. There is a canonical isomorphism $[F1, F1] \simeq F1$, often denoted by *can*. In an entropic variety all operations are homomorphisms (see [11]). An entropic variety $\mathcal{V}$ has at most one nullary operation 0, and every $\mathcal{V}$-algebra A contains $\{0\}$ as a one-element subalgebra (see [11]).

Examples 2 The following varieties are entropic:

1. **Set**; here the tensor product is the binary product and its unit is the singleton 1.
2. $\mathbf{Set}_*$, the category of pointed sets.
3. **SLat** and $\mathbf{SLat_0}$, the category of join-semilattices with zero. Note that the category of distributive lattices, though being a symmetric monoidal subcategory of **SL**, fails to be an entropic variety; in particular, its tensor product is not given by universal bimorphisms.
4. **Ab**, the category of abelian groups and, more generally, all varieties $\mathbf{Mod}_R$ for a commutative ring R.
5. ${}_c\mathbf{Monoids}$, the category of commutative monoids and, more generally, all varieties $\mathbf{SMod}_S$ for a commutative semiring S.

Examples 3 The following functors, related to an entropic variety $\mathcal{V}$, are monoidal (see the Appendix for a definition):

1. The forgetful functor $|-|\colon \mathcal{V} \to \mathbf{Set}$ with unit $1 \xrightarrow{\eta_1} |F1|$ and multiplication $|A| \times |B| \xrightarrow{-\otimes-} |A \otimes B|$.
2. The tensor functor $-\otimes-\colon \mathcal{V}\times\mathcal{V} \to \mathcal{V}$ with unit $F1 \xrightarrow{\simeq} F1\otimes F1$ and multiplication $(A \otimes B) \otimes (A' \otimes B') \xrightarrow{A\otimes\sigma\otimes B'} (A \otimes A') \otimes (B \otimes B')$.

3. The internal hom-functor $[-,-]\colon \mathcal{V}^{\mathsf{op}} \times \mathcal{V} \to \mathcal{V}$ with unit $F1 \xrightarrow{can} [F1, F1]$ and multiplication $[A,B] \otimes [A',B'] \to [A \otimes A', B \otimes B']$ given by $(f,g) \mapsto f \otimes g$.

4. The hom-functor $\mathcal{V}(-,-) = |-| \circ [-,-]\colon \mathcal{V}^{\mathsf{op}} \times \mathcal{V} \to \mathbf{Set}$

5. The *semi-dualization functor* $(-)^* := [-, F1]\colon \mathcal{V}^{\mathsf{op}} \to \mathcal{V}$ with unit $F1 \simeq [F1, F1]$ being the canonical isomorphism and with multiplication $A^* \otimes B^* \xrightarrow{-\otimes-} (A \otimes B)^*$ given by $(f,g) \mapsto (A \otimes B \xrightarrow{f \otimes g} F1 \otimes F1 \simeq F1)$.

Lemma 2 *In an entropic variety $\mathcal{V}$ the following hold.*

1. *$F(X \times Y) \simeq FX \otimes FY$, for arbitrary sets X and Y.*

2. *The tensor product $f \otimes g$ of extremal epimorphisms f and g is an extremal epimorphism.*

3. *The full subcategory ${}_{fg}\mathcal{V}$ of $\mathcal{V}$, spanned by all finitely generated $\mathcal{V}$-algebras, is closed under tensor products and so form a monoidal subcategory of $\mathcal{V}$.*

4. *The full subcategories ${}_{p}\mathcal{V}$ and ${}_{fgp}\mathcal{V}$ of $\mathcal{V}$, spanned by all projective[4] algebras and finitely generated projective algebras, respectively, are closed under tensor products and so form monoidal subcategories of $\mathcal{V}$.*

Proof: 1. and 2. are immediate, since $F1 \otimes F1 \simeq F1$ and all functors $A \otimes -$ preserve coproducts and coequalizers, respectively (use [19, V.3]). The rest follows from 1 and 2 and by functoriality of the tensor product. □

Lemma 3 *The free algebra functor $F\colon \mathbf{Set} \to \mathcal{V}$ of entropic variety $\mathcal{V}$, considered as the opmonoidal left adjoint of $|-|$, is strong. Thus the adjunction $F \dashv |-|$ is a monoidal adjunction.*

Proof: It immediately follows from the definition (see the diagrams in Remarks 4 of the appendix) that the counit of F is $F1 \xrightarrow{id} F1$ and the comultiplication is $F(A \times B) \xrightarrow{(a,b) \mapsto a \otimes b} FA \otimes FB$. It thus remains to show that the homomorphic extension of the map with $(a,b) \mapsto a \otimes b$ is bijective. But this is the isomorphism of item 1 of Lemma 2. □

Lemma 4 *Given finitely generated algebras A and B in an entropic variety $\mathcal{V}$, the algebra $[A,B]$ is finitely generated provided that $\mathcal{V}$ is noetherian and set-like.*

[4] Projectivity is here to be understood (as usually in Universal Algebra) as projectivity with respect to surjective homomorphisms. Thus, more precisely one should rather say *regular projective* in these statements.

Proof: Assume that $A \simeq Fn/\rho$ for some $n \in \mathbb{N}$ and some $\mathcal{V}$-congruence ρ. Then $[A, B] = [Fn/\rho, B]$ is (isomorphic to) $\{f \in \mathrm{Hom}(Fn, B) \mid \rho \subset \ker f\}$, which is a subalgebra of $[Fn, B] \simeq B^n$. Since finite powers of a finitely generated algebra are finitely generated by assumption, $[A, B]$ is finitely generated, since $\mathcal{V}$ is noetherian. $\square$

It is well known that the concept of a pure embedding $S \xrightarrow{i} A$ as used in universal algebra is, for varieties $\mathbf{Mod}_R$, equivalent to the condition that $S \otimes X \xrightarrow{i \otimes X} A \otimes X$ is an (extremal) monomorphism, for every R-module X. For more general entropic varieties nothing seems to be known regarding the relations between these conditions. Motivated by the respective notions for R-modules we introduce the following notions.

Definition 2

1. An extremal monomorphism $S \xrightarrow{i} A$ in an entropic variety $\mathcal{V}$ will be called *entropically pure* and S will be called an *entropically pure subalgebra of* A, provided that $S \otimes X \xrightarrow{i \otimes X} A \otimes X$ is an extremal monomorphism, for every $\mathcal{V}$-algebra X.

2. An algebra X in an entropic variety $\mathcal{V}$ will be called *entropically flat*, provided that $m \otimes X$ is an extremal monomorphism, for every extremal monomorphism m.

3. An entropic variety is called *flat*, provided that for every extremal monomorphism m in $\mathcal{V}$ its tensor square $m \otimes m$ is an extremal monomorphism[5].

The following is an immediate consequence of the definition of the tensor product.

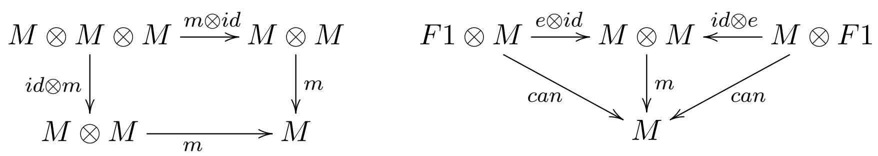
Lemma 5 *If S is an entropically pure subalgebra of A, then $(A \otimes S) \cap (S \otimes A) = S \otimes S$.*

2 Hopf monoids in entropic varieties

2.1 Monoids and comonoids in entropic varieties

2.1.1 The categories $\mathsf{Mon}\mathcal{V}$ and $\mathsf{Comon}\mathcal{V}$

Recall that a *monoid* in $\mathcal{V}$ is a triple $\mathsf{M} = (M, M \otimes M \xrightarrow{m} M, F1 \xrightarrow{e} M)$ in $\mathcal{V}$ such that the following diagrams commute:



With morphisms $\mathsf{M} \xrightarrow{f} \mathsf{M}'$ being those $\mathcal{V}$-homomophisms $M \xrightarrow{f} M'$ such that the following diagrams commute, the monoids in $\mathcal{V}$ form the category $\mathsf{Mon}\mathcal{V}$.

[5]Obviously, an entropic variety $\mathcal{V}$ is flat, if every $\mathcal{V}$-algebra X is entropically flat.

$$
\begin{array}{ccc}
M \otimes M & \xrightarrow{f \otimes f} & M' \otimes M' \\
{\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m'} \\
M & \xrightarrow[f]{} & M'
\end{array}
\qquad\qquad
\begin{array}{ccc}
F1 & \xrightarrow{e} & M \\
 & {\scriptstyle e'}\searrow & \downarrow{\scriptstyle f} \\
 & & M'
\end{array}
$$

As already shown in [11] $\mathsf{Mon}\mathcal{V}$ is a variety, though not an entropic one.

There is a forgetful functor $[-]\colon \mathsf{Mon}\mathcal{V} \to \mathbf{Monoids}$ into the category of (algebraic) monoids, induced by the (monoidal) forgetful functor of $\mathcal{V}$. The monoid $[\mathsf{M}]$ will be called the *underlying monoid* of the $\mathcal{V}$-monoid M. Since $\mathsf{Mon}\mathcal{V}$ is a variety, the functor $[-]$ is an algebraic functor in the sense of [3].

Given a monoid $\mathsf{M} = (M, M \otimes M \xrightarrow{m} M, F1 \xrightarrow{e} M)$, the *opposite monoid of* M is the monoid $\mathsf{M}^{\mathsf{op}} = (M, M \otimes M \xrightarrow{\sigma_{M,M}} M \otimes M \xrightarrow{m} M, F1 \xrightarrow{e} M)$.

A monoid M is *commutative*, provided that $\mathsf{M}^{\mathsf{op}} = \mathsf{M}$. We denote the category of commutative monoids by ${}_c\mathsf{Mon}\mathcal{V}$. As in every symmetric monoidal category $F1$ becomes a commutative monoid $\mathsf{F1}$ in $\mathcal{V}$ with multiplication $F1 \otimes F1 \xrightarrow{can_{F1}} F1$ and unit $F1 \xrightarrow{id} F1$. Its underlying monoid is the monoid described in [11, Prop. 2.2]. ${}_c\mathsf{Mon}\mathcal{V}$ is an entropic variety with $\mathsf{F1}$ as its unital object.

A *comonoid* $\mathsf{C} = (C, C \xrightarrow{\Delta} C \otimes C, C \xrightarrow{\epsilon} F1)$ in $\mathcal{V}$ is a monoid in $\mathcal{V}^{\mathsf{op}}$, and the category $\mathsf{Comon}\mathcal{V}$ of comonoids in $\mathcal{V}$ is defined to be $(\mathsf{Mon}\mathcal{V}^{\mathsf{op}})^{\mathsf{op}}$. The *coopposite comonoid* $\mathsf{C}^{\mathsf{cop}}$ of a comonoid C and the category ${}_{coc}\mathsf{Comon}\mathcal{V}$ of *cocommutative* comonoids are defined dually.

