The Formal Theory of Hopf Algebras

Part I: Hopf monoids in a monoidal category

Hans–E. Porst

Abstract

The category **Hopf**$\mathbb{C}$ of Hopf monoids in a symmetric monoidal category $\mathbb{C}$, assumed to be locally finitely presentable as a category, is analyzed with respect to its categorical properties. Assuming that the functors "tensor squaring" and "tensor cubing" on $\mathbb{C}$ preserve directed colimits one has the following results: (1) If, in $\mathbb{C}$, extremal epimorphisms are stable under tensor squaring, then **Hopf**$\mathbb{C}$ is locally presentable, coreflective in the category of bimonoids in $\mathbb{C}$ and comonadic over the category of monoids in $\mathbb{C}$. (2) If, in $\mathbb{C}$, extremal monomorphisms are stable under tensor squaring, then **Hopf**$\mathbb{C}$ is locally presentable as well, reflective in the category of bimonoids in $\mathbb{C}$ and monadic over the category of comonoids in $\mathbb{C}$.

MSC 2000: Primary 16T05, Secondary 18D10
Keywords: Monoidal category, Hopf monoids, bimonoids, limits, colimits, left and right adjoints

Introduction

The first monograph on Hopf algebra theory (Sweedler 1969 [23]) paid quite some attention to their categorical properties. Somewhat surprisingly more recent successors like e.g. [8] — though using categorical language throughout — hardly touch these questions. The question we here have in mind are, e.g., *Does the category of Hopf algebras have products* (or, more generally, all limits). *How are they constructed? Does it have colimits? Do the naturally occurring functors* (e.g., the embedding of the category of Hopf algebras into that of bialgebras) *have adjoints?* Some scattered results exist. To name a few of them we mention the following

1. Takeuchi proved the existence of free Hopf algebras over coalgebras (claimed to exist but not proved by Sweedler) in [25].
2. He also proved in that paper that coproducts of Hopf algebras exist and can be constructed on the level of bialgebras.
3. The Hopf envelope, i.e., the reflection from bialgebras into Hopf algebras has been constructed in the 1980s.
4. Special instances of (co)free Hopf algebras are better known as, e.g., (co)commutative (co)free Hopf algebras over (co)commutative (co)algebras; this might be due to the fact that the tensor product is the product for cocommutative coalgebras and the coproduct for commutative algebras.

A systematic approach to tackle these problems surfaced only recently (see [15], [16], [17], [18],[19], [20], [21] [22]). This approach requires not only categorical language but quite a bit of category theory, too. The author believes that — after these scattered reports on work in progress — now the time has come to present the complete picture that emerged. As one would expect, this is more than a simple collection of the results presented in those papers: indeed, there are improvements in both, the results and the arguments. For example, the categorical content of Takeuchi's construction of free Hopf algebras has been made clearer by introducing *comonoids, the use of the concept of the monoidal lift of a monoidal functor has been extended to the discussion of convolution, and the restriction of only considering locally presentable categories whose extremal epimorphisms coincide with the regular ones could be avoided.

We call this line of work *formal theory of Hopf algebras* for two reasons. (a) We do not discuss examples and applications of Hopf algebras at all, since this is done extensively in the monographs available. (b) Our starting point for developing the theory is a more formal (rather: a more abstract) one, in that we do not start with module categories but rather with symmetric monoidal ones. (Note that this is well in line with other recent publications such as [4] and [24].).

The reason for this more abstract approach is twofold: First, this way we are able to make the analogy between Hopf algebras and groups precise: R-Hopf algebras are non-cartesian Hopf monoids in $\mathbf{Mod}_R$, while groups are cartesian Hopf monoids in **Set**; secondly, this more abstract starting point enables us to use the tool of categorical dualization, which allows to avoid quite a number of (unnecessary) proofs.

Clearly, this way we cannot use a tool, considered convenient by many researchers working in the field, namely the so-called *Sweedler notation.* But the author believes that it was this notation which has hidden the categorical content of both, results and arguments in Hopf algebra theory, and so prevented the theory to make use of helpful categorical tools. The *Crucial Lemma* in Section 1.5 illustrates this effect in a paradigmatic way.

The formal theory of Hopf algebras requires the following tools:

- The theory of symmetric monoidal categories to *define* these structures in a simple way.
- Elements of the theory of accessible and locally presentable categories. This not only is crucial in order to prove existence of cofree Hopf algebras, but also proves to be extremely convenient otherwise.

  For the reader feeling uneasy with the use of that level of category theory we also show in Section 3.3, how much of it can be avoided, if less general statements are considered satisfactory.
- The theory of factorization structures of morphisms (and, more generally, of sources). Here in particular extremal factorizations are a useful additional tool, necessary to prove the existence of the various adjunctions one wants to have for Hopf algebras.

The ingredients mentioned above will have to interact appropriately. We will therefore assume that we are working over a symmetric monoidal base category $\mathbb{C} = (\mathbf{C}, - \otimes -, I)$, where $\mathbf{C}$ is locally finitely presentable[1], satisfying the following conditions:

1. The functors *tensor squaring* $\otimes^2$ and *tensor cubing* $\otimes^3$, i.e., the functors mapping a morphism $C \xrightarrow{f} D$ to $C \otimes C \xrightarrow{f \otimes f} D \otimes D$ and $C \otimes C \otimes C \xrightarrow{f \otimes f \otimes f} D \otimes D \otimes D$ respectively, preserve directed colimits.
2. Extremal epimorphisms in $\mathbf{C}$ are stable under tensor squaring, i.e., the functor tensor squaring $\otimes^2$ on $\mathbf{C}$ preserves extremal epimorphisms.

[1]We could instead use any locally λ-presentable category and ask in the first of the following conditions that the functors $\otimes^2$ and $\otimes^3$ preserve λ-directed colimits.

Note that every module category $\mathbf{Mod}_R$ satisfies these conditions.

Because of the length of this presentation it has been devided in two parts. Part I is devoted to the general theory of Hopf monoids, while Part II contains the specialization to Hopf algebras over a commutative ring as well as a couple of additional results made possible by our approach.

This paper (Part I) is organized as follows.

Section 1 is devoted to the definitions of bi- and Hopf monoids and is essentially standard. Here we only use that the base category is symmetric monoidal. The chosen abstract approach makes it possible to reduce the definition of bimonoid to two simple constructions, namely forming the dual $\mathbb{C}^{\mathsf{op}}$ and the category of monoids $\mathbf{Mon}\mathbb{C}$ respectively of a monoidal category $\mathbb{C}$.

Section 2 provides the expected properties of the categories of monoids, comonoids, and bimonoids. Here we require in addition that the base category is locally finitely presentable and that condition 1 above is satisfied.

Section 3 provides the main results concerning existence of the universal constructions mentioned at the beginning and of their constructions. This requires the use of all the conditions mentioned above.

The Appendix contains technical definitions and results from category theory, the reader may not be familiar with (some results on extremally monadic functors are even new) as well as some technical arguments omitted from the main text for the sake of readability.

In a sequel to this paper (Part II) we will apply the results obtained here to Hopf algebras over arbitrary (commutative unital) rings. That Part will be organized as follows.

Section 1 contains the explicit translation of the results of Section 3 to the case of Hopf algebras over a commutative ring R and makes clear in particular, which of those require the additional assumption of R being absolutely flat. Particular emphasis will be given to relations between our results and known constructions, in particular to Takeuchi's.

Section 2 presents extensions of the results of Section 3 to relevant subcategories of the category of Hopf algebras.

Section 3 will generalize the discussion of convolution monoids in Section 1.3 to the effect that we can prove the existence of the so called *finite* or *Sweedler dual* of an algebra and, more generally, of universal measuring coalgebras, to arbitrary commutative unital rings.

In Section 4 the question is raised whether the approach of this paper might also work in more general situations. We show in particular, that one hardly loses anything when generalizing from symmetric to braided monoidal categories. We close this section by suggesting a way of dealing with weak Hopf algebras.

Some remarks on the presentation

It is the intention of the author to make this presentation comprehensible for readers who are not necessarily specialists in category theory; it is assumed, however, that they are familiar with categorical language. Therefore categorical concepts beyond the standard ones (category, functor, natural transformation, left and right adjoint, limit) will be defined and more special categorical results and arguments, which are used, are included in this text. This way the reader is not forced to consult the specific categorical literature, except for those cases where a deeper understanding of these arguments is desired.

The resulting conflict between the demand of using a concept or result only after it has been introduced and, on the other side, not to interrupt the main line of thought has been resolved as follows: The necessary categorical definitions and propositions have been put into the Appendix. Thus, the reader may simply consult the appendix instead of browsing through a number of different categorical monographs or papers. The reader is warned at the beginning of a section by a note typeset in sans serif, which part of the appendix will be used therein. This should allow for a linear reading, including the necessary parts of category theory, if desired. Only in two cases a deviation from this rule has been deemed necessary: Since the concepts of *symmetric monoidal category* and of *equifer in a category of functor algebras* are of fundamental importance, these are introduced in the main text before they are used for the first time.

We are throughout considering mathematical objects, which consist of an object A of a category $\mathbf{A}$ and some additional structure. Here we often distinguish between both notationally by choosing different fonts as, e.g., in the following cases: $\mathbb{C} = (\mathbf{C}, -\otimes-, I)$ for a monoidal category, $\mathsf{M} = (M, m, e)$ for a monoid (and similarly for co- and bimonoids).

1 What is a Hopf monoid?

The reader not familiar with the basics of the theory of monoidal categories is advised to read Section 4.2 of the appendix before continuing.

1.1 Monoidal categories and functors

1 Definition A *monoidal category* is a triple $\mathbb{C} := (\mathbf{C}, -\otimes-, I)$, where $\mathbf{C}$ is a category, $-\otimes-\colon \mathbf{C}\times\mathbf{C} \to \mathbf{C}$ is a bifunctor and I is a $\mathbf{C}$-object, equipped with natural isomorphisms (called *constraints*)

1. $a_{ABC}\colon (A\otimes B)\otimes C \to A\otimes(B\otimes C)$, for each triple of $\mathbf{C}$-objects (A, B, C),
2. $r_C\colon C\otimes I \to C$ for each $\mathbf{C}$-object C,
3. $l_C\colon I\otimes C \to C$ for each $\mathbf{C}$-object C,

satisfying certain so-called *coherence conditions* (see Section 4.2 in the Appendix for a definition of these).

$\mathbb{C}$ is called *symmetric*, if there exist natural isomorphisms $s_{BC}\colon B\otimes C \to C\otimes B$ with $s_{BC}\circ s_{CB} = \mathsf{id}_{C\otimes B}$ for each pair of $\mathbf{C}$-objects (B, C), again subject to a certain coherence condition.

In the sequel, if not explicitly stated otherwise, C will always denote a symmetric monoidal category.

A *monoidal functor* $\mathbb{C} \to \mathbb{C}'$ is a triple (F, Φ, ϕ), where $\mathbf{C} \xrightarrow{F} \mathbf{C}'$ is a functor, $(FC\otimes FD) \xrightarrow{\Phi_{C,D}} F(C\otimes D)_{C,D}$ is a natural transformation and $I' \xrightarrow{\phi} FI$ is a $\mathbf{C}'$-morphism, again subject to certain coherence conditions (see Section 4.2 in the Appendix for details, again).

Such a functor is called *strong monoidal* or *strict monoidal*, if Φ and ϕ are (natural) isomorphisms and identities respectively.

2 Remark If $\mathbb{C} := (\mathbf{C}, -\otimes-, I)$ is a symmetric monoidal category with constraints a, l, r and symmetry s, then $(\mathbf{C}^{\mathsf{op}}, -\otimes-, I)$ is a symmetric monoidal category again, with constraints $\bar{a}, \bar{l}, \bar{r}$ and symmetry $\bar{s}$, where $\bar{a}_{ABC} = a^{-1}_{ABC}$ etc. This monoidal category will be denoted by $\mathbb{C}^{\mathsf{op}}$.

If $\mathbb{C}$ and $\mathbb{D}$ are symmetric monoidal categories, then their product is a symmetric monoidal category again, where the monoidal structure is defined component wise, i.e., one has $(C, D)\otimes(C', D') = (C\otimes C', D\otimes D')$. This monoidal category will be denoted by $\mathbb{C}\times\mathbb{D}$.

If $F\colon \mathbb{C} \to \mathbb{C}'$ is strong monoidal, so is $F^{\mathsf{op}}\colon \mathbb{C}^{\mathsf{op}} \to \mathbb{C}'^{\mathsf{op}}$.

