Varieties Without Minimal Generators

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Abstract

While many familiar varieties have a minimal varietal generator, i.e., a regular projective finitely presentable regular generator such that none of its retracts is a regular generator, and even a unique one, we present (a) a variety having no minimal varietal generator at all and (b) a variety having two non-isomorphic minimal varietal generators. Moreover we demonstrate that the same effects can happen with respect to a weaker notion of minimality and are common even in module categories.

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Introduction

It is well known that the varieties $\mathcal{W}$ Morita equivalent to a given variety $\mathcal{V}$ are determined by the *varietal generators* of $\mathcal{V}$, i.e., by those regular generators G of $\mathcal{V}$ which are finitely presentable and regular projective: the Lawvere theory $\mathsf{Th}\mathcal{W}$ of $\mathcal{W}$ is isomorphic to the theory $\mathsf{Th}_{\mathcal{V}}G$ for some varietal generator G in $\mathcal{V}$ (where $\mathsf{Th}_{\mathcal{V}}G$ is the dual of the full subcategory of $\mathcal{V}$ spanned by all finite copowers nG, one for each $n \in \mathbb{N}$). Thus, the possible equational representations of $\mathcal{V}$ are given by its varietal generators. Given varietal generators G and Q, there exists a retraction $r\colon nG \longrightarrow Q$ for some $n \in \mathbb{N}$. Given a retraction $r\colon G \longrightarrow Q$ (with section $s\colon Q \longrightarrow G$) between non-isomorphic varietal generators G and Q of the variety $\mathcal{V}$ the equational representation of $\mathcal{V}$ by means of Q is simpler than the representation given by G in the following sense:

1. for each V in $\mathcal{V}$ the underlying set $\hom_{\mathcal{V}}(Q, V)$ is "smaller" than the underlying set $\hom_{\mathcal{V}}(G, V)$ since $\hom(r, V)$ embeds the first into the latter;

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2. the theory $\mathsf{Th}_{\mathcal{V}}Q$ has "fewer" morphisms than $\mathsf{Th}_{\mathcal{V}}G$, since the theory-morphism $\bar{u}\colon \mathsf{Th}_{\mathcal{V}}G \longrightarrow \mathsf{Th}_{\mathcal{V}}Q$ defined by

$$\bar{u}(jG \xrightarrow{t} kG) = jQ \xrightarrow{js} jG \xrightarrow{t} kG \xrightarrow{kr} kQ$$

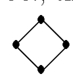
is surjective on morphisms: for each $jQ \xrightarrow{\tau} kQ$ there exists, by projectivity of jG, some $jG \xrightarrow{t} kG$ with $kr \circ t = \tau \circ jr$; then $\bar{u}(t) = \tau$[1] (see [8] for details).

In order to find an as simple as possible equational representation of $\mathcal{V}$ one, therefore, has to look for a varietal generator G of $\mathcal{V}$ which is, in some sense, minimal with respect to the following preorder on (isomorphism classes of) varietal generators:

$$H \sqsubseteq G \quad \text{iff} \quad H \text{ is a retract of } G.$$

Definition. A varietal generator G in a variety $\mathcal{V}$ is called *minimal* provided none of its proper[2] retracts is a regular (hence varietal) generator. G will be called *preminimal* provided that, for each varietal generator H with $H \sqsubseteq G$, one also has $G \sqsubseteq H$.

Note that a varietal generator G is preminimal iff $[G]$ is minimal in the order associated to the preorder $\sqsubseteq$. A minimal varietal generator is preminimal.

In many familiar varieties as, e.g., Ab, Ring, and Bool, the representing objects of the canonical underlying functors ($\mathbb{Z}$, $\mathbb{Z}[X]$, and  respectively) are the only minimal varietal generators (see [9]). In contrast to these observations — and maybe contradicting algebraic intuition — we will show in this note that

- a variety need not to have a (pre)minimal varietal generator, and
- a variety might have even two non-isomorphic (pre)minimal varietal generators.

We are going to use two completely different methods to produce such varieties. The first uses fundamental results of categorical algebra while the second is based on a result of Bergman [5] and shows that (some of) the varieties in question can even be chosen to be categories of R-modules for suitable rings R.