Examples 4

1. $\mathsf{Mon}\mathbf{Set} = \mathbf{Monoids}$. $\mathsf{Comon}\mathbf{Set}$ is isomorphic to $\mathbf{Set}$, since the only comonoid structure an a set X is given by the diagonal $X \xrightarrow{\Delta} X \times X$ and the unique map $X \xrightarrow{!} 1$. The same holds in every category with finite products, which is symmetric monoidal with binary product taken as tensor product and the terminal object as unit object. An example is ${}_{coc}\mathsf{Comon}\mathcal{V}$.

2. $\mathsf{Mon}\mathbf{Mod}_R$ is the usual category $\mathbf{Alg}_R$ of R-algebras while the category $\mathsf{Comon}\mathbf{Mod}_R$ is called the category of *R-coalgebras*, denoted by $\mathbf{Coalg}_R$.

3. As every symmetric monoidal closed category, an entropic variety $\mathcal{V}$ is enriched over itself. In particular, for every $\mathcal{V}$-algebra A its endomorphisms not only form a monoid, but a $\mathcal{V}$-monoid $[A, A]$, whose multiplication c maps a pair (f, g) of endomorphisms to $f \circ g$, that is, the following diagram commutes.

$$
\begin{array}{ccc}
[A,A] \otimes [A,A] \otimes A & \xrightarrow{c \otimes A} & [A,A] \otimes A \\
{\scriptstyle [A,A]\otimes ev}\downarrow & & \downarrow{\scriptstyle ev} \\
[A,A] \otimes A & \xrightarrow[ev]{} & A
\end{array}
$$

   Its unit is the homomorphism with $1 \mapsto id$.
   It is easy to see that, for any pair (A, B) of $\mathcal{V}$-algebras, the $\mathcal{V}$-homomorphism

$\Lambda\colon [A,A]\otimes[B,B] \to [A\otimes B, A\otimes B]$ making $[-,-]$ a monoidal functor is a morphism of $\mathcal{V}$-monoids.

Since the functor $-\otimes-\colon \mathcal{V}\times\mathcal{V}\to\mathcal{V}$ is symmetric monoidal, it induces a functor $-\otimes-\colon \mathsf{Mon}\mathcal{V}\times\mathsf{Mon}\mathcal{V} = \mathsf{Mon}(\mathcal{V}\times\mathcal{V}) \to \mathsf{Mon}\mathcal{V}$. This implies much of the following, for the rest see e.g. [22].

Proposition 1 *For every entropic variety $\mathcal{V}$ the following holds.*

1. $\mathsf{Mon}\mathcal{V}$ *is a symmetric monoidal category.*
2. $\mathsf{Comon}\mathcal{V}$ *is a symmetric monoidal closed category.*
3. ${}_c\mathsf{Mon}\mathcal{V}$ *is reflective in* $\mathsf{Mon}\mathcal{V}$ *and* ${}_{coc}\mathsf{Comon}\mathcal{V}$ *is coreflective in* $\mathsf{Comon}\mathcal{V}$.

Moreover, there is the following generalization of the concept of monoid algebras and group algebras respectively[6]. This is an immediate consequence of the fact that the adjunction $F\dashv|-|$ is a monoidal adjunction (see Lemma 3) and, thus can be lifted to the monoid level.

Proposition 2 *The forgetful functor* $[-]\colon \mathsf{Mon}\mathcal{V}\to\mathbf{Monoids}$ *has a left adjoint* $\mathcal{V}[-]$ *and for each monoid* M *the underlying* $\mathcal{V}$*-algebra of* $\mathcal{V}[\mathsf{M}]$ *is the free* $\mathcal{V}$*-algebra over* M.

Given a monoid M, we call the $\mathcal{V}$-monoid $\mathcal{V}[\mathsf{M}]$ the *$\mathcal{V}$-monoid algebra of* M, and the *$\mathcal{V}$-group algebra of* M , respectively, if M is a group.

2.1.2 Convolution monoids and universal measuring comonoids

The internal hom-functor of $\mathcal{V}$ is a symmetric monoidal functor (see Examples 3) and so induces a functor $[-,-]\colon (\mathsf{Comon}\mathcal{V})^{\mathsf{op}}\times\mathsf{Mon}\mathcal{V}\to\mathsf{Mon}\mathcal{V}$. This is a symmetric monoidal functor (see e.g. [25]).

Given a comonoid C and a monoid A in $\mathcal{V}$, the underlying monoid $Conv(\mathsf{C},\mathsf{A})$ of the $\mathcal{V}$-monoid $[\mathsf{C},\mathsf{A}]$ is called the *convolution monoid* of the pair (C,A). Note that the underlying set of $Conv(\mathsf{C},\mathsf{A})$ is $\mathcal{V}(C,A)$.

The multiplication of the convolution monoid, usually denoted by $-*-$, is given by $(C\xrightarrow{f}A, C\xrightarrow{g}A)\mapsto C\xrightarrow{\mu}C\otimes C\xrightarrow{f\otimes g}A\otimes A\xrightarrow{m}A$, while $C\xrightarrow{\epsilon}F1\xrightarrow{e}A$ is its unit.

Since $\mathcal{V}$ is locally presentable one gets from [24] (see also [13] or [17])

Proposition 3 *Let $\mathcal{V}$ be an entropic variety and* M *be a monoid in $\mathcal{V}$. Then the functor* $[-,\mathsf{M}]\colon(\mathsf{Comon}\mathcal{V})^{\mathsf{op}}\to\mathsf{Mon}\mathcal{V}$ *has a left adjoint* $\mu(-,\mathsf{M})$, *that is, there exists a natural isomorphism*

$$\mathsf{Mon}\mathcal{V}(\mathsf{A},[\mathsf{C},\mathsf{M}])\simeq\mathsf{Comon}\mathcal{V}(\mathsf{C},\mu(\mathsf{A},\mathsf{M})).$$

[6]Recall the the classical construction of the monoid algebra $R[\mathsf{M}]$ over a commutative ring R for some monoid M provides a left adjoint to the forgetful functor $\mathbf{Alg}_R\to\mathbf{Monoids}$. Restricting this functor to **Grp**, the category of groups, gives the group ring functor, and this is left adjoint to the functor $U\circ|-|$, where U is the coreflection of **Monoids** into **Grp**, assigning to a monoid M the group $U\mathsf{M}$ of its invertible elements.

By parametrized adjunctions (see [19, p. 102]) the assignment $(\mathsf{A}, \mathsf{M}) \mapsto \mu(\mathsf{A}, \mathsf{M})$ yields a functor $\mu\colon (\mathsf{Mon}\mathcal{V})^{\mathsf{op}} \times \mathsf{Mon}\mathcal{V} \to \mathsf{Comon}\mathcal{V}$ by . As shown in [17] this functor makes $\mathsf{Mon}\mathcal{V}$ an enriched category over the symmetric monoidal category $\mathsf{Comon}\mathcal{V}$.

Following Sweedler (see [28])), the comonoid $\mu(\mathsf{A}, \mathsf{M})$ is called *universal measuring comonoid*, for any pair of monoids (A, M). These comonoids, thus, exist over every entropic variety, not only over vector spaces as in [28].

2.2 Bimonoids and Hopf monoids in entropic varieties

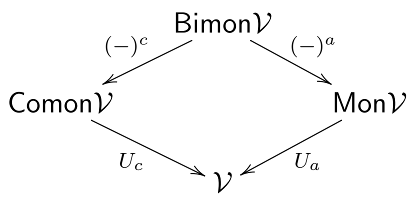
Since the categories $\mathsf{Mon}\mathcal{V}$ and $\mathsf{Comon}\mathcal{V}$ are symmetric monoidal again, one can form the categories of monoids and comonoids, respectively, in these categories. The following then is well known (see e.g. [23]).

1. The categories $\mathsf{MonComon}\mathcal{V}$ and $\mathsf{ComonMon}\mathcal{V}$ are isomorphic.

2. $\mathsf{MonMon}\mathcal{V} = {}_c\mathsf{Mon}\mathcal{V}$ and $\mathsf{ComonComon}\mathcal{V} = {}_{coc}\mathsf{Comon}\mathcal{V}$.

By $\mathsf{Bimon}\mathcal{V}$ we denote the category of *bimonoids in* $\mathcal{V}$, that is of quintuples (B, m, e, μ, ϵ), where $\mathsf{B}^a = (B, m, e)$ is a monoid in $\mathsf{Comon}\mathcal{V}$ (equivalently, where $\mathsf{B}^c = (B, \mu, \epsilon)$ is a comonoid in $\mathsf{Mon}\mathcal{V}$). By slight abuse of notation we write $\mathsf{Bimon}\mathcal{V} = \mathsf{MonComon}\mathcal{V} = \mathsf{ComonMon}\mathcal{V}$. For every bimonoid $\mathsf{B} = (\mathsf{B}^a, \mathsf{B}^c)$ the pairs $\mathsf{B}^{\mathsf{op}} = ((\mathsf{B}^a)^{\mathsf{op}}, \mathsf{B}^c)$, $\mathsf{B}^{\mathsf{cop}} = (\mathsf{B}^a, (\mathsf{B}^c)^{\mathsf{cop}})$ and $\mathsf{B}^{\mathsf{op,cop}} = ((\mathsf{B}^a)^{\mathsf{op}}, (\mathsf{B}^c)^{\mathsf{cop}})$ are bimonoids. A bimonoid B is *(co)commutative*, if B^a is a commutative monoid (B^c is a cocommutative comonoid).

The following diagram depicts the various forgetful functors occuring here.



It again follows from [22] that one has

Proposition 4 *For every entropic variety $\mathcal{V}$ the category ${}_c\mathsf{Bimon}\mathcal{V}$ is reflective in $\mathsf{Bimon}\mathcal{V}$ and the category ${}_{coc}\mathsf{Bimon}\mathcal{V}$ is coreflective in $\mathsf{Bimon}\mathcal{V}$.*

The following conditions are known to be equivalent for a bimonoid $\mathsf{B} = (B, m, e, \mu, \epsilon)$ over any symmetric monoidal category $\mathcal{V}$ (see e.g. [23]):

- The identity id_B has an inverse S in the convolution monoid of $(\mathsf{B}^c, \mathsf{B}^a)$.

- There exists a $\mathcal{V}$-homomorphism $S\colon B \to B$ satisfying the equations

$$(1)\quad B \xrightarrow{\mu} B \otimes B \xrightarrow{S \otimes \mathsf{id}_B} B \otimes B \xrightarrow{m} B = \qquad B \xrightarrow{\epsilon} F1 \xrightarrow{e} B$$

$$(2)\qquad\qquad = \; B \xrightarrow{\mu} B \otimes B \xrightarrow{\mathsf{id}_B \otimes S} B \otimes B \xrightarrow{m} B$$

- There exists a bimonoid morphism $S\colon \mathsf{B} \to \mathsf{B}^{\mathsf{op,cop}}$ satisfying equations (1) and (2).