3 Example 1. The most important symmetric monoidal categories in the sequel are the categories of modules $\mathbf{Mod}_R$ for a commutative unital ring R, equipped with their usual tensor product.

2. Every category $\mathbf{A}$ with finite products is a symmetric monoidal category with binary product $- \times -$ as tensor product and terminal object 1 serving as specified object I. We call this monoidal structure the *cartesian monoidal structure*. The category $\mathbf{Set}$ of sets with cartesian product as tensor product is the simplest example of a cartesian category.

3. A simple example of a monoidal functor is the forgetful functor $|-|\colon \mathbf{Mod}_R \to \mathbf{Set}$ where, for R-modules M and N, $|M| \times |N| \xrightarrow{\Phi_{M,N}} |M \otimes N|$ is the universal bilinear map with $(m, n) \mapsto m \otimes n$ and $1 \xrightarrow{\phi} |R|$ is the map 'picking out' the unit $1 \in |R|$.

4. Let $\mathbb{C}$ be a symmetric monoidal category.

   (a) Denote by

   $$\hom_{\mathbb{C}}(C, D) \times \hom_{\mathbb{C}}(C', D') \xrightarrow{\Psi_{(C,D),(C',D')}} \hom_{\mathbb{C}}(C \otimes C', D \otimes D')$$

   the map with $(f, g) \mapsto f \otimes g$ and with $1 \xrightarrow{\psi} \hom_{\mathbb{C}}(I, I)$ the map 'picking out' the identity id_I. Then $(\hom_{\mathbb{C}}(-,-), \Psi, \psi)$ is a monoidal functor $\mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbf{Set}$.

   (b) The natural isomorphisms

   $$(C \otimes D) \otimes (C' \otimes D') \xrightarrow{C \otimes s_{D,C'} \otimes D'} (C \otimes C') \otimes (D \otimes D')$$

   and the isomorphism $l_I^{-1} = r_I^{-1}\colon I \to I \otimes I$ make the bifunctor $- \otimes -\colon \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ a strong monoidal functor $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$.

Given a symmetric monoidal category $\mathbb{C} := (\mathbf{C}, - \otimes -, I)$ we denote, for any object C, by $C \otimes -\colon \mathbf{C} \to \mathbf{C}$ the functor with $(B \xrightarrow{f} B') \mapsto (C \otimes B \xrightarrow{\mathsf{id}_C \otimes f} C \otimes B')$. The morphism $C \otimes B \xrightarrow{\mathsf{id}_C \otimes f} C \otimes B'$ may also be denoted by $C \otimes B \xrightarrow{C \otimes f} C \otimes B'$. Analogously we have the functors $- \otimes C$.

We also have, for each $n \in \mathbb{N}$, a functor $\otimes^n\colon \mathbf{C} \to \mathbf{C}$, generalizing the construction for $n = 2$, namely $(C \xrightarrow{f} C) \mapsto (C \otimes C \xrightarrow{f \otimes f} C \otimes C)$, in the obvious way.

By a slight abuse of language we say that the monoidal category $\mathbb{C}$ has a certain categorical property P, when the category $\mathbf{C}$ has property P.

1.2 Monoids and comonoids

Monoids

4 Definition Let $\mathbb{C}$ be a monoidal category. A *monoid* in $\mathbb{C}$ is a triple $\mathsf{M} = (M, M \otimes M \xrightarrow{m} M, I \xrightarrow{e} M)$ with M a $\mathbf{C}$-object and m, e $\mathbf{C}$-morphisms, such that the following diagrams commute, which express associativity and unit-axiom respectively:

$$\begin{tikzcd}
M \otimes M \otimes M \arrow[r, "m \otimes M"] \arrow[d, "M \otimes m"'] & M \otimes M \arrow[d, "m"] \\
M \otimes M \arrow[r, "m"'] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
M \otimes I \arrow[r, "M \otimes e"] \arrow[dr, Rightarrow, "r_M"'] & M \otimes M \arrow[d, "m"] & I \otimes M \arrow[l, "e \otimes M"'] \arrow[dl, Rightarrow, "l_M"] \\
 & M &
\end{tikzcd}$$

A *monoid homomorphism* $f\colon (M,m,e) \longrightarrow (M',m',e')$ is a $\mathbf{C}$-morphism $f\colon M \to M'$ making the following diagrams commute:

$$\begin{tikzcd} M\otimes M \arrow[r, "m"] \arrow[d, "f\otimes f"'] & M \arrow[d, "f"] \\ M'\otimes M' \arrow[r, "m'"'] & M' \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "e"] \arrow[dr, "e'"'] & M \arrow[d, "f"] \\ & M' \end{tikzcd}$$

$\mathbf{Mon}\mathbb{C}$ denotes the category of monoids in $\mathbb{C}$, where identities and composition are as in $\mathbf{C}$. The forgetful functor $\mathbf{Mon}\mathbb{C} \to \mathbf{C}$ will be denoted by U_a[2].

A monoid M' is called a *submonoid* of a monoid M, provided there exists a monoid homomorphism $i\colon \mathsf{M}' \to \mathsf{M}$ such that $i\colon M' \to M$ is a monomorphism in $\mathbf{C}$.

5 Fact $\mathbf{Mon}(\mathbb{C}\times\mathbb{D}) = \mathbf{Mon}\mathbb{C} \times \mathbf{Mon}\mathbb{D}$.

The following result, which is easy to prove, will be of frequent use.

6 Proposition *Let $F\colon \mathbb{C} \to \mathbb{C}'$ be a monoidal functor. Then*

$$F^\sharp(M,m,e) = (FM, FM\otimes FM \xrightarrow{\Phi_{M,M}} F(M\otimes M) \xrightarrow{Fm} FM, I' \xrightarrow{\phi} FI \xrightarrow{Fe} FM) \text{ and } F^\sharp f = Ff$$

defines a functor $F^\sharp\colon \mathbf{Mon}\mathbb{C} \to \mathbf{Mon}\mathbb{C}'$, called the monoidal lift of F*, such that the following diagram commutes*

$$\begin{tikzcd} \mathbf{Mon}\mathbb{C} \arrow[r, "F^\sharp"] \arrow[d, "U_a"'] & \mathbf{Mon}\mathbb{C}' \arrow[d, "U_a'"] \\ \mathbf{C} \arrow[r, "F"'] & \mathbf{C}' \end{tikzcd}$$

7 Lemma *Given a symmetric monoidal category $\mathbb{C}$, the symmetry s of $\mathbb{C}$ induces a functorial isomorphism*

$$\begin{array}{rccc} (-)^{\mathsf{op}}\colon & \mathbf{Mon}\mathbb{C} & \longrightarrow & \mathbf{Mon}\mathbb{C} \\ & (M,m,e) & \longmapsto & (M, m\circ s_{MM}, e) \end{array}.$$

8 Definition A monoid M is called *commutative*, if $\mathsf{M} = \mathsf{M}^{\mathsf{op}}$. ${}_c\mathbf{Mon}\mathbb{C}$ denotes the category of commutative monoids.

9 Examples

1. $\mathsf{I} := (I, I\otimes I \xrightarrow{r_I} I, I \xrightarrow{\mathsf{id}_I} I)$ is a (commutative) monoid in $\mathbb{C}$. For every monoid $\mathsf{M} = (M,m,e)$ in $\mathbf{C}$ the morphism e is a monoid homomorphism $e\colon \mathsf{I} \to \mathsf{M}$.

2. (Commutative) monoids in the cartesian monoidal category $\mathbf{Set}$ of sets and mappings are the ordinary (commutative) monoids. Thus, $\mathbf{Mon}(\mathbf{Set}) = \mathbf{Mon}$, the category of (ordinary monoids).

3. (Commutative) monoids in the monoidal category $\mathbf{Mod}_R$ are the (commutative) R-algebras. In particular, for the categories $\mathbf{Alg}_R$ and ${}_c\mathbf{Alg}_R$ of those one has $\mathbf{Alg}_R = \mathbf{Mon}(\mathbf{Mod}_R)$ and ${}_c\mathbf{Alg}_R = {}_c\mathbf{Mon}(\mathbf{Mod}_R)$ respectively.

[2]If the dependence of the base category needs to be stressed we may also write ${}_{\mathbb{C}}U_a$:

4. Monoidal lifts of strict monoidal functors map commutative monoids to commutative monoids, i.e., the functor $F^\sharp$ can be restricted to a functor ${}_cF^\sharp\colon {}_c\mathbf{Mon}\mathbb{C} \to {}_c\mathbf{Mon}\mathbb{C}'$.

The following is elementary (for an elegant prove of 3. use the Eckmann-Hilton argument — see Remark 31).

10 Fact 1. The functor $U_a\colon \mathbf{Mon}\mathbb{C} \to \mathbb{C}$ creates limits.

2. U_a creates colimits of those diagrams D, for which the functors $\otimes^2$ and $\otimes^3$ preserve colimits of U_aD.

3. ${}_c\mathbf{Mon}\mathbb{C}$ is closed in $\mathbf{Mon}\mathbb{C}$ under limits.

Applying the functor $\mathbf{Mon}\mathbb{C} \times \mathbf{Mon}\mathbb{C} \to \mathbf{Mon}\mathbb{C}$ induced by the strong monoidal functor of Example 3 (4.b) above one gets the so-called *tensor product* of monoids as follows.

11 Proposition *Given monoids (M_1, m_1, e_1) and (M_2, m_2, e_2) in $\mathbb{C}$, the triple*

$$(M_1, m_1, e_1) \otimes (M_2, m_2, e_2) \colon = (M_1 \otimes M_2, m, e)$$

with $m = (M_1 \otimes M_2) \otimes (M_1 \otimes M_2) \xrightarrow{M_1 \otimes s \otimes M_2} (M_1 \otimes M_1) \otimes (M_2 \otimes M_2) \xrightarrow{m_1 \otimes m_2} M_1 \otimes M_2$ and $e = I \simeq I \otimes I \xrightarrow{e_1 \otimes e_2} M_1 \otimes M_2$ is a monoid.

This construction is functorial.

12 Remark The following is a coproduct in ${}_c\mathbf{Mon}\mathbb{C}$

$$(M_1, m_1, e_1) \xrightarrow{\iota_1} (M_1, m_1, e_1) \otimes (M_2, m_2, e_2) \xleftarrow{\iota_2} (M_2, m_2, e_2)$$

where $\iota_1 = M_1 \simeq M_1 \otimes I \xrightarrow{M_1 \otimes e_2} M_1 \otimes M_2$ and $\iota_2 = M_2 \simeq I \otimes M_2 \xrightarrow{e_1 \otimes M_2} M_1 \otimes M_2$.

The following well known theorem (see e.g. [11] for a more general result) is fundamental for our approach. It can be proved in an essentially straightforward though somewhat lengthy way. One only needs to check that all the constraints lift to monoid homomorphisms. See Section 4.2 in the appendix for some details.

13 Theorem *By means of the tensor product of monoids defined above* $\mathbf{Mon}\mathbb{C}$ *becomes a symmetric monoidal category with specified object $(\mathsf{I}, r_I, \mathsf{id}_I)$ and constraints as in $\mathbb{C}$. The forgetful functor U_a is a strict monoidal functor* $\mathbf{Mon}\mathbb{C} \to \mathbb{C}$.

Comonoids

14 Definition Let $\mathbb{C}$ be a monoidal category. A *comonoid* in $\mathbb{C}$ is a triple $\mathsf{C} = (C, C \xrightarrow{\mu} C \otimes C,\ C \xrightarrow{\epsilon} I)$ with C a $\mathbf{C}$-object and μ, ϵ $\mathbf{C}$-morphisms such that the following diagrams commute:

$$\begin{tikzcd} C \arrow[r, "\mu"] \arrow[d, "\mu"'] & C \otimes C \arrow[d, "C \otimes \mu"] \\ C \otimes C \arrow[r, "\mu \otimes C"'] & C \otimes C \otimes C \end{tikzcd} \qquad \begin{tikzcd} C \otimes I & C \otimes C \arrow[l, "C \otimes \epsilon"'] \arrow[r, "\epsilon \otimes C"] & I \otimes C \\ & C \arrow[ul, Rightarrow, "r_C^{-1}"] \arrow[u, "\mu"] \arrow[ur, Rightarrow, "l_C^{-1}"'] & \end{tikzcd}$$

A *comonoid homomorphism* $f\colon (C, \mu, \epsilon) \longrightarrow (C', \mu', \epsilon')$ is a $\mathbf{C}$-morphism $f\colon C \to C'$ making the following diagrams commute:

$$\begin{tikzcd} C \arrow[r, "\mu"] \arrow[d, "f"'] & C \otimes C \arrow[d, "f \otimes f"] \\ C' \arrow[r, "\mu'"'] & C' \otimes C' \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\epsilon"] \arrow[d, "f"] & I \\ C' \arrow[ur, "\epsilon'"'] & \end{tikzcd}$$

$\mathbf{Comon}\mathbb{C}$ denotes the category of comonoids in $\mathbb{C}$, where identities and composition are as in $\mathbf{C}$. The forgetful functor $\mathbf{Comon}\mathbb{C} \to \mathbf{C}$ will be denoted by U_c (or ${}_{\mathbb{C}}U_c$, if necessary).