The following immediate consequence of the definitions will be useful here.

Lemma 1 *If P is a preminimal varietal generator in $\mathcal{V}$ then either P is minimal or there exists a preminimal varietal generator Q in $\mathcal{V}$ such that P and Q are not isomorphic but mutually retracts of each other.*

[1] It also holds that $\mathsf{Th}_{\mathcal{V}}Q$ embeds into $\mathsf{Th}_{\mathcal{V}}G$—though not as a subcategory.

[2] We call a retract G' of G *proper*, if G' is not isomorphic to G; observe that a retract might be improper even if the retraction under consideraton is not an isomorphism.

1 Varieties without minimal generators

The varieties in this section are constructed as model categories of finite product sketches. The corresponding facts important for this note are collected as follows: If $\mathcal{A}$ is a small category with some specified set of finite products, we denote by $\mathsf{Mod}\mathcal{A}$ the category of all Set-valued functors preserving all the specified finite products. This category is equivalent to a (many-sorted) variety. $\mathcal{A}^{\mathsf{op}}$ is contained as a full subcategory in $\mathsf{Mod}\mathcal{A}$ via Yoneda embedding. The dual of the closure of $\mathcal{A}^{\mathsf{op}}$ in $\mathsf{Mod}\mathcal{A}$ under finite coproducts is called the *finite product theory* $\mathsf{FP}(\mathcal{A})$ generated by $\mathcal{A}$; it is again a sketch by all its finite products. $\mathsf{FP}(\mathcal{A})^{\mathsf{op}}$ is the full subcategory of $\mathsf{Mod}\mathcal{A}$ consisting of the finitely generated free algebras in $\mathsf{Mod}\mathcal{A}$ w.r.t. to the evaluation functor $\mathsf{Mod}\mathcal{A} \longrightarrow \mathsf{Set}^{ob\mathcal{A}}$. $\mathsf{FP}(\mathcal{A})$ sketches the same variety as $\mathcal{A}$ (i.e., $\mathsf{Mod}\mathcal{A} \simeq \mathsf{Mod}\mathsf{FP}(\mathcal{A})$) and its dual is a regular generator of $\mathsf{Mod}\mathcal{A}$ (see [3, 8.4, Theorem 1]).

A given variety $\mathcal{V}$ can have several (non-equivalent) finite product theories. But all these theories have the same Cauchy-completion which is the dual of the full subcategory $\mathsf{Proj}_{fp}\mathcal{V}$ of $\mathcal{V}$ consisting of its finitely presentable regular projectives. Following [7, 11.10], $(\mathsf{Proj}_{fp}\mathcal{V})^{\mathsf{op}}$ is the only Cauchy-complete finite product theory sketching $\mathcal{V}$ (see [1] and [8] for more details[3]). We can collect the just stated facts as follows.

Theorem 1 *For each small category $\mathcal{A}$ with a specified set of finite products the following hold:*

1. *There are equivalences*
$$\mathsf{Mod}\mathcal{A} \simeq \mathsf{Mod}\mathsf{FP}(\mathcal{A}) \simeq \mathsf{Mod}(\mathsf{Proj}_{fp}\mathsf{Mod}\mathcal{A})^{\mathsf{op}}$$
given by extending the models.

2. *The subcategory $\mathsf{FP}(\mathcal{A})^{\mathsf{op}}$ is a regular (even a dense) generator of $\mathsf{Mod}\mathcal{A}$.*

3. *If $\mathcal{A}$ is Cauchy-complete, so is $\mathsf{FP}(\mathcal{A})$ and $\mathsf{FP}(\mathcal{A}) = (\mathsf{Proj}_{fp}\mathsf{Mod}\mathcal{A})^{\mathsf{op}}$. If, in addition, $\mathcal{A}$ has all finite products one even has $\mathcal{A} = \mathsf{FP}(\mathcal{A}) = (\mathsf{Proj}_{fp}\mathsf{Mod}\mathcal{A})^{\mathsf{op}}$.*

The resulting one-to-one correspondence between Cauchy-complete categories with finite products and varieties can be extended to a dual biequivalence. Morphisms among Cauchy-complete categories with finite products then are finite product preserving functors while morphisms among varieties are right adjoints preserving filtered colimits and regular epimorphisms.