- The $\mathcal{V}$-homomorphism $B \otimes B \xrightarrow{id \otimes \mu} B \otimes B \otimes B \xrightarrow{m \otimes id} B \otimes B$ is an isomorphism.

Definition 3 A bimonoid in $\mathcal{V}$ satisfying the equivalent conditions above is called a *Hopf monoid in* $\mathcal{V}$ with *antipode* S.

The full subcategory of $\mathsf{Bimon}\mathcal{V}$ spanned by all Hopf monoids in $\mathcal{V}$ is the category $\mathsf{Hopf}\mathcal{V}$ of Hopf monoids in $\mathcal{V}$.

This definition of morphisms in $\mathsf{Hopf}\mathcal{V}$ is reasonable, since antipodes of Hopf monoids are, obviously, uniquely determined and preserved by bimonoid morphisms.

Examples 5

1. Hopf monoids in **Set** are groups by Examples 4(1).
2. Hopf monoids in $\mathbf{Mod}_R$ are R-Hopf algebras.
3. $\mathcal{V}$-group algebras are Hopf monoids in $\mathcal{V}$, for any entropic variety $\mathcal{V}$.
   In fact, by Proposition 2 in connection with Remarks 4 the $\mathcal{V}$-monoid algebra functor $\mathcal{V}[-]$ is an opmonoidal functor and, thus, maps comonoids in **Monoids** to comonoids in $\mathsf{Mon}\mathcal{V}$; hence $\mathcal{V}[\mathsf{M}]$ is a bimonoid in $\mathcal{V}$, for each monoid M. By [25, Prop. 31] this functor maps Hopf monoids in **Set**, that is, groups to Hopf monoids in $\mathcal{V}$ as well, since $|\mathcal{V}[-]| \simeq F \circ |-|$. Hence, $\mathcal{V}[\mathsf{G}]$ is a Hopf monoid in $\mathcal{V}$, for every group G.[7]

2.2.1 Free and cofree bialgebras

By specialization of more general facts from [23] one gets the following theorem.

Theorem 1 *For an entropic variety $\mathcal{V}$ the following hold:*

1. *The forgetful functors* $\mathsf{Mon}\mathcal{V} \to \mathcal{V}$ *and* $\mathsf{Bimon}\mathcal{V} \to \mathsf{Comon}\mathcal{V}$ *have left adjoints.*
2. *The forgetful functors* $\mathsf{Comon}\mathcal{V} \to \mathcal{V}$ *and* $\mathsf{Bimon}\mathcal{V} \to \mathsf{Mon}\mathcal{V}$ *have right adjoints.*
3. *The forgetful functor* $\mathsf{Hopf}\mathcal{V} \to \mathsf{Mon}\mathcal{V}$ *has a right adjoint.*
4. $\mathsf{Hopf}\mathcal{V}$ *is a coreflective subcategory of* $\mathsf{Bimon}\mathcal{V}$*.*
5. *The forgetful functor* $\mathsf{Hopf}\mathcal{V} \to \mathsf{Comon}\mathcal{V}$ *has a left adjoint, provided that $\mathcal{V}$ is flat.*
6. $\mathsf{Hopf}\mathcal{V}$ *is reflective in* $\mathsf{Bimon}\mathcal{V}$*, provided that $\mathcal{V}$ is flat.*

Here only the constructions of free $\mathcal{V}$-monoids over $\mathcal{V}$-algebras and of free $\mathcal{V}$-bimonoids over $\mathcal{V}$-comonoids are trivial in the sense that they can be obtained by the standard free monoid construction.

[7] Note that this argument is much simpler then the standard argument for group algebras over commutative rings.

2.2.2 Group-like elements

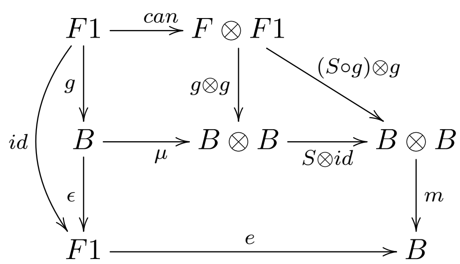
Definition 4 An element g of a $\mathcal{V}$-comonoid C is called *group-like*[8], provided that $\mu(g) = g \otimes g$ and $\epsilon(g) = 1$. By C_g we denote the set of group-like elements of C.

An element $g \in C$ obviously is group-like if and only if $g\colon \mathsf{F1} \to \mathsf{C}$ is a comonoid morphism. We thus have a bijection $\mathsf{C}_g \simeq \mathsf{Comon}\mathcal{V}(\mathsf{F1}, \mathsf{C})$. Moreover one has $\mathsf{C}_g = (\mathsf{C}^{\mathsf{op}})_g$. The following proposition sharpens and simplifies a result of [27].

Proposition 5 *If $\mathcal{V}$ is an entropic variety and B a bimonoid in $\mathcal{V}$, then B^c_g is a submonoid of the convolution monoid $Conv(\mathsf{F1}, \mathsf{B}^c)$.*
If (B, S) is a Hopf monoid, then this monoid is a group with $Sg = g^{-1}$.

Proof: For $g, h \in \mathsf{B}^c_g$ one has $g * h = F1 \simeq F1 \otimes F1 \xrightarrow{g\otimes h} B \otimes B \xrightarrow{m} B$, since the canonical isomorphism $F1 \simeq F1 \otimes F1$ is the comultiplication of $\mathsf{F1}$. Since B is a bimonoid, this is a comonoid morphism. This proves the first claim.
Since $S\colon \mathsf{B}^c \to (\mathsf{B}^c)^{\mathsf{op}}$ is a comonoid morphism, so is $S \circ g\colon \mathsf{F1} \to (\mathsf{B}^c)^{\mathsf{op}}$ for any $g \in \mathsf{B}^c_g$. This proves $Sg \in \mathsf{B}^c_g$.

Since the unit of the convolution monoid is $e \circ \epsilon_{F1} = e \circ id_{F1}$, it remains to prove the equation $(S \circ g) * g = e \circ id_{F1}$, that is commutativity of the outer frame of the following diagram, since the canonical isomorphism $F1 \simeq F1 \otimes F1$ is the comultiplication of $\mathsf{F1}$.



But this is obvious, since g is a comonoid morphism and S is an antipode. □

Examples 6

1. For $\mathcal{V} = \mathbf{Set}$, hence $\mathsf{Comon}\mathcal{V} = \mathbf{Set}$, $\mathsf{Bimon}\mathcal{V} = \mathbf{Monoids}$ and $\mathsf{Hopf}\mathcal{V} = \mathbf{Groups}$, all elements of a comonoid (that is, a set) are group-like, the monoid of group-like elements of a bimonoid B (that is, a monoid) is the monoid B, and the group of group-like elements of a Hopf monoid (B, S) (that is, a group) is this group.

2. In $\mathcal{V}[\mathsf{M}]$, the $\mathcal{V}$-monoid algebra of a monoid M, considered as a bialgebra, all elements $m \in M$ are group-like, since the comultiplication of $\mathcal{V}[\mathsf{M}]$ is $FM \xrightarrow{F\Delta} F(M \times M) \simeq FM \otimes FM$ and the counit is $FM \xrightarrow{F!} F1$.

[8]Some authors call these elements *set-like*.

2.3 Modules and bimodules of monoids

Recall the definitions of modules[9] in a symmetric monoidal category $\mathcal{V}$ as follows.

Definition 5 Let $\mathsf{A} = (A, m, e)$ be a monoid in $\mathcal{V}$.

1. A pair $(M, A \otimes M \xrightarrow{l} M)$ with a $\mathcal{V}$-homomorphism l is called a *left* A- *module*, provided that the following diagrams commute.

$$\begin{array}{ccc} A\otimes A\otimes M & \xrightarrow{A\otimes l} & A\otimes M \\ {\scriptstyle m\otimes M}\downarrow & & \downarrow{\scriptstyle l} \\ A\otimes M & \xrightarrow[l]{} & M \end{array} \qquad \begin{array}{ccc} I\otimes M & \xrightarrow{e\otimes id} & A\otimes M \\ & \searrow^{can} & \downarrow{\scriptstyle l} \\ & & M \end{array}$$

   A *morphism* $(M, l) \xrightarrow{f} (M', l')$ *of left* A-*modules* is a $\mathcal{V}$-homomorphism $M \xrightarrow{f} M'$, such that

$$\begin{array}{ccc} A\otimes M & \xrightarrow{A\otimes f} & A\otimes M' \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle l'} \\ M & \xrightarrow[f]{} & M' \end{array}$$

   commutes. The category so defined will be denoted by ${}_{\mathsf{A}}\mathcal{V}$.

2. A pair $(M, M \otimes A \xrightarrow{r} M)$ with a $\mathcal{V}$-morphism r making the analogous diagrams commute is called a *right* A-*module*. With the obvious definition of morphisms they constitute the category $\mathcal{V}_{\mathsf{A}}$.

3. A triple (M, l, r), where (M, l) is a left and (M, r) is a right A-module, is a called an A-*bimodule*, provided that

$$\begin{array}{ccc} A\otimes M\otimes A & \xrightarrow{A\otimes r} & A\otimes M \\ {\scriptstyle l\otimes A}\downarrow & & \downarrow{\scriptstyle l} \\ M\otimes A & \xrightarrow[r]{} & M \end{array} \tag{3}$$

   commutes. With morphisms which are both, left and right A-module morphisms, the A-bimodules constitute the category ${}_{\mathsf{A}}\mathcal{V}_{\mathsf{A}}$.

Example 7 For every $\mathcal{V}$-monoid A there is a canonical A-bimodule (A^*, l, r), where one defines operations $A \otimes A^* \xrightarrow{l} A^*$ and $A^* \otimes A \xrightarrow{r} A^*$, respectively, by commutativity of the diagrams

$$\begin{array}{ccc} A\otimes A^* & \xrightarrow{ev} & F1 \\ {\scriptstyle A\otimes l}\uparrow & & \uparrow{\scriptstyle ev} \\ A\otimes A\otimes A^* & \xrightarrow[m\otimes A^*]{} & A\otimes A^* \end{array} \qquad \begin{array}{ccc} A^*\otimes A & \xrightarrow{ev} & F1 \\ {\scriptstyle r\otimes A}\uparrow & & \uparrow{\scriptstyle ev} \\ A^*\otimes A\otimes A & \xrightarrow[A^*\otimes m]{} & A\otimes A^* \end{array}$$

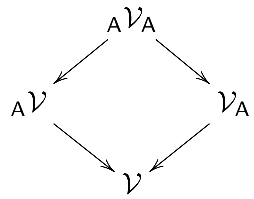
As usual we will, in the sequel, write $af := l(a \otimes f)$ and $fb := r(f \otimes b)$.

[9] See for example [11] or [19] for these notions, where they are called *actions*, however.

Fact 1 The following hold for every entropic variety $\mathcal{V}$.