A comonoid C' is called a *sub-comonoid* of a comonoid C, provided there exists a comonoid homomorphism $i\colon \mathsf{C}' \to \mathsf{C}$ such that $i\colon C' \to C$ is a monomorphism in $\mathbf{C}$.

Comonoids in $\mathbf{Mod}_R$ are called *R-coalgebras*; their category will be denoted by $\mathbf{Coalg}_R$.

15 Remark If $\mathbb{C}$ is a cartesian monoidal category $\mathbf{C}$, then the category $\mathbf{Comon}(\mathbf{C}, -\times-, 1)$ is isomorphic to $\mathbf{C}$. Indeed, for any object C the triple $(C, C \xrightarrow{\Delta} C \times C, C \xrightarrow{!} 1)$ with Δ the diagonal and ! the only morphism into the terminal object is the only comonoid on C and every $\mathbf{C}$-morphism respects these comonoid structures.

16 Fact The following categories and functors coincide:

$$\mathbf{Comon}(\mathbb{C}^{\mathsf{op}}) \xrightarrow{{}_{\mathbb{C}^{\mathsf{op}}}U_c} \mathbf{C}^{\mathsf{op}} = (\mathbf{Mon}\mathbb{C})^{\mathsf{op}} \xrightarrow{({}_{\mathbb{C}^{\mathsf{op}}}U_a)^{\mathsf{op}}} \mathbf{C}^{\mathsf{op}}$$

This simple observation will make it possible to obtain quite a number of results on co-, bi- and Hopf monoids by categorical dualization.

By duality every strong monoidal functor $\mathbb{C} \xrightarrow{F} \mathbb{C}'$[3] induces a functor $\mathbf{Comon}\mathbb{C} \xrightarrow{F^\star} \mathbf{Comon}\mathbb{C}'$, which we also call a monoidal lift of F. That is, we have

17 Proposition *Let $F\colon \mathbb{C} \to \mathbb{C}'$ be a strong monoidal functor. Then*

$$F^*(C, \mu, \epsilon) = (FC, FC \xrightarrow{F\mu} F(C \otimes C) \xrightarrow{\Phi^{-1}_{C,C}} FC \otimes FC, FC \xrightarrow{F\epsilon} FI \xrightarrow{\phi^{-1}} I') \text{ and } F^*f = Ff$$

defines a functor $F^\star\colon \mathbf{Comon}\mathbb{C} \to \mathbf{Comon}\mathbb{C}'$, called the monoidal lift of F*, such that the following diagram commutes*

$$\begin{tikzcd} \mathbf{Comon}\mathbb{C} \arrow[r, "F^*"] \arrow[d, "U_c"'] & \mathbf{Comon}\mathbb{C}' \arrow[d, "U_c'"] \\ \mathbf{C} \arrow[r, "F"'] & \mathbf{C}' \end{tikzcd}$$

By duality we get as well

18 Lemma *Given a symmetric monoidal category $\mathbb{C}$, the symmetry s of $\mathbb{C}$ induces a functorial isomorphism*

$$\begin{array}{rccc} (-)^{\mathsf{cop}}\colon & \mathbf{Comon}\mathbb{C} & \longrightarrow & \mathbf{Comon}\mathbb{C} \\ & (C, \mu, \epsilon) & \longmapsto & (C, s_{CC} \circ \mu, \epsilon) \end{array}.$$

[3]Asking F to be a monoidal functor only does not suffice, since Φ and ϕ do not change their directions. Only in the strong monoidal case one can use their inverses (For the reader more inclined to category theory: The appropriate condition of F is to be an *opmonoidal functor*, which is the dual of a monoidal one; and strong monoidal functors are opmonoidal – see also Section 4.2 in the appendix).

19 Definition A comonoid C is called *cocommutative*, if $\mathsf{C} = \mathsf{C}^{\mathsf{cop}}$. ${}_{coc}\mathbf{Comon}\mathbb{C}$ denotes the category of commutative monoids.

20 Remark Since, obviously,

$$ {}_{coc}\mathbf{Comon}(\mathbb{C}^{\mathsf{op}}) = ({}_{c}\mathbf{Mon}\mathbb{C})^{\mathsf{op}} $$

the dual of Fact 10 above holds.

The monoidal properties of $\mathbf{Comon}\mathbb{C}$ follow by dualization from those of $\mathbf{Mon}\mathbb{C}$, too. In detail:

21 Theorem *Let $\mathbb{C}$ be a symmetric monoidal category.*

1. *Given comonoids (C_1, μ_1, ϵ_1) and (C_2, μ_2, ϵ_2) in $\mathbb{C}$, the triple*
$$ (C_1, \mu_1, \epsilon_1) \otimes (C_2, \mu_2, \epsilon_2) \colon = (C_1 \otimes C_2, \mu, \epsilon) $$
with
$\mu = C_1 \otimes C_2 \xrightarrow{\mu_1 \otimes \mu_2} (C_1 \otimes C_1) \otimes (C_2 \otimes C_2) \xrightarrow{C_1 \otimes s \otimes C_2} (C_1 \otimes C_2) \otimes (C_1 \otimes C_2)$ *and*
$\epsilon = C_1 \otimes C_2 \xrightarrow{\epsilon_1 \otimes \epsilon_2} I \otimes I \simeq I$
is a comonoid, called the tensor product *of (C_1, μ_1, ϵ_1) and (C_2, μ_2, ϵ_2).*

2. $\mathsf{I} := (I, I \xrightarrow{r_I^{-1}} I \otimes I, I \xrightarrow{\mathsf{id}_I} I)$ *is a comonoid.*

3. *By means of this tensor product $\mathbf{Comon}\mathbb{C}$ becomes a symmetric monoidal category with specified object $(I, r_I^{-1}, \mathsf{id}_I)$ such that the forgetful functor U_c is a strict monoidal functor $\mathbf{Comon}\mathbb{C} \to \mathbb{C}$.*

4. *This tensor product is the categorical product in ${}_{coc}\mathbf{Comon}\mathbb{C}$.*

1.3 Convolution monoids

22 Proposition *Let $\mathsf{C} = (C, \mu, \epsilon)$ be a comonoid and $\mathsf{M} = (M, m, e)$ a monoid in $\mathbb{C}$. Then the hom-set $\hom_{\mathbf{C}}(C, M)$ becomes an (ordinary) monoid $\Phi_{\mathbb{C}}(\mathsf{C}, \mathsf{M})$ — called* convolution monoid of (C, M) *— as follows.*

- *Given $f, g \colon C \longrightarrow M$, define their product (called convolution product)*
$f * g = C \xrightarrow{\mu} C \otimes C \xrightarrow{f \otimes g} M \otimes M \xrightarrow{m} M$,
- *chose as unit $C \xrightarrow{\epsilon} I \xrightarrow{e} M$.*

Moreover, this construction is functorial.

Proof By Proposition 6 the monoidal functor $(\hom_{\mathbb{C}}(-,-), \Psi, \psi) \colon \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbf{Set}$ of Example 3 induces a functor $\Phi_{\mathbb{C}} \colon \mathbf{Mon}(\mathbb{C}^{\mathsf{op}} \times \mathbb{C}) \to \mathbf{Mon}(\mathbf{Set})$[4]. Now
$\mathbf{Mon}(\mathbb{C}^{\mathsf{op}} \times \mathbb{C}) = \mathbf{Mon}(\mathbb{C}^{\mathsf{op}}) \times \mathbf{Mon}\mathbb{C} = (\mathbf{Comon}\mathbb{C})^{\mathsf{op}} \times \mathbf{Mon}\mathbb{C}$ and $\mathbf{Mon}(\mathbf{Set}) = \mathbf{Mon}$.
By construction of the induced functor Φ

1. the multiplication $- * -$ of the monoid $\Phi_{\mathbb{C}}(\mathsf{C}, \mathsf{M})$ is
$\hom_{\mathbf{C}}(C, M) \times \hom_{\mathbf{C}}(C, M) \xrightarrow{\Psi} \hom_{\mathbf{C}}(C \otimes C, M \otimes M) \xrightarrow{\hom_{\mathbf{C}}(\mu, m)} \hom_{\mathbf{C}}(C, M)$,
hence $f * g = C \xrightarrow{\mu} C \otimes C \xrightarrow{f \otimes g} M \otimes M \xrightarrow{m} M$, for each pair of morphisms $f, g \colon C \longrightarrow M$, by the definition of Ψ.

[4] We will simply write Φ instead of $\Phi_{\mathbb{C}}$ if stressing the base category is not necessary.

2. the unit of the monoid $\Phi_{\mathbb{C}}(\mathsf{C},\mathsf{M})$ is $1 \xrightarrow{\psi} \hom_{\mathbf{C}}(I,I) \xrightarrow{\hom_{\mathbf{C}}(\epsilon,e)} \hom_{\mathbf{C}}(C,M)$, that is $C \xrightarrow{\epsilon} I \xrightarrow{e} M$, by the definition of ψ. $\square$

See Part II for an extension of this construction.

1.4 Bimonoids

The category Bimon$\mathbb{C}$

23 Lemma *Let (C,m,e,μ,ϵ) be a quintuple, where (C,m,e) is a monoid and (C,μ,ϵ) is a comonoid.*

Then the following statements are equivalent:

1. (a) *$\mu\colon (C,m,e) \to (C,m,e) \otimes (C,m,e)$ is a monoid homomorphism.*
   (b) *$\epsilon\colon (C,m,e) \to (I,r_i,\mathsf{id}_I)$ is a monoid homomorphism.*
2. (a) *$m\colon (C,\mu,\epsilon) \otimes (C,\mu,\epsilon) \to (C,\mu,\epsilon)$ is a comonoid homomorphism.*
   (b) *$e\colon (I,r_I^{-1},\mathsf{id}_I) \to (C,\mu,\epsilon)$ is a comonoid homomorphism.*

Proof 1. (a) holds iff the following diagrams commute:

$$\begin{tikzcd} C\otimes C \arrow[rr, "m"] \arrow[d, "\mu\otimes\mu"'] & & C \arrow[d, "\mu"] \\ \otimes^4 C \arrow[r, "C\otimes s\otimes C"'] & \otimes^4 C \arrow[r, "m\otimes m"'] & C\otimes C \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "e"] \arrow[d, "r_I^{-1}"'] & C \arrow[d, "\mu"] \\ I\otimes I \arrow[r, "e\otimes e"'] & C\otimes C \end{tikzcd}$$

1. (b) holds iff the following diagrams commute:

$$\begin{tikzcd} C\otimes C \arrow[r, "\epsilon\otimes\epsilon"] \arrow[d, "m"'] & I\otimes I \arrow[d, "r_I"] \\ C \arrow[r, "\epsilon"'] & I \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "e"] \arrow[dr, "\mathsf{id}_I"'] & C \arrow[d, "\epsilon"] \\ & I \end{tikzcd}$$

2. (a) holds iff the first and the third diagram commute, while 2. (b) holds iff the second and the fourth diagram commute. $\square$

This observation leads to the following definition.

24 Definition A quintuple (C,m,e,μ,ϵ) is called a *bimonoid in* $\mathbb{C}$, provided that

1. (C,m,e) is a monoid in $\mathbb{C}$,
2. (C,μ,ϵ) is a comonoid in $\mathbb{C}$,
3. the following equivalent conditions are satisfied
   (a) m and e are comonoid homomorphisms,
   (b) μ and ϵ are monoid homomorphisms.

A *bimonoid homomorphism* $(C,m,e,\mu,\epsilon) \to (C',m',e',\mu',\epsilon')$ is a $\mathbb{C}$-morphism, which is both, a monoid homomorphism $(C,m,e) \to (C',m',e')$ and a comonoid homomorphism $(C,\mu,\epsilon) \to (C',\mu',\epsilon')$.

This defines the category **Bimon**$\mathbb{C}$ as well as the forgetful functors[5]

[5] Mostly we will suppress the index $\mathbb{C}$ in the symbols ${}_{\mathbb{C}}(-)^a$ and ${}_{\mathbb{C}}(-)^c$.