[3] A category $\mathcal{A}$ is called *Cauchy-complete* provided every idempotent morphism u in $\mathcal{A}$ splits as $u = sr$ with $rs = 1$; every category $\mathcal{A}$ has a *Cauchy-completion*, i.e, a full embedding E into a Cauchy-complete category $\mathcal{A}^*$ such that any functor F from $\mathcal{A}$ into a Cauchy-complete category $\mathcal{B}$ factors uniquely over E.

Theorem 2 ([1]) *The passage $\mathcal{A} \longrightarrow \mathsf{Mod}\mathcal{A}$ serves as a dual biequivalence between small Cauchy-complete categories with finite products and varieties.*

Remark 1 If, in a Cauchy complete category $\mathcal{A}$ with finite coproducts, there exists some object G such that each $\mathcal{A}$-object A is a retract of some finite copower of G, the variety determined by $\mathcal{A}$ is even one-sorted. In fact $\mathsf{Mod}\mathcal{A}$ is equivalent to the variety $\mathcal{V}$ with Lawvere theory $\mathsf{Th}\mathcal{V} = \mathsf{Th}_{\mathcal{A}^{op}}G$ made up by G and its finite powers in $\mathcal{A}$. $\mathsf{Th}\mathcal{V}$ is, as a category, equivalent to $\mathsf{FP}(\mathcal{A})$.

1.1 A variety without preminimal generators

We are going to sketch a variety as follows:

Let $\mathbb{N}_i = \{n \in \mathbb{N} \mid n \geq i\}$ for $i \in \mathbb{N}$. Denote by μ_i and ν_i the maps $\mathbb{N}_{i+1} \longrightarrow \mathbb{N}_i$ with $\mu_i(i+1) = e_i$, $\nu_i(i+1) = o_i$ and $\mu_i(n+1) = \mu_i(n)+2$, $\nu_i(n+1) = \nu_i(n)+2$, where e_i is smallest even number in $\mathbb{N}_i$ and o_i the smallest odd number in $\mathbb{N}_i$ respectively. Then $\mathbb{N}_{i+1} \xrightarrow{\mu_i} \mathbb{N}_i \xleftarrow{\nu_i} \mathbb{N}_{i+1}$ is a coproduct in Set. There is a codiagonal $\triangle_i \colon \mathbb{N}_i \longrightarrow \mathbb{N}_{i+1}$ defined by $\triangle_i\mu_i = \triangle_i\nu_i = 1$ and a symmetry $s_i : \mathbb{N}_i \longrightarrow \mathbb{N}_i$ defined by $s_i\mu_i = \nu_i$ and $s_i\nu_i = \mu_i$.

Let $\mathcal{F}$ be the subcategory of Set generated by the sets and maps just defined. Then the following facts are easily seen to be true:

Fact 1 *For each $i \in \mathbb{N}$ the diagram*

$$\mathbb{N}_{i+1} \xrightarrow{\mu_i} \mathbb{N}_i \xleftarrow{\nu_i} \mathbb{N}_{i+1}$$

is a coproduct in $\mathcal{F}$.

Fact 2 *For $i, j \in \mathbb{N}$, $i < j$, $\mathbb{N}_i$ is not a retract of $\mathbb{N}_j$ in $\mathcal{F}$.*

Fact 3 *For $i, j \in \mathbb{N}$, $i \neq j$, the objects $\mathbb{N}_i$ and $\mathbb{N}_j$ are not isomorphic in $\mathcal{F}$.*

Fact 4 *$\mathcal{F}$ is a Cauchy complete category.*

$\mathcal{F}^{\mathsf{op}}$, together with the products specified in Fact 1 now sketches a (possibly many-sorted) variety $\mathcal{V}$.