1. ${}_{\mathsf{A}}\mathcal{V}$, ${}_{\mathsf{A}}\mathcal{V}$ and ${}_{\mathsf{A}}\mathcal{V}_{\mathsf{A}}$ are varieties; these are entropic, provided that A is a commutative monoid. $\mathcal{V}_{\mathsf{A}}$ is isomorphic to ${}_{\mathsf{A}^{\mathsf{op}}}\mathcal{V}$; ${}_{\mathsf{F1}}\mathcal{V}$ is isomorphic to $\mathcal{V}$ (see [11]).

2. The forgetful functors depicted in the diagram below, therefore, are algebraic functors in the sense of [3] and, thus, have left adjoints.



For example, the free A-module over a $\mathcal{V}$-algebra X is $(A \otimes X, A \otimes A \otimes X \xrightarrow{m \otimes X} A \otimes X)$ (see [19]). In particular, the free A-module over $\mathsf{F1}$ (and, thus, the free A-module over the singleton 1, if ${}_{\mathsf{A}}\mathcal{V}$ is considered as a variety) is $(A, A \otimes A \xrightarrow{m} A)$ with $F1 \xrightarrow{e} A$ its universal morphism.

Fact 2 ([20]) For every left A-module $(M, A \otimes M \xrightarrow{l} M)$ in an entropic variety the following hold.

1. $M \otimes C$ is a left A-module, for every $\mathcal{V}$-algebra C. This defines a functor $M \otimes -: \mathcal{V} \to {}_{\mathsf{A}}\mathcal{V}$.

2. The functor $M \otimes -$ has a right adjoint ${}_{\mathsf{A}}[M, -]$. For a left A-module N the $\mathcal{V}$-algebra ${}_{\mathsf{A}}[M, N]$ is ${}_{\mathsf{A}}\mathcal{V}(M, N)$, considered as a subalgebra of $[M, N]$ or, equivalently, as a subalgebra of M^N.

3. ${}_{\mathsf{A}}[M, M]$ is a $\mathcal{V}$-monoid. Its underlying monoid is the endomorphism monoid $\mathrm{End}(M, l)$ of the left A-module (M, l). It is a $\mathcal{V}$-submonoid of $[M, M]$.

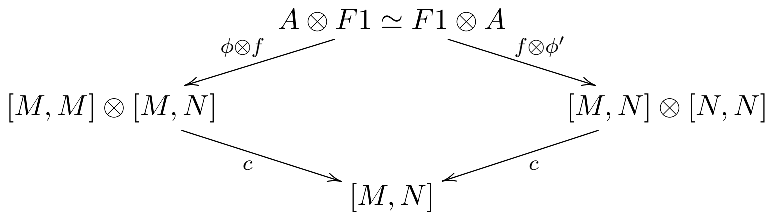
By this construction ${}_{\mathsf{A}}\mathcal{V}$ becomes a $\mathcal{V}$-category. The "composition"-homomorphisms will be denoted by c.

Proposition 6 *For any two homomorphisms $A \otimes M \xrightarrow{l} M$ and $A \xrightarrow{\phi} [M, M]$ in a symmetric monoidal closed category $\mathcal{V}$ corresponding by adjunction, that is, with $l = A \otimes M \xrightarrow{\phi \otimes M} [M, M] \otimes M \xrightarrow{ev} M$, the following are equivalent.*

1. *$(M, A \otimes M \xrightarrow{l} M)$ is a left A-module.*

2. *$A \xrightarrow{\phi} [M, M]$ is a $\mathcal{V}$-monoid morphism, called the* canonical monoid morphism of the module (M, l).

If $\mathcal{V}$ is an entropic variety, then a homomorphism $M \xrightarrow{f} N$ is left A-module homomorphism $(M,l) \to (M',l')$ if an only of the following diagram commutes



where ϕ and ϕ' are the canonical monoid morphisms of of (M,l) and (M',l'), respectively.

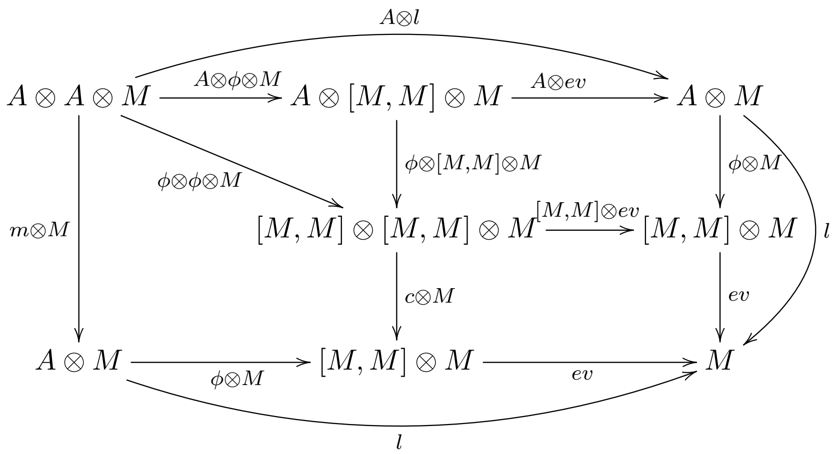
Proof: ϕ preserves units if and only if l satisfies the second module axiom as is easily seen.

The first monoid morphism condition on ϕ is, by the couniversal property of the evaluations, equivalent to the equality

$$ev \circ (\phi \otimes M) \circ (m \otimes M) = ev \circ (c \otimes M) \circ (\phi \otimes \phi \otimes M),$$

which is equivalent to commutativity of the outer frame of the diagram below, that is, commutativity of Diagram (3), the first module condition on l (see, in particular, item 5 of Examples 4). This proves the equivalence of (1) and (2).

The final statement follows from the definition of the multiplication of the endomorphism monoid (see Examples 4 (3)) and the fact, that the tensor product is given by universal bimorphisms.



□

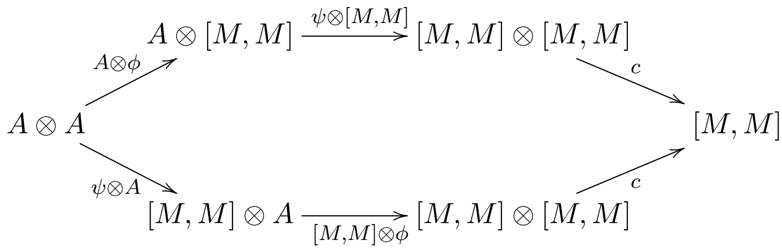
Remark 1 The canonical monoid morphism of the left A-modul $(A, A \otimes A \xrightarrow{m} A)$ is given by $a \mapsto (b \mapsto m(a \otimes b))$; this, indeed, is a $\mathcal{V}$-homomorphism, since the tensor product is given by universal bimorphisms.

Corollary 1

1. *If* $M \otimes A \xrightarrow{r} M$ *and* $A \xrightarrow{\psi} [M,M]$ *correspond by adjunction, that is, if* $r = M \otimes A \xrightarrow{M\otimes\psi} M \otimes [M,M] \xrightarrow{ev} M$, *then* $\mathsf{A}^{\mathsf{op}} \xrightarrow{\phi} [M,M]$ *(equivalently,* $\mathsf{A} \xrightarrow{\phi} [M,M]^{\mathsf{op}}$*) is a* $\mathcal{V}$*-monoid morphism if and only if* $(M, M \otimes A \xrightarrow{r} M))$ *is a right* A*-module.*

2. *Given a pair of* $\mathcal{V}$*-monoid morphisms* $\mathsf{A} \xrightarrow{\phi} [M,M]$ *and* $\mathsf{A}^{\mathsf{op}} \xrightarrow{\psi} [M,M]$ *for some* $\mathcal{V}$*-algebra* M*, then this defines an* A*-bimodule, provided that the following diagram commutes.*

(4)



The following facts, where the extension of $\mathcal{V}$-monoid morphism $\mathsf{A} \xrightarrow{\phi} [M,M]$ to a $\mathcal{V}[\mathsf{A}]$-morphism in the sense of Proposition 2[10] is denoted by $\phi^\sharp$, seem not be known even in the case of R-modules.

Proposition 7 *Let* A *be a* $\mathcal{V}$*-monoid. Assigning, for* $\mathcal{V}$*-algebras* M*, to* $\mathcal{V}$*-monoid morphisms* $\mathsf{A} \xrightarrow{\phi} [M,M]$ *their extensions* $\mathcal{V}[\mathsf{A}] \xrightarrow{\phi^\sharp} [M,M]$ *defines full (and faithful) algebraic functors* ${}_{\mathsf{A}}\mathcal{V} \xrightarrow{{}_{\mathsf{A}}E} {}_{\mathcal{V}[\mathsf{A}]}\mathcal{V}$, $\mathcal{V}_{\mathsf{A}} \xrightarrow{E_{\mathsf{A}}} \mathcal{V}_{\mathcal{V}[\mathsf{A}]}$ *and* ${}_{\mathsf{A}}\mathcal{V}_{\mathsf{A}} \xrightarrow{{}_{\mathsf{A}}E_{\mathsf{A}}} {}_{\mathcal{V}[\mathsf{A}]}\mathcal{V}_{\mathcal{V}[\mathsf{A}]}$.

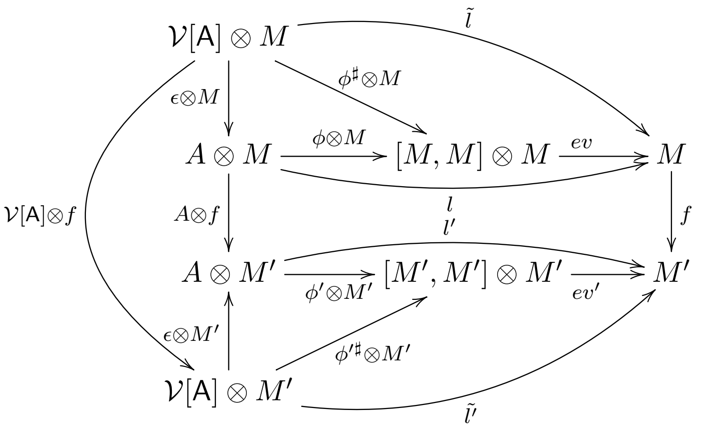
In other words, each of the categories ${}_{\mathsf{A}}\mathcal{V}$, $\mathcal{V}_{\mathsf{A}}$ *and* ${}_{\mathsf{A}}\mathcal{V}_{\mathsf{A}}$ *of* A*-modules is equivalent to a full reflective subcategory of its corresponding category of* $\mathcal{V}[\mathsf{A}]$*-modules.*

Proof: In view of the proposition and its corollary above one obtains functors ${}_{\mathsf{A}}\mathcal{V} \to {}_{\mathcal{V}[\mathsf{A}]}\mathcal{V}$ and, thus, $\mathcal{V}_{\mathsf{A}} \to \mathcal{V}_{\mathcal{V}[\mathsf{A}]}$, provided that, for every left A-module morphism $f\colon (M,l) \to (M',l')$, f is $\mathcal{V}[\mathsf{A}]$-module homomorphism as well.