1. ${}_{\mathbb{C}}(-)^a\colon \mathbf{Bimon}\mathbb{C} \to \mathbf{Mon}\mathbb{C}$ with $\mathsf{C} = (C, m, e, \mu, \epsilon) \mapsto \mathsf{C}^a := (C, m, e)$

2. ${}_{\mathbb{C}}(-)^c\colon \mathbf{Bimon}\mathbb{C} \to \mathbf{Comon}\mathbb{C}$ with $\mathsf{C} = (C, m, e, \mu, \epsilon) \mapsto \mathsf{C}^c := (C, \mu, \epsilon)$

A bimonoid $\mathsf{C} = (C, m, e, \mu, \epsilon)$ then often is denoted as $(\mathsf{C}^a, \mathsf{C}^c)$.

A bimonoid $\mathsf{B} = (\mathsf{B}^a, \mathsf{B}^c)$ is called *commutative*, iff B^a is commutative, and *cocommutative*, iff B^c is cocommutative. By ${}_c\mathbf{Bimon}\mathbb{C}$, ${}_{coc}\mathbf{Bimon}\mathbb{C}$ und ${}_{c,coc}\mathbf{Bimon}\mathbb{C}$ we denote the categories of commutative, cocommutative and commutative and cocommutative bimonoids respectively.

A bimonoid B' is called a *sub-bimonoid* of a bimonoid B, provided there exists a bimonoid homomorphism $i\colon \mathsf{B}' \to \mathsf{B}$ such that $i\colon B' \to B$ is a monomorphism in $\mathbf{C}$, i.e., that i is both, a submonoid embedding $(\mathsf{B}')^a \hookrightarrow \mathsf{B}^a$ and a sub-comonoid embedding $(\mathsf{B}')^c \hookrightarrow \mathsf{B}^c$.

Bimonoids in $\mathbf{Mod}_R$ are called *R-bialgebras*; their category will be denoted by $\mathbf{Bialg}_R$.

Since both categories, $\mathbf{Mon}\mathbb{C}$ and $\mathbf{Comon}\mathbb{C}$, are symmetric monoidal categories by Propositions 13 and 21, $\mathbf{Mon}(\mathbf{Comon}\mathbb{C})$ and $\mathbf{Comon}(\mathbf{Mon}\mathbb{C})$ are defined. The observation of Lemma 23 therefore can be rephrased as follows:

25 Proposition *For any symmetric monoidal category $\mathbb{C}$ the categories $\mathbf{Mon}(\mathbf{Comon}\mathbb{C})$, $\mathbf{Comon}(\mathbf{Mon}\mathbb{C})$ and $\mathbf{Bimon}\mathbb{C}$ are isomorphic.*

We will identify these categories in what follows. The forgetful functors occurring are depicted as follows:

$$\begin{tikzcd} & \mathbf{Bimon}\mathbb{C} \arrow[dl, "{}_{\mathbb{C}}(-)^c"'] \arrow[dr, "{}_{\mathbb{C}}(-)^a"] & \\ \mathbf{Comon}\mathbb{C} \arrow[dr, "{}_{\mathbb{C}}U_c"'] & & \mathbf{Mon}\mathbb{C} \arrow[dl, "{}_{\mathbb{C}}U_a"] \\ & \mathbb{C} & \end{tikzcd}$$

26 Remark For every bimonoid $\mathsf{B} = (B, m, e, \mu, \epsilon) = (\mathsf{B}^a, \mathsf{B}^c)$ also $\mathsf{B}^{\mathsf{op}} := ((\mathsf{B}^a)^{\mathsf{op}}, \mathsf{B}^c)$, $\mathsf{B}^{\mathsf{cop}} := (\mathsf{B}^a, (\mathsf{B}^c)^{\mathsf{cop}})$ and $\mathsf{B}^{\mathsf{op,cop}} := ((\mathsf{B}^a)^{\mathsf{op}}, (\mathsf{B}^c)^{\mathsf{cop}})$ are bimonoids. This yields functorial isomorphisms $(-)^{\mathsf{op}}, (-)^{\mathsf{cop}}, (-)^{\mathsf{op,cop}}\colon \mathbf{Bimon}\mathbb{C} \to \mathbf{Bimon}\mathbb{C}$,

27 Remark 1. $\mathbf{Bimon}(\mathbb{C}^{\mathsf{op}}) = (\mathbf{Bimon}\mathbb{C})^{\mathsf{op}}$ and the following diagrams coincide:

$$\begin{tikzcd} & (\mathbf{Bimon}\mathbb{C})^{\mathsf{op}} \arrow[dl, "\left({}_{\mathbb{C}}(-)^a\right)^{\mathsf{op}}"'] \arrow[dr, "\left({}_{\mathbb{C}}(-)^c\right)^{\mathsf{op}}"] & \\ (\mathbf{Mon}\mathbb{C})^{\mathsf{op}} \arrow[dr, "\left({}_{\mathbb{C}}U_a\right)^{\mathsf{op}}"'] & & (\mathbf{Comon}\mathbb{C})^{\mathsf{op}} \arrow[dl, "\left({}_{\mathbb{C}}U_c\right)^{\mathsf{op}}"] \\ & \mathbb{C}^{\mathsf{op}} & \end{tikzcd} \qquad \begin{tikzcd} & \mathbf{Bimon}(\mathbb{C}^{\mathsf{op}}) \arrow[dl, "{}_{\mathbb{C}^{\mathsf{op}}}(-)^c"'] \arrow[dr, "{}_{\mathbb{C}^{\mathsf{op}}}(-)^a"] & \\ \mathbf{Comon}(\mathbb{C}^{\mathsf{op}}) \arrow[dr, "{}_{\mathbb{C}^{\mathsf{op}}}U_c"'] & & \mathbf{Mon}(\mathbb{C}^{\mathsf{op}}) \arrow[dl, "{}_{\mathbb{C}^{\mathsf{op}}}U_a"] \\ & \mathbb{C}^{\mathsf{op}} & \end{tikzcd}$$

2. (a) The functor ${}_{\mathbb{C}}(-)^a\colon \mathbf{Bimon}\mathbb{C} \to \mathbf{Mon}\mathbb{C}$ is the monoidal lift of ${}_{\mathbb{C}}U_c$ in the sense of Proposition 6.

   (b) The functor ${}_{\mathbb{C}}(-)^c\colon \mathbf{Bimon}\mathbb{C} \to \mathbf{Comon}\mathbb{C}$ is the monoidal lift of ${}_{\mathbb{C}}U_a$ in the sense of Proposition 17.

The following fact, which is easy to prove, will be used occasionally.

28 Lemma *Assume that for every embedding i of a subcomonoid in* $\mathbf{Comon}\mathbb{C}$ *the morphism $i \otimes i$ is a monomorphism in* $\mathbf{C}$. *Then the following holds for every bimonoid* B. *Given a quintuple (C, m, e, μ, ϵ) and a* $\mathbf{C}$*-morphism $i \colon C \to B$, where (C, m, e) is a monoid, (C, μ, ϵ) is a comonoid and $(C, m, e) \xrightarrow{i} \mathsf{B}^a$ as well as $(C, \mu, \epsilon) \xrightarrow{i} \mathsf{B}^c$ are (co)monoid embeddings. Then (C, m, e, μ, ϵ) is a sub-bimonoid of* B.

The monoidal structure of $\mathbf{Bimon}\mathbb{C}$

Since $\mathbf{Bimon}\mathbb{C}$ equals $\mathbf{Comon}(\mathbf{Mon}\mathbb{C})$, it inherits a monoidal structure from $\mathbf{Mon}\mathbb{C}$ by Theorem 21. Since $\mathbf{Bimon}\mathbb{C}$ also equals $\mathbf{Mon}(\mathbf{Comon}\mathbb{C})$, it inherits one from $\mathbf{Comon}\mathbb{C}$, too, by Theorem 13. By simple inspection one observes that these monoidal structures coincide and, thus, the following holds.

29 Proposition *With the tensor product given by $(\mathsf{B}^a, \mathsf{B}^c) \otimes (\mathsf{C}^a, \mathsf{C}^c) = (\mathsf{B}^a \otimes \mathsf{C}^a, \mathsf{B}^c \otimes \mathsf{C}^c)$* $\mathbf{Bimon}\mathbb{C}$ *is a symmetric monoidal category such that the functors $(-)^a$ and $(-)^c$ are strict monoidal.*

The Eckmann-Hilton argument

As we could form the categories $\mathbf{Mon}(\mathbf{Comon}\mathbb{C})$ and $\mathbf{Comon}(\mathbf{Mon}\mathbb{C})$ above, one might ask what the other possible constructions of this kind yield. The answer is surprisingly simple:

30 Proposition *Let $\mathbb{C}$ be a symmetric monoidal category. Then*

1. $\mathbf{Mon}(\mathbf{Mon}\mathbb{C}) = {}_c\mathbf{Mon}\mathbb{C}$.
2. $\mathbf{Comon}(\mathbf{Comon}\mathbb{C}) = {}_{coc}\mathbf{Comon}\mathbb{C}$.

The origin of this result is the observation by Eckmann and Hilton, that statement 1 holds for ordinary monoids, i.e., in case $\mathbb{C}$ is the cartesian category of sets. A proof of the general result (see also [4]) is put into the Appendix. 2. follows from 1. by duality.

A first simple application of this observation is:

31 Remark As has been shown in Remark 10 the category ${}_c\mathbf{Mon}\mathbb{C}$ is closed in $\mathbf{Mon}\mathbb{C}$ under limits, or in other words, that the embedding ${}_c\mathbf{Mon}\mathbb{C} \hookrightarrow \mathbf{Mon}\mathbb{C}$ creates limits. By the above this is equivalent to saying that the forgetful functor $\mathbf{Mon}(\mathbf{Mon}\mathbb{C}) \to \mathbf{Mon}\mathbb{C}$ creates limits; and this holds by Remark 10 (1).

1.5 Hopf monoids

Recall that, by Remark 15, a bimonoid in a cartesian monoidal category $\mathbb{C}$, thus in $\mathbf{Set}$ in particular, is nothing but an internal monoid (B, m, e) in $\mathbf{C}$ (equipped with its diagonal Δ and its unique morphism ! to the terminal object of $\mathbf{C}$). A group thus, thought of as a monoid in which every element x has an inverse $S(x)$, is a bimonoid B in $\mathbf{Set}$ such that the following diagram commutes.

$$\begin{tikzcd}
B \otimes B \arrow[rr, "S \times \mathsf{id}_B"] & & B \times B \arrow[d, "m"] \\
B \arrow[u, "\Delta"] \arrow[d, "\Delta"'] \arrow[r, "!"] & 1 \arrow[r, "e"] & B \\
B \otimes B \arrow[rr, "\mathsf{id}_B \times S"'] & & B \times B \arrow[u, "m"']
\end{tikzcd}$$

Monoids of this type in an arbitrary monoidal category $\mathbb{C}$, not just a cartesian one (where they are called *internal groups* in $\mathbf{C}$), are called Hopf monoids. Or: a Hopf monoid is a non-cartesian group. More precisely:

32 Definition A *Hopf monoid* in $\mathbb{C}$ is a $\mathbb{C}$-bimonoid $\mathsf{H} = (H, m, e, \mu, \epsilon)$ equipped with a $\mathbf{C}$-morphism $S\colon H \to H$ making the following diagram commute

$$\begin{tikzcd} H \otimes H \arrow[rr, "S\otimes \mathsf{id}_H"] & & H \otimes H \arrow[d, "m"] \\ H \arrow[u, "\mu"] \arrow[d, "\mu"'] \arrow[r, "\epsilon"] & I \arrow[r, "e"] & H \\ H \otimes H \arrow[rr, "\mathsf{id}_H\otimes S"'] & & H \otimes H \arrow[u, "m"'] \end{tikzcd} \tag{1}$$

S is called the *antipode* of the Hopf monoid, which will be denoted by (H, S) or $(\mathsf{H}, S_{\mathsf{H}})$.

A *homomorphism of Hopf monoids* $f\colon (\mathsf{H}, S) \to (\mathsf{H}', S')$ is a bimonoid homomorphism $f\colon \mathsf{H} \to \mathsf{H}'$ satisfying $f \circ S = S' \circ f$. This defines, with composition and identities as in $\mathbf{Bimon}\mathbb{C}$, the category $\mathbf{Hopf}\mathbb{C}$ and a faithful functor $E\colon \mathbf{Hopf}\mathbb{C} \to \mathbf{Bimon}\mathbb{C}$.