Proposition 1 *$\mathcal{V}$ is a one-sorted variety without preminimal varietal generators.*

Proof. Let us observe first that each object $\mathbb{N}_i$ is — considered as a $\mathcal{V}$-object — a varietal generator of $\mathcal{V}$. Clearly, each $\mathbb{N}_i$ belongs to $\mathsf{Proj}_{fp}\mathcal{V}$. $\mathbb{N}_0$ is a varietal generator of $\mathcal{V}$. Indeed, since $\mathsf{FP}(\mathcal{F}^{\mathsf{op}})^{\mathsf{op}}$ is a regular generator of $\mathcal{V}$, for each $\mathcal{V}$-object V there exists a regular epimorphism $\coprod_J \mathbb{N}_{i_j} \xrightarrow{q} V$. Now, for each $k \in \mathbb{N}$, by definition of $\mathcal{F}$ there exists a retraction $r_k\colon \mathbb{N}_0 \longrightarrow \mathbb{N}_k$. Hence there results a regular epimorphism

$$J \cdot \mathbb{N}_0 \xrightarrow{\coprod_J r_{i_j}} \coprod_J \mathbb{N}_{i_j} \xrightarrow{q} V.$$

Thus, $\mathbb{N}_0$ is a regular generator of $\mathcal{V}$, and $\mathcal{V}$ is one-sorted.

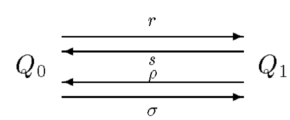
But then also each $\mathbb{N}_i$ is a regular (hence varietal) generator; in fact each regular quotient $X \cdot \mathbb{N}_0 \to V$ is a regular quotient $Y\mathbb{N}_i \to V$ since, for each $i \in \mathbb{N}$, $\mathbb{N}_0 = 2^i \mathbb{N}_i$.

Assume now that G is a preminimal varietal generator. By Theorem 1 and Fact 4 $G = \coprod_{j=1}^{k} \mathbb{N}_{i_j}$. With $m = \min\{n_{i_j}\}$ and $n = \max\{n_{i_j}\} + 1$ there are retractions $G \to \mathbb{N}_m$ and $G \to \mathbb{N}_n$. Since G is preminimal G has to be a retract of $\mathbb{N}_n$. It follows that $\mathbb{N}_m$ is a retract of $\mathbb{N}_n$ which contradicts Fact 2 since $m < n$. $\diamond$

1.2 A universal variety with two non-isomorphic preminimal generators

We define a Cauchy complete category as follows.

Let $\mathcal{F}$ be the free category over the graph



subject to the equations

$$rs = 1_{Q_1} \text{ and } \rho\sigma = 1_{Q_0}.$$

Lemma 2 *Let v be a morphism in $\mathcal{F}$, v not an identity. Then v is of precisely one of the following types:*

Type 1 *a composition of finitely many copies of r and ρ;*

Type 2 *a composition of finitely many copies of s and σ;*

Type 3 *a composition $w_2 \circ w_1$, where w_1 is of type 1 and w_2 is of type 2.*

Proof. Since none of the morphisms r, s, ρ, σ is an isomorphism the obove types exclude each other.

Assume v is neither of type 1 nor of type 2. v can be written as a composition of r, s, ρ, σ containing no sequence rs and $\rho\sigma$. Decompose v as

$$v = w_n \circ \ldots \circ w_1$$

such that each w_i is a composition of either retractions r and ρ or sections s and σ and, for each $i < n$, if w_i consists of sections (retractions), w_{i+1} consists of retractions (sections).

Now only one of the segments w_i consists of sections only. For, if $w_i = s \circ \ldots$, w_{i+1} must be of the form $w_{i+1} = \ldots \circ r$ since the composition $w_{i+1} \circ w_i$ is defined. But this is impossible since v contains no sequence rs. Similarly for $w_i = \sigma \circ \ldots$.

Hence $v = w_2 \circ w_1$ where w_1 consists of retractions and w_2 consists of sections only. ◇

Fact 5 *Q_0 and Q_1 are not isomorphic in $\mathcal{F}$.*

Proof. None of the types of morphisms listed in the Lemma allows for an isomorphism. ◇

Fact 6 *$\mathcal{F}$ is Cauchy complete.*

Proof. Any idempotent v—say $v\colon Q_0 \to Q_0$—can, by the Lemma, be decomposed as $v = w_2 \circ w_1$.