The commutative diagram below, where ϕ and ϕ' denote the canonical monoid morphisms of (M,l) and (M',l'), respectively, $\tilde{l}$ and $\tilde{l}'$ the module actions corresponding to $\phi^\sharp$ and $\phi'^\sharp$, respectively, and ϵ the counit of the adjunction for $\mathcal{V}[-]$ (see Proposition 2) shows, that f is an A-module homomorphism if and only if it is a $\mathcal{V}[\mathsf{A}]$-module homomorphism. Thus, we not only obtain a functor, which is a concrete (hence, faithful) functor over $\mathcal{V}$, but even a full functor. Since this functor then is concrete over **Set** as well, it is algebraic and therefore it has a left adjoint.

By extending the canonical monoid morphisms ϕ and ψ of an A-bimodule to $\mathcal{V}[\mathsf{A}]$, one obtains a $\mathcal{V}[\mathsf{A}]$-bimodule, since, if ϕ and ψ make Diagram (4) commute, so do their extensions to $\mathcal{V}[\mathsf{A}]$.

[10]More precisely we should have written $\mathcal{V}[[\mathsf{A}]]$-module.



□

3 Generalized finite duals

3.1 Semi-dualization and the dual monoid functor

Let $\mathcal{V}$ be an entropic variety. Recall the semi-dualization functor $(-)^* = [-, F1] \colon \mathcal{V}^{\mathsf{op}} \to \mathcal{V}$, introduced in Section 1.3. Depending on the variety $\mathcal{V}$, the semi-dualization functor may be interesting, as for $\mathcal{V} = \mathbf{Mod}_R$, where it is the usual linear dualization functor, or uninteresting, as for $\mathcal{V} = \mathbf{Set}$ and $\mathcal{V} = \mathbf{Set}_*$, where it is the constant functor with value a singleton. By specialization from [21] or [25] we obtain (see also Examples 3(5))

Proposition 8 *Let $\mathcal{V}$ be an entropic variety. Then the following hold.*

1. $\mathcal{V} \xrightarrow{(-)^*} \mathcal{V}^{\mathsf{op}}$ *is left adjoint to* $\mathcal{V}^{\mathsf{op}} \xrightarrow{(-)^*} \mathcal{V}$*, in other words, the semi-dualization functor and its opposite form a dual adjunction.*

2. *The semi-dualization functor is a symmetric monoidal functor.*

We recall that the multiplication $\Lambda_{A,B} \colon A^* \otimes B^* \to (A \otimes B)^*$ of the semi-dualization functor is given by the bimorphism $(f, g) \mapsto can_{F1} \circ (f \otimes g)$ for morphisms $f \colon A \to F1$ and $g \colon B \to F1$, that is, $(\Lambda_{A,B}(f \otimes g))(a \otimes b) = fa \cdot fb$; its unit $F1 \to F1^*$ is the isomorphism given by $x \mapsto id$.

Definition 6 A monoidal subcategory $\mathcal{W}$ of $\mathcal{V}$ will be called *-*strong*, provided that $\Lambda_{A,B} \colon A^* \otimes B^* \to (A \otimes B)^*$ is an isomorphism, for all $A, B \in \mathcal{W}$.

Examples 8 *The finite dimensional vector spaces form a *-strong subcategory of* $\mathbf{Vect}_k$ *and, more generally, the finitely generated projective submodules form a *-strong subcategory of* $\mathbf{Mod}_R$.

Since monoidal functors map monoids to monoids, the semi-dualization functor induces a functor $D\colon (\mathsf{Comon}\mathcal{V})^{\mathsf{op}} = \mathsf{Mon}\mathcal{V}^{\mathsf{op}} \to \mathsf{Mon}\mathcal{V}$, called the *dual monoid* functor. For $\mathcal{V} = \mathbf{Mod}_R$ this functor is the so-called *dual algebra functor* $\mathbf{Coalg}_R^{\mathsf{op}} \longrightarrow \mathbf{Alg}_R$. One obtains from [25]

Fact 3 For every entropic variety its dual monoid functor is a symmetric monoidal functor.

3.2 Left adjoints of the dual monoid functors

As is well known (see e.g. [28]), for $\mathcal{V} = \mathbf{Vect}_k$, the dual algebra functor $D\colon \mathbf{Coalg}_k^{\mathsf{op}} \to \mathbf{Alg}_k$ has a left adjoint $(-)^\bullet\colon \mathbf{Alg}_k \to \mathbf{Coalg}_k^{\mathsf{op}}$, called the *finite dual functor.* This functor has the following properties, where we denote, for a k-algebra A, by A° the underlying vector space of the coalgebra $\mathsf{A}^\bullet$:

1. A° is a subspace of A^*.

2. $\mathsf{A}^\circ = A^*$, provided that A is finite dimensional. In fact, more is true: The dualization functor $\mathbf{Vect}_k^{\mathsf{op}} \xrightarrow{(-)^*} \mathbf{Vect}_k$ induces a dual monoidal equivalence between the subcategories of finite dimensional vector spaces, which in turn induces a dual equivalence ${}_{fd}\mathbf{Coalg}_k{}^{\mathsf{op}} \simeq {}_{fd}\mathbf{Alg}_k$ between the subcategories of finite dimensional algebras and coalgebras respectively. The adjunction $(-)^\bullet \vdash D$ extends this equivalence.

3. If A is the underlying algebra of a Hopf algebra then the coalgebra $\mathsf{A}^\bullet$ carries a Hopf algebra structure.

Left adjoints of the dual monoid functors exist more generally. In fact, one has, as a special instance of Proposition 3, the following result.

Proposition 9 *The dual monoid functor of an entropic variety $\mathcal{V}$ has a left adjoint, called the* generalized finite dual functor*.*

To what extent the generalized finite dual shares the properties Sweedler's finite dual functor mentioned above will be discussed in the next section.

3.3 Properties of generalized finite dual functors

Denoting the unit of the dual adjunction, given by the semi-dualization functor $(-)^*$, by λ, the generalized finite dual by $(-)^\bullet$, the unit of its adjunction by η and by $(-)^\circ$ the composition $\mathsf{Mon}\mathcal{V} \xrightarrow{(-)^\bullet} (\mathsf{Comon}\mathcal{V})^{\mathsf{op}} \xrightarrow{||-||} \mathcal{V}^{\mathsf{op}}$, there is a unique natural transformation $\kappa\colon (-)^\circ \Rightarrow (-)^* \circ |-|$ characterized by commutativity of the following diagram, where $|-|$ denotes the forgetful functors of $\mathsf{Mon}\mathcal{V}$ (see [25]).

$$\begin{array}{ccc} |\mathsf{M}| & \xrightarrow{\lambda_{|\mathsf{M}|}} & (|\mathsf{M}|^*)^* \\ & {\scriptstyle |\eta_{\mathsf{M}}|}\searrow & \downarrow {\scriptstyle \kappa^*_{\mathsf{M}}} \\ & & |(\mathsf{M}^\bullet)^*| = (\mathsf{M}^\circ)^* \end{array}$$

For $\mathcal{V} = \mathbf{Vect}_k$ $(-)^*$ and $\mathsf{A} = (A, m, e)$ a k-algebra, κ_A is the embedding of the underlying vector space of the finite dual A° into the linear dual A^*, mentioned at the beginning of this section.

In order to describe the algebraic properties of $\mathsf{A}^\bullet = (\mathsf{A}^\circ, m^\bullet, e^\bullet)$, for some $\mathcal{V}$-monoid $\mathsf{A} = (A, m, e)$, we recall the following concept from [25].

Definition 7 Let $\mathsf{A} = (A, m, e)$ be a $\mathcal{V}$-monoid and $\bar{\mathsf{A}} \xrightarrow{\bar{m}} \bar{\mathsf{A}} \otimes \bar{\mathsf{A}}$ and $\bar{A} \xrightarrow{\bar{e}} F1$ be $\mathcal{V}$-morphisms such that the following diagram commutes for some ψ.

$$
\begin{array}{ccccc}
\bar{\mathsf{A}} & \xrightarrow{\psi\otimes\psi} & \bar{\mathsf{A}} \otimes \bar{\mathsf{A}} & \xrightarrow{\bar{m}} & A^* \otimes A^* \\
{\scriptstyle\psi}\downarrow & & & & \downarrow{\scriptstyle\Lambda_{A,A}} \\
A^* & & \xrightarrow{\quad m^* \quad} & & (A \otimes A)^*
\end{array}
\qquad
\begin{array}{ccc}
\bar{\mathsf{A}} & \xrightarrow{\bar{e}} & F1 \\
{\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\lambda} \\
A^* & \xrightarrow{e^*} & F1^*
\end{array}
$$

Then the triple $(\bar{\mathsf{A}}, \bar{m}, \bar{e})$ is called *induced from* A by ψ. If $(\bar{\mathsf{A}}, \bar{m}, \bar{e})$ is a comonoid, then this comonoid is called an *induced quotient of* A *by* ψ[11].

Remark 2 An alternative description of induced quotients can be given as follows (see [25]). For $\mathcal{V}$-algebras M and N, the map $F1^M \times F1^N \xrightarrow{\chi} F1^{M\times N}$ with $\chi(f, g)(x, y) = fx \cdot gy$, where $- \cdot -$ is the multiplication of $F1$, is a bimorphism and so determines a homomorphism $\Pi_{M,N} \colon F1^M \otimes F1^N \to F1^{M\times N}$. Analogously, one obtains a homomorphism $\Pi_{L,M,N} \colon F1^L \otimes F1^M \otimes F1^N \to F1^{L\times M\times N}$.

Then a homomorphism $\mathsf{A}^\circ \xrightarrow{m^\bullet} \mathsf{A}^\circ \otimes \mathsf{A}^\circ$ is induced from A by κ_A if and only if the outer frame of the following diagram commutes, where t is the universal bimorphism and θ the inclusion $A^* \hookrightarrow F1^A$. Note that the upper right hand cell (and a similar diagram for $\Pi_{A,A,A}$) commutes by definition of $\Lambda_{A,A}$ and $\Lambda_{A,A,A}$, respectively, and that $F1^t \circ \theta$ is injective.