Hopf monoids in $\mathbf{Mod}_R$ are called *R-Hopf algebras*. Their category will be denoted by $\mathbf{Hopf}_R$:

33 Remark Equivalently, a Hopf monoid in $\mathbb{C}$ is a $\mathbb{C}$-bimonoid H whose convolution monoid $\Phi(\mathsf{H}^c, \mathsf{H}^a)$ (see Section 1.3) has an inverse of id_H.

34 Remark Assigning the pair $(\mathsf{H}^{\mathsf{op},\mathsf{cop}}, S)$ to a Hopf monoid (H, S) defines a functorial isomorphism $(-)^{\mathsf{op},\mathsf{cop}}\colon \mathbf{Hopf}\mathbb{C} \to \mathbf{Hopf}\mathbb{C}$.

Since inverses in monoids are unique, the first of the following statements is obvious.

35 Lemma $\mathbf{Hopf}\mathbb{C}$ *is a full subcategory of* $\mathbf{Bimon}\mathbb{C}$ *with* E *as a full embedding, i.e., the following holds:*

1. *The antipode of a Hopf monoid is uniquely determined.*
2. *Given Hopf monoids* (H, S) *and* (H', S')*, any bimonoid homomorphism* $f\colon \mathsf{H} \to \mathsf{H}'$ *commutes with the antipodes, i.e.,* S *satisfies the condition* $f \circ S = S' \circ f$.

Proof For proving 2. consider $\phi := \Phi(\mathsf{id}_H, f)\colon \hom(H, H) \to \hom(H, H')$, which is a monoid homomorphism by Propostion 22. Let u, $\bar{u}$ and u' be the units in $\Phi(\mathsf{H}, \mathsf{H})$, $\Phi(\mathsf{H}, \mathsf{H}')$ and $\Phi(\mathsf{H}', \mathsf{H}')$ respectively. Then

$$\bar{u} = \phi(u) = \phi(\mathsf{id}_H * S) = \phi(\mathsf{id}_H) * \phi(S) = f * (f \circ S)$$

With $\psi := \Phi(f, \mathsf{id}_{H'})\colon \hom(H', H') \to \hom(H, H')$ applied to the unit $\bar{u} \in \Phi(\mathsf{H}', \mathsf{H}')$ one gets, analogously, $\bar{u} = \psi(u') = f * (S' \circ f)$. Using the second antipode equation one gets $\bar{u} = (f \circ S) * f$ and $\bar{u} = (f \circ S') * f$. Thus both, $f \circ S$ and $S' \circ f$ are inverse to f in $\Phi(\mathsf{H}, \mathsf{H}')$, and this proves the claim. $\square$

An important property of the antipode is the following; its rather technical proof (see also [24]) is put to the appendix.

36 Proposition *If* (H, S) *is a Hopf monoid , then its antipode is a bimonoid homomorphism* $S\colon \mathsf{H} \to \mathsf{H}^{\mathsf{op},\mathsf{cop}}$ *(equivalently* $\mathsf{H}^{\mathsf{op},\mathsf{cop}} \to \mathsf{H}$*) and, thus, a morphism* $(\mathsf{H}, S) \to (\mathsf{H}, S)^{\mathsf{op},\mathsf{cop}}$ *in* $\mathbf{Hopf}\mathbb{C}$.

37 Corollary *The antipode S of a Hopf monoid (H, S) is an epimorphism and a monomorphism in the category* $\mathbf{Hopf}\mathbb{C}$.

Proof That S is a morphism $(\mathsf{H}, S) \to (\mathsf{H}^{\mathsf{op},\mathsf{cop}}, S)$ in $\mathbf{Hopf}\mathbb{C}$ follows trivially from the proposition above. By the proof of Lemma 35, for every $\mathbf{Hopf}\mathbb{C}$-morphism $f\colon (\mathsf{H}, S) \to (\mathsf{H}', S')$ the composition $f \circ S$ is the inverse of f in the convolution monoid $\Phi(\mathsf{H}, \mathsf{H}')$. Thus, $f \circ S = g \circ S$ implies $f = g$ for all pairs of $\mathbf{Hopf}\mathbb{C}$-morphisms $f, g\colon \mathsf{H} \to \mathsf{H}'$, i.e., S is an epimorphism. That it is a monomorphism, too, follows dually. □

The next result will be of crucial importance.

38 The Crucial Lemma ([21]) *Let $\mathsf{B} := (B, m, e, \mu, \epsilon)$ be a bimonoid and $S\colon \mathsf{B}^a \to (\mathsf{B}^a)^{\mathsf{op}}$ a homomorphism of monoids.*

If $(E, \eta\colon E \to B)$ is the (multiple) equalizer of $S * \mathsf{id}$, $\mathsf{id} * S$ and $e \circ \epsilon$ in $\mathbf{C}$, then E carries a (unique) monoid structure such that η becomes the embedding of a submonoid E of B^a.

The rather technical proof is put to the Appendix. In Part II we will make the meaning of the statement above explicit for the cases $\mathbf{Mod}_R$ and $\mathbf{Mod}_R^{\mathsf{op}}$, where these become important technical lemmas in classical Hopf algebra theory which, however, require completely independent (non-trivial) proofs due to the absence of the tool of categorical dualization.

The following corollary will be used, which in case 1 of the fact above specializes to the familiar fact that the antipode equations only need to checked on a generating set of the respective algebra. (For the notion of extremal epimorphism see Section 4.3 in the appendix.)

39 Corollary *With data and notation as in the Crucial Lemma assume that there exists a free monoid $\mathsf{C}^\sharp$ over some C in $\mathbf{C}$ with universal morphism $C \xrightarrow{u} U_a\mathsf{C}^\sharp$ and an extremal epimorphism $\mathsf{C}^\sharp \xrightarrow{q} \mathsf{B}^a$ such that*

$$e \circ \epsilon \circ (U_a q \circ u) = [m \circ (\mathsf{id} \otimes S) \circ \mu] \circ (U_a q \circ u) = [m \circ (S \otimes \mathsf{id}) \circ \mu] \circ (U_a q \circ u). \qquad (2)$$

Then (B, S) is a Hopf monoid.

Proof By Equation 2 the morphism $U_a q \circ u$ factors over $U\eta$ and, thus, induces a monoid homomorphism $\mathsf{C}^\sharp \xrightarrow{g} \mathsf{E}$ with $\eta \circ g = q$. Since q is an extremal epimorphism, η is an isomorphism, as has to be shown. □

A simple calculation shows

40 Proposition *With the tensor product given by $(\mathsf{H}, S_H) \otimes (\mathsf{C}, S_C) = (\mathsf{H} \otimes \mathsf{C}, S_H \otimes S_C)$,* $\mathbf{Hopf}\mathbb{C}$ *is a symmetric monoidal category such that the embedding of* $\mathbf{Hopf}\mathbb{C}$ *into* $\mathbf{Bimon}\mathbb{C}$ *is a strict monoidal functor.*

We close this section with some examples, where **Mon** (**Grp**, **Ab**, $\mathbf{AffGrp}_R$, $\mathbf{FormGrp}_R$) denote the categories of (usual) monoids (groups, abelian groups, affine group schemes over R, formal groups over R). One has $\mathbf{AffGrp}_R = (\mathbf{Hopf}({}_c\mathbf{Alg}_R^{\mathsf{op}}))^{\mathsf{op}}$ (see e.g. [8, 24]) and $\mathbf{FormGrp}_R = {}_{coc}\mathbf{Hopf}_R$ (see e.g. [8]).

$\mathbb{C}$	$\mathbf{Mon}\mathbb{C}$	$\mathbf{Comon}\mathbb{C}$	$\mathbf{Bimon}\mathbb{C}$	$\mathbf{Hopf}\mathbb{C}$
Set	**Mon**	**Set**	**Mon**	**Grp**
Grp	**Ab**	**Grp**	**Ab**	**Ab**
Ab	**Ab**	**Ab**	**Ab**	**Ab**
${}_c\mathbf{Alg}_R^{\mathsf{op}}$	${}_c\mathbf{Bialg}_R^{\mathsf{op}}$	${}_c\mathbf{Alg}_R^{\mathsf{op}}$	${}_c\mathbf{Bialg}_R^{\mathsf{op}}$	$\mathbf{AffGrp}_R^{\mathsf{op}}$
${}_{coc}\mathbf{Coalg}_R$	${}_{coc}\mathbf{Bialg}_R$	${}_{coc}\mathbf{Coalg}_R$	${}_{coc}\mathbf{Bialg}_R$	$\mathbf{FormGrp}_R$
$\mathbf{Mod}_R$	$\mathbf{Alg}_R$	$\mathbf{Coalg}_R$	$\mathbf{Bialg}_R$	$\mathbf{Hopf}_R$
$\mathbf{Mod}_R^{\mathsf{op}}$	$\mathbf{Coalg}_R^{\mathsf{op}}$	$\mathbf{Alg}_R^{\mathsf{op}}$	$\mathbf{Bialg}_R^{\mathsf{op}}$	$\mathbf{Hopf}_R^{\mathsf{op}}$

The examples above the horizontal line are cartesian cases (recall that the tensor product in ${}_{coc}\mathbf{Coalg}_R$ is the categorical product and in ${}_{c}\mathbf{Alg}_R$ the categorical coproduct by Theorem 21 and Remark 12 respectively, that is, here $\mathbf{Hopf}\mathbb{C}$ is always the category of internal groups in $\mathbf{C}$.

2 Properties of categories of monoids, comonoids and bimonoids

2.1 Functor (co)algebras and equifiers

The following categorical notions will prove to be extremely helpful in order to obtain categorical properties of the categories defined so far.

41 Definition Let $\mathbf{A} \xrightarrow{T} \mathbf{A}$ be a functor. The category $\mathbf{Alg}T$ has as objects, called *T–algebras*, all pairs (A, α) where A is an object of $\mathbf{A}$ and $\alpha\colon TA \to A$ is a morphism. Morphisms $f\colon (A, \alpha) \to (B, \beta)$ of $\mathbf{Alg}T$, called *T–algebra homomorphisms*, are morphisms $f\colon A \to B$ in $\mathbf{A}$ such that the square

$$\begin{tikzcd} TA \arrow[r, "\alpha"] \arrow[d, "Tf"'] & A \arrow[d, "f"] \\ TB \arrow[r, "\beta"'] & B \end{tikzcd}$$

commutes. Composition and identities in $\mathbf{Alg}T$ are those of $\mathbf{C}$.

The category $\mathbf{Coalg}T$ of *T-coalgebras* is the dual of $\mathbf{Alg}T^{\mathsf{op}}$. Its objects thus are pairs (A, α) with $\alpha\colon A \to TA$, while a homomorphism $f\colon (A, \alpha) \to (B, \beta)$ of coalgebras is any $f\colon A \to B$ with $Tf \circ \alpha = \beta \circ f$.

42 Definition Let $F_\kappa, G_\kappa\colon \mathbf{A} \to \mathbf{B}_t$ $(\kappa \in K)$ be functors and $\phi_\kappa, \psi_\kappa\colon F_\kappa \Longrightarrow G_\kappa$ $(\kappa \in K)$ natural transformations.

The full subcategory $\mathbf{Eq}\big((\phi_\kappa, \psi_\kappa)_{\kappa\in K}\big)$ of $\mathbf{A}$, spanned by all $\mathbf{A}$-objects A with $\phi_{\kappa A} = \psi_{\kappa A}$ for all $\kappa \in K$ is called the *equifier* of the family $\big((\phi_\kappa, \psi_\kappa)_{\kappa\in K}\big)$.

43 Example A paradigmatic example of these notions is the category $\mathbf{Grp}$ of groups. A group is a quadruple (G, m, i, e) with G a set and maps $G^2 \xrightarrow{m} G$, $G \xrightarrow{i} G$, $1 \xrightarrow{e} G$, satisfying the obvious equations. Denoting by $T\colon \mathbf{Set} \to \mathbf{Set}$ the functor with $X \mapsto X^2 + X + 1$ where $+$ is the coproduct in $\mathbf{Set}$ (i.e., disjoint union), this its equivalent to saying a group is T-algebra $(G, TG \xrightarrow{\gamma} G)$, where γ is the map from $G^2 + G + 1$ to G induced by the three maps m, i, e, satisfying the respective equations.

Considering the forgetful functor $|-|\colon \mathbf{Alg}T \to \mathbf{Set}$ one has natural transformations $\phi, \psi\colon |-| \Rightarrow |-|$ with $\phi_{(A,\alpha)} = A \xrightarrow{\Delta} A \times A \xrightarrow{A\times i} A \times A \xrightarrow{m} A$ and $\psi_{(A,\alpha)} = A \xrightarrow{!} 1 \xrightarrow{e} A$.