Case 1: cod w_1 = dom $w_2 = Q_0$

$$\text{Then} \quad \begin{array}{rcccl} w_1 & = & \rho r \ldots \rho r & = & (\rho r)^n \\ w_2 & = & s\sigma \ldots s\sigma & = & (s\sigma)^m \end{array}$$

Now $v^2 = v$ means

$$(s\sigma)^m (\rho r)^n (s\sigma)^m (\rho r)^n = (s\sigma)^m (\rho r)^n$$

which implies (since sections are monomorphisms and retractions epimorphisms)

$$(\rho r)^n (s\sigma)^m = 1.$$

Consequently $n = m$ since otherwise r or s would be an isomorphism, contradicting Fact 5. Thus the decomposition $w_2 \circ w_1$ is a splitting of v.

Case 2: cod w_1 = dom $w_2 = Q_1$

$$\text{Then} \quad \begin{array}{rcccl} w_1 & = & r\rho \ldots \rho r & = & (r\rho)^n r \\ w_2 & = & s\sigma \ldots \sigma s & = & s(\sigma s)^m \end{array}$$

Here $v^2 = v$ means

$$s(\sigma s)^m (\rho r)^n rs(\sigma s)^m (\rho r)^n r = s(\sigma s)^m (\rho r)^n r$$

and one can conclude as above that $m = n$ since $rs = 1$. Again, $w_2 \circ w_1$ is a splitting of v. ◇

Remark 2 Instead of proving Facts 5 and 6 directly one might alternatively use the general methods developed in [10].

$\mathcal{F}$ (without any specified products) sketches a (possibly many-sorted) variety $\mathcal{W}$ with unary operations only; i.e. $\mathcal{W} = \mathsf{Set}^{\mathcal{F}}$.

Let $\hat{\mathcal{F}} = \mathsf{FP}(\mathcal{F})$. Then the following holds where we implicitely refer to Theorem 1. Note in particular that the varietal generators of $\mathcal{W}$ (if they exist) are precisely those $\hat{\mathcal{F}}$-objects G such that every $\hat{\mathcal{F}}$-object is a retract of some finite copower mG. Note also that the coproducts in $\hat{\mathcal{F}}$ are coproducts in $\mathcal{W}$ and that $\hat{\mathcal{F}}$, hence $\mathcal{W}$, contains $\mathcal{F}$ as a full subcategory.

Fact 7 *Every object of $\hat{\mathcal{F}}$, different from the initial one, is a varietal generator of $\mathcal{W}$; in particular $\mathcal{W}$ is a one-sorted variety.*

Proof. Let $G = \coprod_I Q_i$, $H = \coprod_j Q_j$ be in $\hat{\mathcal{F}}$, hence $Q_i,\ Q_j \in \{Q_0,\ Q_1\}$, with $I \neq \emptyset$. Choose $i_0 \in I$ and, for each $k \in I \cup J$, a retraction

$$Q_k \xrightarrow{s_k} Q_{i_0} \xrightarrow{r_k} Q_k = 1$$

with $r_{i_0} = 1$. This is possible by the definition of $\mathcal{F}$. Then one gets retractions

$$\coprod_J Q_{i_0} \xrightarrow{\coprod_J r_j} \coprod_J Q_j = H$$

and

$$\coprod_I Q_i \xrightarrow{[r_i]=r} Q_{i_0}$$

(defined by $r \circ \mu_i = r_i$ for $i \in I$; observe $r\mu_{i_0} = r_{i_0} = 1$). Thus,

$$\coprod_J \Big(\coprod_I Q_i\Big) \xrightarrow{J \cdot r} \coprod_J Q_{i_0} \xrightarrow{\coprod_J r_j} H$$

is a retraction. ◇

Fact 8 *Every object G of $\hat{\mathcal{F}}$, different from the initial one, has a retract Q in $\hat{\mathcal{F}}$, different from the initial one, which is not isomorphic to G.*

Proof. Let $G = \coprod_I Q_i$ be in $\hat{F}$, $I \neq \emptyset$. By the proof of Fact 7 Q_0 or Q_1 is a retract of G which certainly is not isomorphic to G provided that $|I| > 1$ according to the construction of $\hat{\mathcal{F}}$. Finally, the case $|I| = 1$ is settled by Fact 5 since Q_0 is a retract of Q_1 and Q_1 is a retract of Q_0. ◇

We summarize these results as

Proposition 2 *$\mathcal{W}$ is a finitary one-sorted variety which has no minimal but precisely two non-isomorphic preminimal varietal generators.*

Remark 3 In view of Lemma 1 the variety $\mathcal{W}$ constructed above has the smallest possible number of preminimal varietal generators which are not minimal.