(5)
$$
\begin{array}{ccccccc}
\mathsf{A}^\circ & \xrightarrow{m^\bullet} & \mathsf{A}^\circ \otimes \mathsf{A}^\circ & \xrightarrow{\kappa_\mathsf{A}\otimes\kappa_\mathsf{A}} & A^* \otimes A^* & \xrightarrow{\qquad\theta_\mathsf{A}\otimes\theta_\mathsf{A}\qquad} & F1^A \otimes F1^A \\
{\scriptstyle\kappa_\mathsf{A}}\downarrow & & & & \vdots{\scriptstyle\Lambda_{A,A}} & & \downarrow{\scriptstyle\Pi_{A,A}} \\
A^* & & \xrightarrow{\qquad m^* \qquad} & & (A \otimes A)^* & \xrightarrow{\theta_{A\otimes A}} F1^{A\otimes A} \xrightarrow{F1^t} & F1^{A\times A} \\
{\scriptstyle\theta_\mathsf{A}}\downarrow & & & & \downarrow{\scriptstyle\theta_{A\otimes A}} & \nearrow{\scriptstyle F1^t} & \\
F1^A & & \xrightarrow{\qquad F1^m \qquad} & & F1^{A\otimes A} & &
\end{array}
$$

Lemma 6 ([25]) *If a left adjoint* $(-)^\bullet$ *of the dual monoid functor exists, then* $\mathsf{A}^\bullet$ *is an induced quotient of* A *by* κ_A*, for each* $\mathcal{V}$*-monoid* A*.*

Specializing results from [25] we get:

Proposition 10 *Let* $\mathcal{V}$ *be an entropic variety. Then the following hold:*

[11] Note that $(\bar{\mathsf{A}}, \bar{m}, \bar{e})$ is not necessarily a quotient of A in the strict sense of the word. We will even use this term by abuse of language if the morphism $LA \to A'$ is not epic.

1. *The left adjoint* $(-)^\bullet$ *of the dual monoid functor is an opmonoidal functor and, thus, induces a functor* $\mathsf{Bimon}\mathcal{V} \to (\mathsf{Bimon}\mathcal{V})^{\mathsf{op}}$.

2. *This functor has a right adjoint* $(\mathsf{Bimon}\mathcal{V})^{\mathsf{op}} \to \mathsf{Bimon}\mathcal{V}$, *which we denote by* $(-)^\bullet$ *as well.*

3. *If all maps* κ_{A} *are injective, the following holds.*

   (a) *The functor* $(-)^\bullet \colon \mathsf{Bimon}\mathcal{V} \to (\mathsf{Bimon}\mathcal{V})^{\mathsf{op}}$ *can be restricted to a functor* $\mathsf{Hopf}\mathcal{V} \to (\mathsf{Hopf}\mathcal{V})^{\mathsf{op}}$.

   (b) *This functor has a right adjoint, provided that* $\mathcal{V}$ *is a flat variety.*

3.4 Constructing generalized finite duals

In order to *construct* a left adjoint of the dual monoid functor, one needs, by the above, first to construct a functor $(-)^\circ \colon \mathsf{Mon}\mathcal{V} \to \mathcal{V}^{\mathsf{op}}$ equipped with a natural transformation $\kappa \colon (-)^\circ \Rightarrow (-)^* \circ |-|$. Since we want the left adjoint to map, as in the classical case, Hopf algebras to Hopf algebras, we assume in view of Proposition 10 moreover, that κ is monomorphic. We call such a pair $((-)^\circ, \kappa)$ a *Sweedler functor.*

In a second step one has to make sure that $((-)^\circ, \kappa)$ is *liftable (to a functor F)* in the sense that there exists a functor $\mathsf{Mon}\mathcal{V} \xrightarrow{F} (\mathsf{Comon}\mathcal{V})^{\mathsf{op}}$ with $||-|| \circ F = (-)^\circ$ such that $F\mathsf{A}$ is an induced quotient of A by κ_{A}, for each $\mathcal{V}$-monoid A.

Finally, one has to make sure that this lifted functor F is left adjoint to the dual monoid functor.

3.4.1 Constructing Sweedler functors

Recall the following construction of a Sweedler functor from [25], where also the naturality of his construction is discussed.

Consider, for every monoid $\mathsf{A} \in \mathsf{Mon}\mathcal{V}$, the family of all monoid morphisms $f \colon \mathsf{A} \to \mathsf{K}_f$ with K_f in ${}_{fg}\mathcal{V}$, the category of finitely generated algebras, and denote by $\mathcal{S}_{\mathsf{A}}$ the family of the duals $f^* \colon K_f^* \to A^*$. Form the (extremal episink, mono)-factorization of $\mathcal{S}_{\mathsf{A}}$ as

$$K_f^* \xrightarrow{f^*} A^* = K_f^* \xrightarrow{e_f} \mathsf{A}^\circ \xrightarrow{\kappa_{\mathsf{A}}} A^*$$

A° thus is the smallest subalgebra of A^* through which all maps f^* factor or, in other words, the subalgebra of A^*, generated by the union of all images of the homomorphism f^*. This defines a Sweedler functor $\mathsf{Mon}\mathcal{V} \xrightarrow{(-)^\circ} \mathcal{V}^{\mathsf{op}}$ where, in particular, $\mathsf{K}^\circ \xrightarrow{f^\circ} \mathsf{A}^\circ = \mathsf{K}^* \xrightarrow{e_f} \mathsf{A}^\circ$.

Note that one has $\mathsf{A}^\circ = \mathsf{A}^*$ if and only if $A \in {}_{fg}\mathcal{V}$[12]. The following sharpening of a statement in [25] holds as a look at the proof given there shows.

[12]More precisely one should have said instead of $\mathsf{A}^\circ = \mathsf{A}^*$ that κ_{A} is an isomorphism.

Lemma 7 *There exists a natural family* $\Phi_{\mathsf{A},\mathsf{B}} \colon \mathsf{A}^\circ \otimes \mathsf{B}^\circ \to (\mathsf{A} \otimes \mathsf{B})^\circ$ *such that the following diagram commutes.*

$$\begin{array}{ccc} \mathsf{A}^\circ \otimes \mathsf{B}^\circ & \xrightarrow{\Phi_{\mathsf{A},\mathsf{B}}} & (\mathsf{A} \otimes \mathsf{B})^\circ \\ {\scriptstyle \kappa_\mathsf{A} \otimes \kappa_\mathsf{B}} \downarrow & & \downarrow {\scriptstyle \kappa_{\mathsf{A}\otimes\mathsf{B}}} \\ A^* \otimes B^* & \xrightarrow[\Lambda_{A,B}]{} & (A \otimes B)^* \end{array}$$

The morphism $\Phi_{\mathsf{A},\mathsf{B}}$ *is an isomorphism, provided that the morphism* $\Lambda_{A,B} \circ (\kappa_\mathsf{A} \otimes \kappa_\mathsf{B})$ *is a monomorphism.*

Next we compare this construction with similar ones known from the case of vector spaces (see e.g. [28], [2]). We call a congruence ρ on a $\mathcal{V}$-monoid A (with respect to the variety $\mathsf{Mon}\mathcal{V}$) *cofinite*, provided that the quotient A/ρ is finitely generated as a $\mathcal{V}$-algebra. Thus, for $\mathcal{V} = \mathbf{Vect}_k$, a congruence is cofinite if and only if the k-algebra congruence on A is given by a cofinite ideal. For a $\mathcal{V}$-monoid A we define a subset $\mathsf{A}^\diamond$ of A^* by

$$\mathsf{A}^\diamond = \{h \in A^* \mid \text{there exists a cofinite congruence } \rho \subset \ker h\}$$

Proposition 11 *Let* A *be a monoid in the entropic variety* $\mathcal{V}$*. Then*

1. $\mathsf{A}^\diamond \subset \mathsf{A}^\circ$.
2. $\mathsf{A}^\diamond$ *is a subalgebra of* A^* *and, hence, of* A°*, provided that* $\mathcal{V}$ *is set-like.*
3. $\mathsf{A}^\circ \subset \mathsf{A}^\diamond$ *and, hence,* $\mathsf{A}^\circ = \mathsf{A}^\diamond$*, provided that* $\mathcal{V}$ *is set-like and noetherian.*

Proof: If $h \in \mathsf{A}^\diamond$, that is, if $A \xrightarrow{h} F1$ factors as $A \xrightarrow{q_h} A/\rho \xrightarrow{k_h} F1$, where q_h is the quotient morphism (in $\mathsf{Mon}\mathcal{V}$) of a cofinite congruence ρ, then $h = q_h^*(k_h)$ and, hence $h \in \mathsf{A}^\circ$. This proves item (1).

Let t be an n-ary operation and $h_1, \dots h_n \in \mathsf{A}^\diamond$. Then $t(h_1, \dots, h_n)(a) = t(h_1(a), \dots h_n(a))$, for every $a \in A$, and $(a,b) \in \ker t(h_1, \dots, h_n) \subset A \times A$ if and only if $\big((h_i(a)_i, (h_i(b)_i\big) \in \ker t$. Assume that $\rho_1, \dots, \rho_n$ are $\mathsf{Mon}\mathcal{V}$-congruences on A with $\rho_i \subset \ker h_i$ for each i, then $\rho := \cap \rho_i$ is a congruence contained in $\ker t(h_1, \dots, h_n)$. The obvious homomorphisms $A/\rho \to A/\rho_i$ induce an injective homomorphism $A/\rho \to \prod_{i=1}^n A/\rho_i$. If all ρ_i are cofinite, hence all $\mathcal{V}$-quotients A/ρ_i are finitely generated, then $\prod_{i=1}^n A/\rho_i$ is finitely generated, since $\mathcal{V}$ is set-like. Thus, $\ker t(h_1, \dots, h_n)$ contains the cofinite congruence ρ and $t(h_1, \dots, h_n) \in \mathsf{A}^\diamond$ follows. This proves item (2).

Recalling the definition of A° one sees that $\ker f$ is cofinite for every f^* belonging to $\mathcal{S}_\mathsf{A}$, provided that $\mathcal{V}$ is noetherian; hence the kernel of every $h \in \operatorname{Im} f^*$, that is, of every $h \in A^*$ which factors as $h = g \circ f$ with some $g \in K_f^*$, contains the cofinite congruence $\ker f$. Consequently, $\operatorname{Im} f^* \subset \mathsf{A}^\diamond$. Since A° is the smallest such subalgebra of A^* one can conclude $\mathsf{A}^\circ \subset \mathsf{A}^\diamond$, since $\mathsf{A}^\diamond$ is a subalgebra of A^* by item (2); now equality follows. □

Another description of A° is obtained by considering certain submodules of the A-bimodule A^* introduced in Example 7.

Since A is the free left A-module over a singleton one gets, for every $f \in A^*$, an A-module morphism $\bar{f}\colon A \to A^*$ with $1 \mapsto f$. Its image will be denoted by $\mathsf{A}f$. Obviously, as a set one has $\mathsf{A}f := \{af \mid a \in A\}$. Equivalently, $\mathsf{A}f$ is the left A-submodule of A^*, generated by f. Similarly there is a right A-submodule $f\mathsf{A}$ of A^*. We now define

$$\begin{aligned} {}^{\triangleleft}\mathsf{A} &= \{f \in A^* \mid f\mathsf{A} \text{ is finitely generated as a } \mathcal{V}\text{-algebra}\} \\ \mathsf{A}^{\triangleright} &= \{f \in A^* \mid \mathsf{A}f \text{ is finitely generated as a } \mathcal{V}\text{-algebra}\} \end{aligned}$$

The following result generalizes important observations in [28] and [2].