Now a T-algebra $(G, TG \xrightarrow{\alpha} G)$ satisfies the equation $\forall x \in G:\ m(x, i(x)) = 1$ iff (G, α) belongs to the equifier $\mathbf{Eq}\big((\phi, \psi)\big)$. Similarly one can express satisfaction of the other group equations. Thus, the category of groups is a subcategory of $\mathbf{Alg}T$, which is an equifer.

2.2 (Co)monoids and Hopf monoids as equifiers

Monoids and comonoids

Given a monoidal category $\mathbb{C}$ with finite coproducts, we denote by $\otimes^2 + I$, or short by T_+, the functor with $C \mapsto (C \otimes C) + I$. A $(\otimes^2 + I)$-algebra is nothing but a triple $(M, M \otimes M \xrightarrow{m} M, I \xrightarrow{e} M)$; each monoid in $\mathbb{C}$, thus, is a $(\otimes^2 + I)$-algebra.

Similarly, if the monoidal category $\mathbb{C}$ has finite products, and if we denote by $\otimes^2 \times I$, or short by $T_\times$, the functor with $C \mapsto (C \otimes C) \times I$, a $(\otimes^2 \times I)$-coalgebra is nothing but a triple $(C, C \xrightarrow{\mu} C \otimes C, C \xrightarrow{\epsilon} I)$; each comonoid in $\mathbb{C}$, thus, is a $(\otimes^2 \times I)$-coalgebra.

Note that $(\otimes^2 \times I) \neq (\otimes^2 + I)^{\mathsf{op}}$. The duality principle at work here becomes clearer, when applying a more precise notation as follows: The functors $(\otimes^2 + I)$ and $(\otimes^2 \times I)$ used, depend on $\mathbb{C}$; so we should rather write $F_{\mathbb{C}} := (\otimes^2 + I)$ and $G_{\mathbb{C}} := (\otimes^2 \times I)$. Then $F_{\mathbb{C}}^{\mathsf{op}} = G_{\mathbb{C}^{\mathsf{op}}}$.

In order to characterize the categories of interest within the functor categories just considered we need the concept of equifers. We illustrate this by using the example of comonoids.

1. Considering the forgetful functor $|-|\colon \mathbf{Coalg}T_\times \longrightarrow \mathbf{C}$ one has (with notation as above) the following natural transformations:

   (a) $\varphi^1, \psi^1 \colon |-| \longrightarrow \otimes^3 \circ |-|$
   $$\varphi^1_{(C,[\mu,\epsilon])} \colon = (\mu \otimes C) \circ \mu, \qquad \psi^1_{(C,[\mu,\epsilon])} \colon = (C \otimes \mu) \circ \mu$$
   (b) $\varphi^2, \psi^2 \colon |-| \longrightarrow |-| \otimes I \simeq |-|$
   $$\varphi^2_{(C,[\mu,\epsilon])} \colon = (C \otimes \epsilon) \circ \mu, \qquad \psi^2_{(C,[\mu,\epsilon])} \colon = r_C^{-1}$$
   (c) $\varphi^3, \psi^3 \colon |-| \longrightarrow I \otimes |-| \simeq |-|$
   $$\varphi^3_{(C,[\mu,\epsilon])} \colon = (\epsilon \otimes C) \circ \mu, \qquad \psi^3_{(C,[\mu,\epsilon])} \colon = l_C^{-1}$$

   Then, obviously, $\mathbf{Comon}\mathbb{C} = \mathbf{Eq}\big((\varphi_\kappa, \psi_\kappa)_{t=1,2,3}\big)$.

2. $\mathbf{Mon}\mathbb{C}$ can be described as an equifier dually.

3. Consequently, $\mathbf{Bimon}\mathbb{C}$ can be described as an equifier as well, since it is $\mathbf{Mon}(\mathbf{Comon}\mathbb{C})$.

*-Bimonoids

Proposition 36 allows us to understand a Hopf monoid not only as a bimonoid with an additional property, but rather as a functor algebra over $\mathbf{Bimon}\mathbb{C}$ satisfying a certain equation. This will be of interest later on.

To make this idea explicit we consider the functor $T := (-)^{\mathsf{op},\mathsf{cop}} \colon \mathbf{Bimon}\mathbb{C} \to \mathbf{Bimon}\mathbb{C}$ introduced in Remark 26. Obviously one has $\mathbf{Alg}T = \mathbf{Coalg}T$.

44 Definition A *-*bimonoid in* $\mathbb{C}$ is a T-algebra $(\mathsf{H}, T\mathsf{H} \xrightarrow{S} \mathsf{H})$ for $T = (-)^{\mathsf{op},\mathsf{cop}} \colon \mathbf{Bimon}\mathbb{C} \to \mathbf{Bimon}\mathbb{C}$. We write $^\star\mathbf{Bimon}\mathbb{C}$ instead of $\mathbf{Alg}T$ $(= \mathbf{Coalg}T)$.[6]

Obviuosly, $^\star\mathbf{Bimon}(\mathbb{C}^{\mathsf{op}}) = (^\star\mathbf{Bimon}\mathbb{C})^{\mathsf{op}}$.

45 Remark With $|-|\colon {}^\star\mathbf{Bimon}\mathbb{C} \to \mathbf{C}$ the forgetful functor, there are natural transformations

1. $\big(\lambda_{(\mathsf{B},S))}\big) \colon |(\mathsf{B}, S)| \xrightarrow{S * \mathsf{id}_B} |(\mathsf{B}, S)|$

[6]These structures have been called *near Hopf monoids* in [21]; we believe that *-bimonoid is a more appropriate terminology in view of the established notion of *-algebra elsewhere in algebra.

2. $\left(\mu_{(\mathsf{B},S))}\right)\colon |(\mathsf{B},S)| \xrightarrow{\mathsf{id}_B * S} |(\mathsf{B},S)|$

3. $\left(\rho_{(\mathsf{B},S))}\right)\colon |(\mathsf{B},S)| \xrightarrow{e\circ\epsilon} |(\mathsf{B},S)|$

and $\mathbf{Hopf}\mathbb{C}$ is the equifier of the pairs $(\lambda,\rho),(\mu,\rho)$.

We will need the following lemma.

46 Lemma *Let $f\colon (B,m,e,\mu,\epsilon,S)\to(B',m',e',\mu',\epsilon',S')$ be a homomorphism of *-bimonoids. Then*

1. $f\circ(S*\mathsf{id}_B)=(S'*\mathsf{id}_{B'})\circ f$
2. $f\circ(e\circ\epsilon)=(e'\circ\epsilon')\circ f$

Proof Everything follows from commutativity of the diagrams

$$\begin{tikzcd}
B \arrow[r, "\mu"] \arrow[d, "f"'] & B\otimes B \arrow[r, "S\otimes B"] \arrow[d, "f\otimes f"] & B\otimes B \arrow[r, "m"] \arrow[d, "f\otimes f"] & B \arrow[d, "f"] \\
B' \arrow[r, "\mu'"'] & B'\otimes B' \arrow[r, "S'\otimes B'"'] & B'\otimes B' \arrow[r, "m'"'] & B'
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "\epsilon"] \arrow[d, "f"'] & I \arrow[r, "e"] \arrow[d, "I"] & B \arrow[d, "f"] \\
B' \arrow[r, "\epsilon'"'] & I \arrow[r, "e'"'] & B'
\end{tikzcd}$$

□

2.3 First conclusions

This section requires notions and results from Sections 4.4 and 4.5 in the appendix the reader might want to be familiar with before continuing.

We now assume that $\mathbb{C}=(\mathbf{C},-\otimes-,I)$ is a symmetric monoidal category, where $\mathbf{C}$ is locally finitely presentable and the functors $\otimes^2$ and $\otimes^3$ preserve directed colimits; then the functors $\otimes^2$, T_+ and $T_\times$ preserve directed colimits as well. Examples of this situation are all categories $\mathbf{Mod}_R$ (more generally, every symmetric monoidal closed category, which is locally finitely presentable) and every locally finitely presentable category with its cartesian structure.

By means of the descriptions of monoids, comonoids and Hopf monoids above we can easily deduce the following results using the properties of equifers presented in the Appendix (see Sections 4.4 and 4.5). Most of this is from [16].

47 Proposition *Let $\mathbb{C}=(\mathbf{C},-\otimes-,I)$ be a symmetric monoidal category, where $\mathbf{C}$ is locally finitely presentable and the functors $\otimes^2$ and $\otimes^3$ preserve directed colimits. Then*

1. (a) *$\mathbf{Mon}\mathbb{C}$ is a locally presentable category.*
   (b) *$U_a\colon \mathbf{Mon}\mathbb{C}\to\mathbf{C}$ has a left adjoint and, thus, is monadic.*
   (c) *U_a creates directed colimits.*
2. (a) *$\mathbf{Comon}\mathbb{C}$ is a locally presentable category.*
   (b) *$U_c\colon \mathbf{Comon}\mathbb{C}\to\mathbf{C}$ has a right adjoint and thus is comonadic.*
3. (a) *$\mathbf{Bimon}\mathbb{C}$ is a locally presentable category.*
   (b) *$(-)^c\colon \mathbf{Bimon}\mathbb{C}\to\mathbf{Comon}\mathbb{C}$ has a left adjoint and thus is monadic.*
   (c) *$(-)^c$ creates directed colimits.*
   (d) *$(-)^a\colon \mathbf{Bimon}\mathbb{C}\to\mathbf{Mon}\mathbb{C}$ has a right adjoint and thus is comonadic.*

4. (a) ***Bimon**$\mathbb{C}$ *is a locally presentable category.*
   (b) *Its forgetful functor into* **Bimon**$\mathbb{C}$ *has a left and a right adjoint and, thus, is monadic and comonadic.*
5. (a) **Hopf**$\mathbb{C}$ *is an accessible category.*
   (b) **Hopf**$\mathbb{C}$ *is closed in* **Bimon**$\mathbb{C}$ *under directed colimits.*

Proof Since the functors $\otimes^2$ and $\otimes^3$ preserve directed colimits, so do the functors T_+ and $T_\times$ (for the latter see Proposition 89 (5)) and, hence, the forgetful functors $\mathbf{Alg}T_+ \xrightarrow{|-|} \mathbf{A}$ and $\mathbf{Coalg}T_\times \xrightarrow{|-|} \mathbf{A}$ as well. By Corollary 94 the categories $\mathbf{Alg}T_+$ and $\mathbf{Coalg}T_\times$, thus, are locally presentable and the categories **Mon**$\mathbb{C}$ and **Comon**$\mathbb{C}$ are equifiers of families of pairs of natural transformations between accessible functors. By Proposition 92 both categories are accessible. By Fact 10 **Mon**$\mathbb{C}$ is complete; **Comon**$\mathbb{C}$ is cocomplete by Remark 20; thus, both of these categories are locally presentable by Proposition 89.

By Fact 10 U_a creates directed colimits due to the general assumptions of this proposition. Since U_a is strictly monoidal one concludes that the functors $\otimes^2$ and $\otimes^3$ on **Mon**$\mathbb{C}$ preserve directed colimits as well. Thus, the same argument as above shows that **Bimon**$\mathbb{C}$ is a locally presentable category and that the functor $(-)^c$ creates directed colimits. Consequently, ***Bimon**$\mathbb{C}$ is locally presentable by its very definition due to Corollary 94, since the functor $T = (-)^{\mathsf{op,cop}}$ is an isomorphism, hence accessible in particular.

Finally, in the description of **Hopf**$\mathbb{C}$ as an equifer (see Remark 45) the functor $|-|$ mentioned there is a composition of accessible functors, thus accessible. Hence **Hopf**$\mathbb{C}$ is an accessible category; and it closed under directed colimits in ***Bimon**$\mathbb{C}$ by Lemma 84. This proves all statements labelled (a) as well as 1 (c), 3 (c) and 5(b).

Concerning the existence of adjoints we need to argue differently: The adjoints in 4. are given by the free algebra construction of Lemma 85 and its dual. The right adjoint of U_c in 2. (b) exists by the Special Adjoint Functor Theorem, since **Comon**$\mathbb{C}$, being locally presentable, has a generator and U_c preserves colimits by the dual of Fact 10. The left adjoint of U_a in 1.(b) exists by the General Adjoint Functor Theorem, since it preserves limits by Fact 10 and satisfies the solution set condition by Proposition 93. Now these adjunctions can be lifted to the adjunctions claimed to exist in 3.(a) as monoidal lifts of adjunctions according to Proposition 65 and its dual (Remark 66).