This variety is, moreover, universal in the following sense: Let $\mathcal{W}'$ be a variety with two non-isomorphic preminimal varietal generators P_0, P_1. By Lemma 1 there exist retractions $P_1 \xrightarrow{\bar{s}} P_0 \xrightarrow{\bar{r}} P_1 = id$ and $P_0 \xrightarrow{\bar{\sigma}} P_1 \xrightarrow{\bar{\varrho}} P_0 = id$. Hence there exists a unique functor $S\colon \mathcal{F} \longrightarrow \mathcal{W}'$ sending Q_i to P_i and the morphisms in our starting graph to the corresponding barred ones. It is not difficult to see that S in fact embeds $\mathcal{F}$ into the canonical algebraic theory of $\mathcal{W}'$. Thus, $\mathcal{F}$ is a subtheory of the canonical algebraic theory of $\mathcal{W}'$ for each variety $\mathcal{W}'$ with two non-isomorphic preminimal varietal generators; in other words (see Theorem 2): $\mathcal{W}$ is a quotient of each such variety.

2 Module categories with non-isomorphic (pre-)minimal generators

In this section we show that varieties with non-isomorphic preminimal (or even minimal) varietal generators can even appear as categories R-Mod of left R-modules for some ring R.

Concerning varietal generators in categories of modules the following is obvious:

Lemma 3 *A varietal generator G in R-Mod is*

- *minimal iff none of its proper direct summands is a varietal generator;*
- *preminimal iff each varietal generator H has G as a direct summand, provided H is a direct summand of G.*
  ($G = H \oplus H' \Rightarrow H = G \oplus G'$)

Now denote, for a given ring R, by $\mathcal{P}(R)$ its monoid of projectives, i.e., the monoid of isomorphism classes $|P|$ of finitely generated projective left R-modules P with the operation $|P| + |Q| = |P \oplus Q|$. $\mathcal{P}(R)$ is always conical, (i.e., $x + y = 0 \Rightarrow x = 0 = y$) and has an order-unit, i.e., an element $e \neq 0$ such that for each x there is y and $n \in \mathbb{N}$ such that $x + y = ne$ (order-units correspond to varietal generators).

Our construction is based on the following result of Bergman:

Theorem 3 ([5, 6.2, 6.4]) *For any conical commutative monoid M with an order-unit there exists a ring R_M such that $\mathcal{P}(R_M)$ and M are isomorphic. This ring is even hereditary (i.e., submodules of projective R_M-modules are projective) provided that M is finitely generated.*

Since finite sums of varietal generators are varietal generators and, for each module category R-Mod, the varietal generators are precisely the order units of the monoid $\mathcal{P}(R)$, the order units of $\mathcal{P}(R)$ form a subsemigroup $\mathcal{V}(R)$. The following is then a direct consequence of Lemma 3.

Lemma 4 *If, for some ring R, the monoid $\mathcal{V}(R)$ is a non-trivial group, the category R-*Mod *has no minimal varietal generator and* card$\mathcal{V}(R)$ *non-isomorphic pre-minimal varietal generators.*

In order to produce conical monoids with order units the following observation is useful: for any commutative monoid M, let M^* be the monoid obtained by adding a new zero-element 0, i.e., $M^* = M \cup \{0\}$ and $m + 0 = 0 + m = m$ for all $m \in M$. Then M^* is conical and order units of M remain to be order units in M^*. If in particular M is a group, each element of M is an order unit in M^*.