Proposition 12 *For every monoid* A *in a noetherian and set-like entropic variety* $\mathcal{V}$ *one has*

$$\mathsf{A}^{\circ} = \mathsf{A}^{\triangleright} = {}^{\triangleleft}\mathsf{A} = \mathsf{A}^{\diamond}.$$

In particular, all of these are $\mathcal{V}$*-algebras.*

Proof: Let, for $f \in \mathsf{A}^{\diamond}$, $\rho \subset \ker f$ be a cofinite congruence. Then, for each $af \in \mathsf{A}f$ there exists a $\mathcal{V}$-homomorphism $\phi(af)\colon A/\rho \to F1$ with $\phi(af) \circ q = af$, where q is the quotient homomorphism $A \to A/\rho$, since $\rho \subset \ker f \subset \ker af$. The map $\phi\colon \mathsf{A}f \to (A/\rho)^* = [A/\rho, F1]$ so defined is obviously injective; it is a $\mathcal{V}$-homomorphism since by definition $q^* \circ \phi$ is the embedding of $\mathsf{A}f$ into A^*. Thus, $\mathsf{A}f$ is isomorphic to a subalgebra of $[A/\rho, F1]$. Since $[A/\rho, F1]$ is finitely generated by Lemma 4, the inclusion $\mathsf{A}^{\diamond} \subset \mathsf{A}^{\triangleright}$ follows.

Assume $f \in \mathsf{A}^{\triangleright}$. Since, for $a, b \in A$, $af = bf$ implies $f(a) = f(b)$ one has $\ker \phi_f \subset \ker f$[13] where $\mathsf{A} \xrightarrow{\phi_f} [\mathsf{A}f, \mathsf{A}f]$ is the canonical monoid morphism of the A-module $\mathsf{A}f$. Since $\mathcal{V}$ is noetherian and set-like and $\mathsf{A}f$ is finitely generated, the image of ϕ_f is finitely generated by Lemma 4, such that $\ker \phi_f$ is a cofinite congruence. Thus, $f \in \mathsf{A}^{\diamond}$.

By Proposition 11 one gets $\mathsf{A}^{\circ} = \mathsf{A}^{\triangleright} = \mathsf{A}^{\diamond}$. The missing equation now follows by the observations that $\mathsf{A}^{\circ} = (\mathsf{A}^{\mathsf{op}})^{\circ}$ and $f\mathsf{A} = (\mathsf{A}^{\mathsf{op}})f$. □

3.4.2 Lifting a Sweedler functor

As shown by Sweedler, the A° construction allows a lift to the desired left adjoint in the case where $\mathcal{V} = \mathbf{Vect}_k$. There is, however, no reason to assume that this can be lifted in general; see [25] for a counterexample.

We now give sufficient conditions for a lift, where one first gets by specialization from [25]

Proposition 13 *Assume that* ${}_{fg}\mathcal{V}$ *is a* **-strong subcategory of* $\mathcal{V}$*. Then for each* $\mathcal{V}$*-monoid* A *there exist homomorphisms* $\mathsf{A}^{\circ} \xrightarrow{m^{\bullet}} \mathsf{A}^{\circ} \otimes \mathsf{A}^{\circ}$ *and* $\mathsf{A}^{\circ} \xrightarrow{e^{\bullet}} F1$ *induced from* A *by* κ_{A}*, provided that* $\Lambda_{A,A} \circ (\kappa_{\mathsf{A}} \otimes \kappa_{\mathsf{A}})$ *is a monomorphism. This construction defines a lift* $(-)^{\bullet}$ *of* $((-)^{\circ}, \kappa)$*, provided that the morphisms* $\Lambda_{A,A,A} \circ (\kappa_{\mathsf{A}} \otimes \kappa_{\mathsf{A}} \otimes \kappa_{\mathsf{A}})$ *are monomorphisms.*

[13] Note that $\ker \phi_f$ is the kernel congruence taken in $\mathsf{Mon}\mathcal{V}$, while the kernel congruence $\ker f$ is taken in $\mathcal{V}$.

The shortcoming of this result is the restrictive assumption, that ${}_{fg}\mathcal{V}$ has to be *-strong. Following an idea of [2] we therefore note that by commutativity of Diagram (5) the assumptions of [25, Lemma 15] are satisfied, provided that for each $\mathcal{V}$-monoid A

1. the maps $\Pi_{A,A}$ and $\Pi_{A,A,A}$ (see Remark 2) are injective and
2. the $\mathcal{V}$-algebra A° is an entropically pure subalgebra of A^*.

One now obtains the following alternative to Proposition 13 where, for simplicity, we consider the embeddings κ_A and θ_A of Remark 2 as inclusions and denote the respective restrictions of $\Pi_{A,A}$ simply by Π.

Proposition 14 *Assume that, for a $\mathcal{V}$-monoid A,*

1. *for each $f \in \mathsf{A}^\circ$ one has $m^*(f) \in \Pi[F1^A \otimes \mathsf{A}^\circ] \cap \Pi[\mathsf{A}^\circ \otimes F1^A]$,*
2. *the morphisms $\Pi_{A,A}$ and $\Pi_{A,A,A}$ are injective,*
3. *the algebra A° is entropically pure in $F1^A$.*

Then there exist a monoid $\mathsf{A}^\bullet = (\mathsf{A}^\circ, \mathsf{A}^\circ \xrightarrow{m^\bullet} \mathsf{A}^\circ \otimes \mathsf{A}^\circ, \mathsf{A}^\circ \xrightarrow{e^\bullet} F1)$ induced from A by κ_A.

Proof: By Lemma 5 one has $m^*(f) \in \Pi[\mathsf{A}^\circ \otimes \mathsf{A}^\circ]$ by assumptions 1 and 3, for each $f \in \mathsf{A}^\circ$. By taking preimages one obtains a homomorphism $\pi[\mathsf{A}^\circ \otimes \mathsf{A}^\circ] \xrightarrow{\phi} \mathsf{A}^\circ \otimes \mathsf{A}^\circ$ and, consequently, the homomorphism $m^\bullet = \mathsf{A}^\circ \to \pi[\mathsf{A}^\circ \otimes \mathsf{A}^\circ] \xrightarrow{\phi} \mathsf{A}^\circ \otimes \mathsf{A}^\circ$, which is induced from A by κ_A by commutativity of Diagram (5). Now the claim follows by [25, Lemma 15] and assumption 3. This argument is the same as in [2]. □

Remark 3 In the variety $\mathcal{V} = \mathbf{Mod}_R$ conditions 1 and 2 are satisfied for every noetherian ring R, while condition 3 is satisfied, provided that R in addition is hereditary (see [2]).

3.4.3 The maximality condition

Recall from [25] that a lift $(-)^\bullet$ of a Sweedler functor $((-)^\circ, \kappa)$ is a left adjoint of the dual monoid functor, provided that $\mathsf{A}^\bullet$ not only is a subcomonoid of A^* induced by A, but the largest such. We, thus, have to show that a comonoid (C, μ, ϵ), which is a subcomonoid of A^* via a $\mathcal{V}$-embedding $C \xrightarrow{\psi} A^*$, is a subcomonoid of $\mathsf{A}^\bullet$, provided that the following diagram commutes.

(6)
$$\begin{array}{ccccc} C & \xrightarrow{\mu} & C \otimes C & \xrightarrow{\psi\otimes\psi} & A^* \otimes A^* \\ \downarrow{\scriptstyle \psi} & & & & \downarrow{\scriptstyle \Lambda_{A,A}} \\ A^* & & \xrightarrow{m^*} & & (A \otimes A)^* \end{array}$$

Lemma 8 *Let $\mathcal{V}$ be a noetherian and set-like entropic variety and assume that the Sweedler functor $((-)^\circ, \kappa)$ is liftable. Then, for a $\mathcal{V}$-monoid A, every subcomonoid C of A^* induced by A is a subcomonoid of $\mathsf{A}^\bullet$.*

Proof: Assume that $C \subset A^*$ and that Diagram (6) commutes. With $X = \operatorname{Im} \Lambda_{A,A} \circ (\kappa_\mathsf{A} \otimes \kappa_\mathsf{A})$ we thus have $m^* f \in X$ for each $f \in C$. Using Proposition 12 it suffices to prove the implication $m^* f \in X \implies \mathsf{A}f \in {}_{fg}\mathcal{V}$.

Since $\mathcal{V}$ is noetherian, it suffices to find a finitely generated subalgebra S of A^* with $af \in S$ for all $a \in A$. Since $af(b) = f(m(b \otimes a)) = m^* f(b \otimes a)$ by definition and $m^* f = \Lambda_{C,C}(t(g_i \otimes h_i))$ with $g_i, h_i \in C^*$ with $1 \leq i \leq n$ and an n-ary term term t by assumption, it is enough to show, that there is a map h in $S := < g_1, \ldots, g_n >$, the subalgebra of A^*, generated by the g_i, with $h(b) = \Lambda_{C,C}(t(g_i \otimes h_i))(b \otimes a)$ for all $b \in A$.

Let $s((a \otimes g_i) \otimes f_i)$ correspond to $a \otimes t(g_i \otimes h_i)$ by the isomorphism $C \otimes (C \otimes C^*) \simeq (C \otimes C^*) \otimes C^*$ and consider the map $h := can_{C^*} s(g_i \otimes h_i a) \in C^* \simeq C^* \otimes F1$. Then h belongs to S and we have $h(b) = ev(b \otimes can_{C^*} s(g_i \otimes h_i a)) = \Lambda_{C,C}(t(g_i \otimes h_i))(b \otimes a)$ by definition of Λ. This proves the claim. □

3.4.4 Summary

We summarize the results obtained above as follows.

Theorem 2 *Let $\mathcal{V}$ be a noetherian and set-like variety satisfying one of the following conditions*

1. *${}_{fg}\mathcal{V}$ is a *-strong subcategory of $\mathcal{V}$ and one of the following conditions is satisfied*
   - *(a) $\Lambda_{A,A} \circ (\kappa_\mathsf{A} \otimes \kappa_\mathsf{A})$ and $\Lambda_{A,A,A} \circ (\kappa_\mathsf{A} \otimes \kappa_\mathsf{A} \otimes \kappa_\mathsf{A})$ are monomorphisms, for each $\mathcal{V}$-monoid A.*
   - *(b) $\Lambda_{B,A} \circ (\kappa_\mathsf{B} \otimes \kappa_\mathsf{A})$ is a monomorphism, for each pair of $\mathcal{V}$-monoids (A, B).*
2. *For each $\mathcal{V}$-monoid A*
   - *(a) one has $m^*(f) \in \Pi[F1^A \otimes \mathsf{A}^\circ] \cap \Pi[\mathsf{A}^\circ \otimes F1^A]$, for each $f \in \mathsf{A}^\circ$,*
   - *(b) the morphisms $\Pi_{A,A}$ and $\Pi_{A,A,A}$ are injective,*
   - *(c) the algebra A° is entropically pure in $F1^A$.*

Then

- *I. The Sweedler functor $(-)^\circ$ can be lifted to a functor $(-)^\bullet \colon \mathsf{Mon}\mathcal{V} \to \mathsf{Comon}\mathcal{V}^{\mathsf{op}}$ and this is a left adjoint of the dual monoid functor; it is an opmonoidal functor.*

  Under hypothesis 1. $(-)^\bullet$ even is a strong opmonoidal functor.
- *II. $(-)^\bullet$ induces a dual adjunction on $\mathsf{Bimon}\mathcal{V}$.*
- *III. $(-)^\bullet$ induces a functor $\mathsf{Hopf}\mathcal{V} \to (\mathsf{Hopf}\mathcal{V})^{\mathsf{op}}$; this functor part of a dual adjunction on $\mathsf{Hopf}\mathcal{V}$, provided that $\mathcal{V}$ is a flat variety.*

Proof: Condition 1 (a) implies statement I. by [25, Lemma 15], Proposition 13 and Remark 4. That, under hypothesis 1, $(-)^\bullet$ is strong opmonoidal, follows from Lemma 7 since Φ is easily seen to be the comultiplication of $(-)^\bullet$.