The monadicity statements follow by means of the Beck-Paré criterion (see e.g. [12]): all functors U under consideration create U-absolute coequalizers, which is shown in the same way as creation of directed colimits. The comonadicity statements follow dually. □

We add the following facts for further use. They are easy to prove[7].

48 Fact 1. Every embedding $i\colon \mathsf{M}' \to \mathsf{M}$ of a submonoid is U_a-initial.

2. Every monoid homomorphism $q\colon \mathsf{M}' \to \mathsf{M}$, which is an epimorphism in $\mathbf{C}$, is U_a-final, provided that $q \otimes q$ is an epimorphism in $\mathbf{C}$.

3 Main results

The previous proposition still leaves open the most interesting questions in this context concerning the category **Hopf**$\mathbb{C}$:

1. Is **Hopf**$\mathbb{C}$ a locally presentable category?

[7] For the definition of U-initial and U-final morphisms see Section 4.3 in the appendix.

2. Is $\mathbf{Hopf}\mathbb{C}$ reflective in $\mathbf{Bimon}\mathbb{C}$?

3. Is $\mathbf{Hopf}\mathbb{C}$ coreflective in $\mathbf{Bimon}\mathbb{C}$?

4. Does the forgetful functor $V_a\colon \mathbf{Hopf}\mathbb{C} \to \mathbf{Mon}\mathbb{C}$ have a right adjoint?

5. Does the forgetful functor $V_c\colon \mathbf{Hopf}\mathbb{C} \to \mathbf{Comon}\mathbb{C}$ have a left adjoint?

These questions are related as follows.

49 Proposition ([19]) *Let $\mathbb{C}$ be a locally finitely presentable symmetric monoidal category, where the functors $\otimes^2$ and $\otimes^3$ preserve directed colimits. Then*

1. *The following are equivalent and imply that $\mathbf{Hopf}\mathbb{C}$ is a locally presentable category.*
   - *(a) $\mathbf{Hopf}\mathbb{C}$ is coreflective in $\mathbf{Bimon}\mathbb{C}$.*
   - *(b) $\mathbf{Hopf}\mathbb{C}$ is closed under colimits in $\mathbf{Bimon}\mathbb{C}$.*
   - *(c) The forgetful functor $V_a\colon \mathbf{Hopf}\mathbb{C} \to \mathbf{Mon}\mathbb{C}$ has a right adjoint and, thus, is comonadic.*

2. *The following are equivalent and imply that $\mathbf{Hopf}\mathbb{C}$ is a locally presentable category.*
   - *(a) $\mathbf{Hopf}\mathbb{C}$ is reflective in $\mathbf{Bimon}\mathbb{C}$.*
   - *(b) $\mathbf{Hopf}\mathbb{C}$ is closed under limits in $\mathbf{Bimon}\mathbb{C}$.*
   - *(c) The forgetful functor $V_c\colon \mathbf{Hopf}\mathbb{C} \to \mathbf{Comon}\mathbb{C}$ has a left adjoint and, thus, is monadic.*

Proof 1. (a) implies 1. (c) by composition of adjoints. Comonadicity again follows by the Beck-Paré-Theorem.

1. (c) implies 1. (b) since both forgetful functors, $(-)^a$ and V_a, being comonadic, create colimits.

1. (b) implies that $\mathbf{Hopf}\mathbb{C}$ is locally presentable (use Proposition 47 (5) in connection with Proposition 89). But then the embedding has a right adjoint by the Special Adjoint Functor Theorem.

2. follows mostly dually, except that for the implication (b) $\implies$ (a) one uses, instead of the Special Adjoint Functor Theorem, the General Adjoint Functor Theorem in connection with Proposition 93. □

50 Remark In the statements (a) and (b) above one can replace $\mathbf{Bimon}\mathbb{C}$ by ${}^\star\mathbf{Bimon}\mathbb{C}$, since the forgetful functor ${}^\star\mathbf{Bimon}\mathbb{C} \to \mathbf{Bimon}\mathbb{C}$ is monadic and comonadic.

In the sequel we will provide two different proofs to an affirmative answer of the five questions at the beginning of this section. The first one will — by an explicit construction of (co)limits — show that $\mathbf{Hopf}\mathbb{C}$ is closed under (co)limits in $\mathbf{Bimon}\mathbb{C}$. Thus, the affirmative answers (as sheer existence results) follow by the previous Proposition 49.

The second one will proceed by proving (co)reflectivity of $\mathbf{Hopf}\mathbb{C}$ in $\mathbf{Bimon}\mathbb{C}$ by using the familiar characterization of E-reflective categories (see Proposition 74) and composition of adjunctions; it thus will be based on standard categorical constructions only. We finally give an alternative approach to the second proof, using the same characterization — but in its slightly more intuitive form (see Remark 75). This then, in particular, explains the familiar construction of the Hopf envelope from a categorical perspective.

All proofs will depend on the extremal factorization structure on the base category $\mathbb{C}$ (or its dual), which has to be compatible with the monoidal structure in the sense of the introduction, i.e., the functor tensor squaring $\otimes^2$ has to preserve extremal epimorphisms (or monomorphisms). Moreover, the Crucial Lemma (Lemma 38) will play an important role. The following results are slight improvements of [21].

3.1 Constructing limits and colimits in $\mathbf{Hopf}\mathbb{C}$

The reader not familiar with factorizations structures for morphisms should definitely consult Section 4.3 in the Appendix before reading this section.

51 Lemma *Let $\mathbb{C}$ satisfy the assumptions of Proposition 47. Then the following hold.*

1. *If, in addition, in $\mathbf{C}$ extremal epimorphisms are stable under tensor squaring, then*
   (a) *$\mathbf{Mon}\mathbb{C}$ is an $(ExtrEpi, MonoSource)$-category.*
   (b) *$U_a\colon \mathbf{Mon}\mathbb{C} \to \mathbf{C}$ is extremally monadic.*
2. *If, in addition, in $\mathbf{C}$ extremal monomorphisms are stable under tensor squaring, then*
   (a) *$\mathbf{Comon}\mathbb{C}$ is an $(EpiSink, ExtrMono)$-category.*
   (b) *$U_c\colon \mathbf{Comon}\mathbb{C} \to \mathbf{C}$ is extremally comonadic[8].*

Proof It is easily seen, that with $\otimes^2$ also $T_+ = \otimes^2 + I$ preserves extremal epimorphisms and, thus, $\mathbf{Alg}T_+$ is an $(ExtrEpi, MonoSource)$-category by Proposition 81 (4.) and Corollary 91. $\mathbf{Mon}\mathbb{C}$ is closed in $\mathbf{Alg}T_+$ under monosources by Lemma 84, hence it is extremally epireflective in $\mathbf{Alg}T_+$ by Proposition 74. Proposition 76 now implies statement 1.(a) as well as the fact that the embedding of $\mathbf{Mon}\mathbb{C}$ into $\mathbf{Alg}T_+$ preserves extremal epimorphisms. Since the forgetful functor of $\mathbf{Alg}T_+$ preserves these as well by Proposition 81, this proves 1.(b). 2. is dual to this. $\square$

52 Proposition *Assume that $U_a\colon \mathbf{Mon}\mathbb{C} \to \mathbf{C}$ is an extremally monadic functor. Then the category $\mathbf{Hopf}\mathbb{C}$ is closed in $\mathbf{Bimon}\mathbb{C}$ under colimits.*

Proof Let $D\colon \mathbf{I} \to \mathbf{Hopf}\mathbb{C}$ be a diagram and $\big(\mathsf{A}, (\mathsf{D}_i \xrightarrow{\lambda_i} \mathsf{A})_i\big)$ its colimit in $\mathbf{Bimon}\mathbb{C}$. In order to make A a $\star$-bimonoid, observe that, in the diagram below, the colimit property yields a unique homomorphism S of bimonoids making it commute:

$$\begin{tikzcd} \mathsf{A} \arrow[r, dotted, "S"] & \mathsf{A}^{\mathsf{op,cop}} \\ \mathsf{D}_i \arrow[u, "\lambda_i"] \arrow[r, "S_i"'] & \mathsf{D}_i^{\mathsf{op,cop}} \arrow[u, "\lambda_i"'] \end{tikzcd} \tag{3}$$

It remains to check that we have indeed got a Hopf monoid.

In the following we omit the forgetful functors $\mathbf{Bimon}\mathbb{C} \to \mathbf{Mon}\mathbb{C} \to \mathbf{C}$. Since $\mathbf{Bimon}\mathbb{C} \to \mathbf{Mon}\mathbb{C}$ preserves colimits and $\mathbf{Mon}\mathbb{C} \to \mathbf{C}$ is extremally monadic, we get from Proposition 80 that each colimit map λ_i in $\mathbf{Mon}\mathbb{C}$ is the composition

$$\lambda_i = (D_i \xrightarrow{\kappa_i} C \xrightarrow{u} TC \xrightarrow{q} A)$$

where $\big(C, (\kappa_i)\big)$ is a colimit of D in $\mathbf{C}$, u is the universal morphism from C into the free monoid TC over C, and q is an extremal epimorphism in $\mathbf{Mon}\mathbb{C}$.

[8]See Section 4.3 for a definition and properties of these functors.

Since the functor $\mathbf{Bimon}\mathbb{C} \to \mathbf{Mon}\mathbb{C}$ even creates colimits (see the dual of Fact 10), the comonoid structure $\mu,\ \epsilon$ is given by commutativity of the diagrams

$$\begin{tikzcd} D_i \arrow[r, "\lambda_i"] \arrow[d, "\mu_i"'] & A \arrow[d, "\mu"] \\ D_i \otimes D_i \arrow[r, "\lambda_i \otimes \lambda_i"'] & A \otimes A \end{tikzcd} \qquad \begin{tikzcd} D_i \arrow[r, "\lambda_i"] \arrow[dr, "\epsilon_i"'] & A \arrow[d, "\epsilon"] \\ & R \end{tikzcd}$$

It thus follows that the following diagrams commute:

$$\begin{tikzcd} D_i \arrow[r, "\kappa_i"] \arrow[d, "\mu_i"'] & C \arrow[r, "u"] & TC \arrow[r, "q"] & A \arrow[d, "\mu"] \\ D_i \otimes D_i \arrow[rrr, "\lambda_i \otimes \lambda_i"] \arrow[d, "D_i \otimes S_i"'] & & & A \otimes A \arrow[d, "A \otimes S"] \\ D_i \otimes D_i \arrow[rrr, "\lambda_i \otimes \lambda_i"] \arrow[d, "m_i"'] & & & A \otimes A \arrow[d, "m"] \\ D_i \arrow[r, "\kappa_i"'] & C \arrow[r, "u"'] & TC \arrow[r, "q"'] & A \end{tikzcd} \qquad \begin{tikzcd} D_i \arrow[r, "\kappa_i"] \arrow[d, "\epsilon_i"'] & C \arrow[r, "u"] & TC \arrow[r, "q"] & A \arrow[d, "\epsilon"] \\ R \arrow[rrr, equal] \arrow[d, "e_i"'] & & & R \arrow[d, "e"] \\ D_i \arrow[r, "\kappa_i"'] & C \arrow[r, "u"'] & TC \arrow[r, "q"'] & A \end{tikzcd}$$

From $e_i \circ \epsilon_i = m_i \circ (D_i \otimes S_i) \circ \mu_i$ for all i it thus follows that

$$e \circ \epsilon \circ (q \circ u) = [m \circ (A \otimes S) \circ \mu] \circ (q \circ u).$$

Similarly,

$$e \circ \epsilon \circ (q \circ u) = [m \circ (S \otimes A) \circ \mu] \circ (q \circ u).$$

Since q is an extremal epimorphism in $\mathbf{C}$, now (A, S) indeed is a Hopf monoid by the corollary to the Crucial Lemma (38). □

The above proposition can be dualized. We thus have the following result, too.

53 Proposition *Assume that $U_c\colon \mathbf{Comon}\mathbb{C} \to \mathbf{C}$ is an extremally comonadic functor. Then the category $\mathbf{Hopf}\mathbb{C}$ is closed in $\mathbf{Bimon}\mathbb{C}$ under limits.*

3.2 All adjunctions exist

By means of Proposition 49 the results of the previous subsection on limits and colimits immediately yield our main existence results as follows.