Proposition 3 *For any cardinal number $\kappa > 1$ there exists a ring R_κ such that the variety R_κ-*Mod *has no minimal varietal generator, but κ non-isomorphic pre-minimal ones.*

Proof. Chose an Abelian group M of cardinality κ. Then R_{M^*} has the required properties by Lemma 4. ◇

Proposition 4 *There exists a ring R such that the variety R-*Mod *has two non-isomorphic minimal varietal generators.*

Proof. Let M be the free commutative monoid with two generators p, g satisfying $p^2 = g^2$. Then M consists of elements 1, $p^n g^m$, $n, m \in \mathbb{N}$. M is conical and both, p and g are order units because

$$\begin{array}{lllllll} (p^n g^m) g^m & = & p^n g^{2m} & = & p^n p^{2m} & = & p^{n+2m} \\ p^n (p^n g^m) & = & p^{2n} g^m & = & g^{2n} g^m & = & g^{2n+m} \end{array}$$

Moreover $p \nmid g$ and $g \nmid p$. Hence R_M-Mod has two non-isomorphic minimal varietal generators corresponding to p and g. ◇

3 When minimal generators do exist

Here we present sufficient conditions for the existence of minimal varietal generators.

Proposition 5 *Let $\mathcal{V}$ be a variety where finitely generated objects are finitely presentable or regular projective objects are closed under directed colimits. Then $\mathcal{V}$ has a minimal varietal generator.*

Proof. Assume that $\mathcal{V}$ does not have a minimal varietal generator. Then there is a descending chain

$$P_0 \xrightarrow{r_0} P_1 \xrightarrow{r_1} P_2 \longrightarrow \dots$$

of finitely presentable regular projectives and proper retractions. Let P_ω be a colimit of this chain. Since the colimit component $p_0 \colon P_0 \longrightarrow P_\omega$ is a regular epimorphism (recall that the colimiting cone (p_i) is jointly surjective), P_ω is finitely generated. If it is finitely presentable, some of the $p_i \colon P_i \longrightarrow P_\omega$ has a right inverse s; hence $p_i = p_i \circ s \circ p_i$. Due to the construction of directed colimits in varieties, for each $a \in P_i$, there is some $j > i$ such that $r_{ij}(a) = r_{ij}(sp_i(a))$ where $r_{ij} = r_{j-1} \circ \dots \circ r_i$. Since P_i is finitely generated one even has, for a suitable j, $r_{ij} = r_{ij} \circ sp_i = r_{ij} \circ sp_j \circ r_{ij}$, which implies $r_{ij} \circ sp_j = 1$. Therefore p_j is an isomorphism. Hence r_j is an isomorphism, which is impossible.

If P_ω is regular projective then it is again finitely presentable because finitely generated regular projectives are finitely presentable. In fact, a finitely generated regular projective P admits a retraction $P \xrightarrow{s} F \xrightarrow{r} P = id$ with a finitely generated free object F and, hence a coequalizer diagram

$$F \overset{sr}{\underset{1}{\rightrightarrows}} F \xrightarrow{\ r\ } P$$

Thus P is finitely presentable. ◇

Corollary 1 *Let R be either a left Noetherian ring or a left perfect ring. Then R-*Mod *has a minimal varietal generator.*

Proof. If R is left Noetherian (i.e., left ideals of R are finitely generated) then any finitely generated flat left R-module is projective ([6, 11.31]). Hence any finitely generated directed colimit of projective modules is projective ([6, 11.32]). By the proof of Proposition 5, R-Mod has a minimal varietal generator.

If R is left perfect (i.e., satisfies d.c.c. on principal right ideals), every flat left module is projective (see [4]). Hence projective left R-modules are closed under directed colimits. ◇

Corollary 2 ([9]) *The categories* Grp *of groups,* Bool *of Boolean algebras and* cRng *of commutative unital rings have a (unique) minimal varietal generator.*

Proof. For groups this follows from the fact that projective groups are free. The free group on one generator is the unique minimal varietal generator in Grp. In the remaining categories under consideration the finitely generated objects are finitely presentable: in Bool every finitely generated algebra is finite; in cRng it is a quotient of $\mathbb{Z}[X_1, \dots X_n]$ for some natural number n. But each congruence on $\mathbb{Z}[X_1, \dots X_n]$ is finitely generated. For the unicity arguments see [9]. ◇

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