On order to show that condition 1(b) implies this as well, it suffices to show that $\Lambda_{A,A,A} \circ (\kappa_\mathsf{A} \otimes \kappa_\mathsf{A} \otimes \kappa_\mathsf{A})$ is a monomorphism, since then Lemma [25, Lemma 15] applies. Now this condition implies, with $\mathsf{B} = \mathsf{A} \otimes \mathsf{A}$, that $\Lambda_{A\otimes A,A} \circ (\kappa_{\mathsf{A}\otimes\mathsf{A}} \otimes \kappa_\mathsf{A})$ is a monomorphism, while one gets, with $\mathsf{B} = \mathsf{A}$ and Lemma 7, that $\Phi_{\mathsf{A},\mathsf{A}}$ and, thus, $(\Phi_{\mathsf{A},\mathsf{A}} \otimes \mathsf{A}^\circ)$ is an isomorphism. Hence the claim follows, since the following diagram commutes.

$$
\begin{array}{ccccc}
\mathsf{A}^\circ \otimes \mathsf{A}^\circ \otimes \mathsf{A}^\circ & \xrightarrow{\kappa_\mathsf{A}\otimes\kappa_\mathsf{A}\otimes\kappa_\mathsf{A}} & A^* \otimes A^* \otimes A^* & \xrightarrow{\Lambda_{A,A,A}} & (A \otimes A \otimes A)^* \\
\downarrow{\scriptstyle \Phi_{\mathsf{A},\mathsf{A}}\otimes\mathsf{A}^\circ} & & \downarrow{\scriptstyle \Lambda_{A,A}\otimes A^*} & \nearrow{\scriptstyle \Lambda_{A\otimes A,A}} & \\
(\mathsf{A} \otimes \mathsf{A})^\circ \otimes \mathsf{A}^\circ & \xrightarrow[\kappa_{\mathsf{A}\otimes\mathsf{A}}\otimes\kappa_\mathsf{A}]{} & (A \otimes A)^* \otimes A^* & &
\end{array}
$$

Using, in the argument above, Proposition 14 instead of Proposition 13 one gets statement I. as well.

Now II. and III. follow from Proposition 10. $\square$

These results, with hypothesis 1, apply for example in the case where $\mathcal{V} = \mathbf{Mod}_R$ for a Dedekind ring R (see [7]). Under hypothesis 2 they apply to $\mathcal{V} = \mathbf{Mod}_R$ for hereditary noetherian rings (see [2]).

It would be interesting to find conditions on more general noetherian varieties to satisfy one of the hypotheses of this theorem. An attempt has been made in [1] to show that condition 2(b) is satisfied in every variety of semimodules over a commutative noetherian semiring, where 1(b) here holds as for modules. There is missing an argument, however, for the fact that every finitely generated subsemimodule of a free semimodule is (in the language of that paper) *uniformly finitely generated*.

4 Some open problems

The following problems occurred naturally in the presentation above. We consider them as interesting problems, independet of the context of this paper.

1. Find (necessary and) sufficient conditions on (the theory of) a variety to be noetherian.
   Note that the attempt to simply generalize the case of R-modules would be misleading here: It is not true that $\mathcal{V}$ is noetherian if and only if $F1$ is a noetherian algebra in the sense that every chain of subalgebras of $F1$ is stationary. For a counterexample see [6, Example 2.3].

2. Find (necessary and) sufficient conditions on an extremal monomorphism in an entropic variety to be entropically pure.

3. Find (necessary and) sufficient conditions on an object of an entropic variety to be entropically flat.

4. Find (necessary and) sufficient conditions on (the theory of) an entropic variety, such that the multiplication of the semi-dualization functor is monomorphic.

5. Find (necessary and) sufficient conditions on (the theory of) an entropic variety $\mathcal{V}$, such that its subcategory ${}_{fg}\mathcal{V}$ of finitely generated algebras is *-strong.

Appendix: Monoidal categories and functors

Throughout $\mathcal{C} = (\mathcal{C}, - \otimes -, I)$ denotes a symmetric monoidal category. If necessary, the associativity, left and right unit constraints will simply be denoted by *can*; the symmetry will be denoted by $\sigma = (C \otimes D \xrightarrow{\sigma_{CD}} D \otimes C)_{C,D}$.

By $\mathcal{C}^{\mathsf{op}}$ we denote the dual of $\mathcal{C}$ with tensor product and unit as in $\mathcal{C}$.

Recall that $\mathcal{C}$ is called *monoidal closed*, provided that, for each $\mathcal{C}$-object C the functor $C \otimes -$ has a right adjoint $[C, -]$. The counits $C \otimes [C, X] \to X$ of these adjunctions will be denoted by *ev*. By paramatrized adjunctions (see [19]) one thus has the *internal hom-functor* $[-,-]\colon \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathcal{C}$.

A monoidal functor from $\mathcal{C}$ to $\mathcal{C}'$ will be denoted by $G = (G, \Gamma, \gamma)$ with $G\colon \mathcal{C} \to \mathcal{C}'$ its functor, its multiplication Γ, which is a natural transformation $G_1 \Rightarrow G_2$ between the functors $G_1 = \mathcal{C} \times \mathcal{C} \xrightarrow{G \times G} \mathcal{D} \times \mathcal{D} \xrightarrow{-\otimes-} \mathcal{D}$ and $G_2 = \mathcal{C} \times \mathcal{C} \xrightarrow{-\otimes-} \mathcal{C} \xrightarrow{G} \mathcal{D}$ and a morphism $\gamma\colon I' \to GI$, its unit.

A monoidal functor F between symmetric monoidal categories is called a *symmetric monoidal functor* if, the following diagram commutes.

$$\begin{array}{ccc} FC \otimes FD & \xrightarrow{\Phi_{C,D}} & F(C \otimes D) \\ \sigma \downarrow & & \downarrow F\sigma \\ FD \otimes FC & \xrightarrow[\Phi_{D,C}]{} & F(D \otimes C) \end{array}$$

A monoidal functor is called *strong monoidal*, if Γ and γ are isomorphisms, *strict monoidal*, if Γ and γ are identities.

An *opmonoidal functor* from $\mathcal{C}$ to $\mathcal{C}'$ is a monoidal functor from $\mathcal{C}^{\mathsf{op}}$ to $\mathcal{C}'^{\mathsf{op}}$.

By $\mathsf{Mon}\mathcal{C}$ and $\mathsf{Comon}\mathcal{C}$ we denote the categories of monoids $\mathsf{M} = (M, M \otimes M \xrightarrow{m} M, I \xrightarrow{e} M)$ in $\mathcal{C}$ and of comonoids $\mathsf{C} = (C, C \xrightarrow{\Delta} C \otimes C, C \xrightarrow{\epsilon} I)$ in $\mathcal{C}$, respectively. Obviously $\mathsf{Mon}\mathcal{C}^{\mathsf{op}} = (\mathsf{Comon}\mathcal{C})^{\mathsf{op}}$.

Remarks 4 ([15],[16],[5])

1. The composition of monoidal functors is a monoidal functor.

2. If (G, Γ, γ) is a monoidal functor and F is left adjoint to G (with unit η and counit ϵ), then (F, Φ, ϕ) is opmonoidal, where ϕ corresponds by adjunction to γ and $\Phi_{C,D}$ corresponds by adjunction to $\Gamma_{FC,FD} \circ (\eta_C \otimes \eta_D)$. (F, Φ, ϕ) will be called the *opmonoidal mate* of (G, Γ, γ). The connection between a monoidal functor and its opmonoidal mate, thus, is characterized by commutativity of the diagrams

$$\begin{array}{ccc} C\otimes D & \xrightarrow{\eta_{C\otimes D}} & GF(C\otimes D) \\ {\scriptstyle \eta_C\otimes\eta_D}\downarrow & & \downarrow{\scriptstyle G\Phi_{C,D}} \\ GFC\otimes GFD & \xrightarrow[\Gamma_{FC,FD}]{} & G(FC\otimes FD) \end{array} \qquad \begin{array}{ccc} cocommutativeFI & & \\ {\scriptstyle F\gamma}\downarrow & \searrow^{\phi} & \\ FGI & \xrightarrow{\epsilon_I} & I \end{array}$$

or, equivalently

$$\begin{array}{ccc} F(GC\otimes GD) & \xrightarrow{\Phi_{GC,GD}} & FGC\otimes FGD \\ {\scriptstyle F\Gamma_{C,D}}\downarrow & & \downarrow{\scriptstyle \epsilon_C\otimes\epsilon_D} \\ FG(C\otimes D) & \xrightarrow[\epsilon_{C\otimes D}]{} & C\otimes D \end{array} \qquad \begin{array}{ccc} I & \xrightarrow{\eta_I} & GFI \\ & {\scriptstyle\gamma}\searrow & \downarrow{\scriptstyle G\phi} \\ & & GI \end{array}$$

If (G, Γ, γ) is symmetric monoidal, then its mate (F, Φ, ϕ), if it exists, is symmetric opmonoidal.

3. If F is a left adjoint of a monoidal functor and its opmonoidal structure is strong, then the unit and the counit of the adjunction are monoidal transformations; such adjunctions are called *monoidal adjunctions.* The left adjoint of a monoidal adjunction is always strong.

4. Monoidal functors map monoids to monoids. This is a consequence of 1., since a monoid in $\mathcal{C}$ is a monoidal functor $\mathbf{1} \to \mathcal{C}$ with $\mathbf{1}$ the terminal category.

   In more detail: Let $G := (G, \Gamma, \gamma)\colon \mathcal{C} \to \mathcal{D}$ be a monoidal functor. Then, for any monoid (M, m, e) in $\mathcal{C}$,

   $$(GM, GM \otimes GM \xrightarrow{\Gamma_{M,M}} G(M \otimes M) \xrightarrow{Gm} GM, I' \xrightarrow{\psi} GI \xrightarrow{Ge} GM)$$

   is a monoid in $\mathcal{D}$ and this construction — acting on monoid morphisms as G — is functorial. We call the resulting functor $\mathsf{Mon}\mathcal{C} \to \mathsf{Mon}\mathcal{D}$ the *functor induced by* G and often denote it by slight abuse of notation by G as well.

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