54 Theorem *Let $\mathbf{C}$ be a symmetric monoidal category, where $\mathbf{C}$ is locally finitely presentable and the functors $\otimes^2$ and $\otimes^3$ preserve directed colimits. Then the following hold.*

1. *If the functor $\otimes^2$ preserves extremal epimorphisms*
   - *(a) $\mathbf{Hopf}\mathbb{C}$ is coreflective in $\mathbf{Bimon}\mathbb{C}$*
   - *(b) The forgetful functor $V_a\colon \mathbf{Hopf}\mathbb{C} \to \mathbf{Mon}\mathbb{C}$ has a right adjoint.*
2. *If the functor $\otimes^2$ preserves extremal monomorphisms.*
   - *(a) $\mathbf{Hopf}\mathbb{C}$ is reflective in $\mathbf{Bimon}\mathbb{C}$*
   - *(b) The forgetful functor $V_c\colon \mathbf{Hopf}\mathbb{C} \to \mathbf{Comon}\mathbb{C}$ has a left adjoint.*

In both cases $\mathbf{Hopf}\mathbb{C}$ is a locally presentable category.

3.3 Hopf reflections and coreflections by standard constructions

Hopf reflections and coreflections can also be obtained using two simple categorical standard constructions, though with a somewhat limited scope in the case of coreflections. These are the *free algebra construction* already used in the proof of Proposition 47 4 (b) and the familiar *characterization of E-reflective subcategories* (see Proposition 74). To apply the latter we need to be able to lift a factorization structure (E, M) from $\mathbf{C}$ to $\mathbf{Bimon}\mathbb{C}$. And for this we need a stronger compatibility condition. We therefore define

55 Definition Let the symmetric monoidal category $\mathbb{C}$ have the $(ExtrEpi, ExtrMono)$-factorization structure for morphisms. We then call this structure the *monoidal extremal factorization structure*, provided that both classes, $ExtrEpi(\mathbf{C})$ and $ExtrMono(\mathbf{C})$ are stable under tensor squaring and that $\mathbf{C}$ is extremally cowellpowered and extremally wellpowered.

The paradigmatic example of a monoidal category with monoidal extremal factorization structure is $\mathbf{Mod}_R$, where R is an absolutely flat ring.

Note that, if $\mathbf{C}$ is complete and cocomplete, the monoidal extremal factorization structure makes $\mathbb{C}$ into an $(ExtrEpi, MonoSource)$- as well as an $(EpiSink, ExtrMono)$-category by Remark 72.

56 Lemma *Let $\mathbb{C}$ be a symmetric monoidal category with the monoidal extremal factorization structure (E, M). Then the forgetful functor $|-|\colon \mathbf{Bimon}\mathbb{C} \to \mathbf{C}$ lifts this factorization structure to a factorization structure $(\bar{E}, \bar{M})$ with $e \in \bar{E}$ iff $|e| \in E$ and $m \in \bar{M}$ iff $|m| \in M$.*

The factorization structure $(\bar{E}, \bar{M})$ then is lifted to the category of *-bimonoids, by its forgetful functor into $\mathbf{Bimon}\mathbb{C}$ *as well and provides* $^\star\mathbf{Bimon}\mathbb{C}$ *with a factorization structure $(\tilde{E}, \tilde{M})$.*

$\mathbf{Hopf}\mathbb{C}$ *inherits this factorization structure.*

Proof For $\mathbf{Bimon}\mathbb{C}$ use Lemma 28 as well as Fact 48 and its dual. The lifting to $^\star\mathbf{Bimon}\mathbb{C}$ then follows by a diagonal fill in. $(\tilde{E}, \tilde{M})$ now can be restricted to a factorization structure on $\mathbf{Hopf}\mathbb{C}$, since $\mathbf{Hopf}\mathbb{C}$ is closed under $\tilde{M}$-subobjects. □

Using this more restrictive condition on the factorization structure of the base category we get the following result which moreover, in the case of Hopf algebras, will allow us to better understand the classical constructions (see Part II).

57 Proposition *Let $\mathbb{C}$ be a locally finitely presentable symmetric monoidal category, such that the functor $\otimes^2$ preserves extremal epimorphisms as well as extremal monomorphisms. Then the following hold:*

1. $\mathbf{Hopf}\mathbb{C}$ *is coreflective in* $^\star\mathbf{Bimon}\mathbb{C}$ *with coreflection morphisms which are extremal monomorphisms in* $\mathbf{C}$.

2. $\mathbf{Hopf}\mathbb{C}$ *is reflective in* $^\star\mathbf{Bimon}\mathbb{C}$ *with reflection morphisms which are extremal epimorphisms in* $\mathbf{C}$.

Proof For 1. simply apply the familiar characterization of E-reflective subcategories (see Proposition 74) with respect to the lifted monoidal extremal factorization structure on $\mathbf{Bimon}\mathbb{C}$ (see Lemma 56). This can be done, since $\mathbf{Hopf}\mathbb{C}$ is closed under subobjects whose embedding is an extremal monomorphism in $\mathbf{C}$ (obvious by Lemma 84) and products by Proposition 53. U_a is extremally monadic as required by Proposition 47.

2. follows by duality. □

We thus get, using Proposition 47, the following result as a corollary. Note, that the existence of the Hopf coreflection as in Theorem 54 holds more generally than the construction given here.

58 Theorem *Let* $\mathbf{C}$ *be a symmetric monoidal category, where* $\mathbf{C}$ *is locally finitely presentable and the functors* $\otimes^2$ *and* $\otimes^3$ *preserve directed colimits as well as extremal epimorphisms and extremal monomorphisms. Then*

1. $\mathbf{Hopf}\mathbb{C}$ *is reflective in* $\mathbf{Bimon}\mathbb{C}$*. The Hopf reflection of a bimonoid* B *can be obtained by first constructing the free* **-bimonoid* $(\tilde{\mathsf{B}}, S)$ *over* B *by the free algebra construction of Lemma 85 and then reflecting this into* $\mathbf{Hopf}\mathbb{C}$ *according to the Proposition above.*

2. $\mathbf{Hopf}\mathbb{C}$ *is coreflective in* $\mathbf{Bimon}\mathbb{C}$*. The coreflection of a bimonoid* B *can be obtained dually.*

59 Remark There is no reason to assume that, in the Theorem above, one could expect the coreflection maps into $\mathbf{Hopf}\mathbb{C}$ to be extremal monomorphisms or, dually, the reflection maps to be extremal epimorphisms. Nevertheless, this *might* happen, as the most simple example of $\mathbb{C} = \mathbf{Set}$ with its cartesian structure shows: Here $\mathbf{Bimon}\mathbb{C}$ is the category $\mathbf{Mon}$ of monoids and $\mathbf{Hopf}\mathbb{C}$ is the category $\mathbf{Grp}$ of groups. And the coreflection of $\mathbf{Mon}$ into $\mathbf{Grp}$ sends a monoid to its subgroup of invertible elements. See Part II for a similar discussion in the case of Hopf algebras.

An alternative description

By Remark 75 the Hopf-coreflection of a *-bimonoid (H, S) as in Proposition 57 is the largest sub*-bimonoid (C, S_C) of (H, S), whose embedding i is an extremal monomorphism in $\mathbf{C}$ and which is a Hopf monoid. The latter condition is equivalent to saying that the embedding i factors over the equalizer $E \xrightarrow{\eta} H$ of $S * \mathsf{id}_H, \mathsf{id}_H * S, e \circ \epsilon$ in $\mathbf{C}$, shortly that C *is contained in* E. Now the following lemma, whose simple but technical proof again will be put to the appendix, shows that one can construct C already on the level of comonoids.

60 Lemma *Let* $\mathbb{C}$ *be a symmetric monoidal category with the coextremal factorization structure, where* $\otimes^2$ *preserves extremal monomorphisms. Then the following holds:*

Let (H, S) *be a* **-bimonoid and* E *an extremal submonoid of* H^a*. The largest extremal subcomonoid* C *of* H^c *contained in* E *is a sub-bimonoid of* H*. If* E *is the equalizer of* $S * \mathsf{id}_H, \mathsf{id}_H * S, e \circ \epsilon$ *in* $\mathbf{C}$*, this becomes a sub-***bimonoid of* (H, S) *and even a Hopf monoid* $(\mathsf{C}, \tilde{S})$*.*

61 Remark We thus get an alternative proof of Proposition 57 as follows: Let (H, S) be a *-bimonoid. Lemma 71 guarantees by our assumptions on $\mathbb{C}$, that the subcomonoids of H^c form a complete lattice. In particular there exists a largest extremal subcomonoid C of H^c contained in the equalizer E of $S * \mathsf{id}_H, \mathsf{id}_H * S, e \circ \epsilon$ in $\mathbf{C}$. By the previous Lemma C becomes a Hopf monoid (C, H_C) and this clearly is the largest sub-*bimonoid of (H, S) belonging to $\mathbf{Hopf}\mathbb{C}$. Since $\mathbf{Hopf}\mathbb{C}$ is closed in $^\star\mathbf{Bimon}\mathbb{C}$ under subobjects carried by (extremal) monomorphims in $\mathbb{C}$, the embeddding $(\mathsf{C}, H_C) \hookrightarrow (\mathsf{H}, S)$ is a coreflection by the dual of Proposition 74, the familiar characterization of E-reflective subcategories.

An alternative construction of the reflection can be obtained by dualization, since the duals of all arguments used above are valid as well: One constructs the largest extremal quotient monoid M of H^a with the property that M is an extremal quotient of the coequalizer $H \xrightarrow{\rho} Q$ of $S * \mathsf{id}_H, \mathsf{id}_H * S, e \circ \epsilon$ in $\mathbf{C}$.

3.4 Constructing free and cofree Hopf monoids

Free and cofree Hopf monoids have been obtained in Theorem 54 by composition of adjunctions. This is in detail (see also [22]):

62 Fact Let $\mathbb{C} = (\mathbf{C}, -\otimes-, I)$ be a symmetric monoidal category, where $\mathbf{C}$ is locally finitely presentable and the functors $\otimes^2$ and $\otimes^3$ preserve directed colimits.

1. If the functor tensor squaring $\otimes^2$ preserves extremal monomorphisms, then the forgetful functor $V_c\colon \mathbf{Hopf}\mathbb{C} \to \mathbf{Comon}\mathbb{C}$ has a left adjoint, and the free Hopf monoid over a comonoid C can be constructed stepwise as follows:

   (a) Form $\mathsf{C}^\sharp$, the free bimonoid over C, which is the monoidal lift of the free monoid over C;

   (b) then form the free *-bimonoid $(\widetilde{\mathsf{C}^\sharp}, S)$ over $\mathsf{C}^\sharp$ according to Lemma 85 (see proof of Propostion 47 4.(b)), that is, adjoin freely a potential antipode;

   (c) finally, form the Hopf reflection $Env(\widetilde{\mathsf{C}^\sharp}, S)$ of $(\widetilde{\mathsf{C}^\sharp}, S)$ (see Remark 50). If the functor $\otimes^2$ preserves extremal epimorphisms as well, this step can be done according to Remark 61 above.

   Then $Env(\widetilde{\mathsf{C}^\sharp}, S)$ is the free Hopf monoid over C.

2. If the functor tensor squaring $\otimes^2$ preserves extremal epimorphisms, then the forgetful functor $V_a\colon \mathbf{Hopf}\mathbb{C} \to \mathbf{Mon}\mathbb{C}$ has a right adjoint, and the cofree Hopf monoid over a monoid M can dually be constructed stepwise as follows:

   (a) Form M^*, the cofree bimonoid over M, which is the monoidal lift of the cofree comonoid over M;

   (b) then form the cofree *-bimonoid $(\overline{\mathsf{M}^*}, S)$ over M^* according to the dual Lemma 85, that is, adjoin cofreely a potential antipode;

   (c) finally form the Hopf coreflection $Cov(\overline{\mathsf{M}^*}, S)$ of $(\overline{\mathsf{M}^*}, S)$ (see Remark 50). If the functor $\otimes^2$ preserves extremal monomorphisms as well, this can be done according to Remark 61 again.

   Then $Cov(\overline{\mathsf{M}^*}, S)$ is the cofree Hopf monoid over M.

There is a shortcoming to these constructions concerning step (b). Adjoining (co)freely a potential antipode to a bimonoid requires the use of products and coproducts in $\mathbf{Bimon}\mathbb{C}$ (see Lemma 85) — and working with those is rather cumbersome as their construction in the proof of Proposition 80 shows. So the question arises, whether this step can rather be done in $\mathbf{Mon}\mathbb{C}$ and $\mathbf{Comon}\mathbb{C}$ respectively. In fact, Takeuchi in [25] precisely did this (see Part II), and we will there provide the respective general (and therefore dualizable) procedure.

4 Appendix

Besides standard categorical methods our approach to Hopf algebras also makes use of certain areas of category theory, which may be lesser known to the general mathematical reader of this text. In this appendix we therefore devote a section to each of the following subjects.

Monoidal categories

We discuss the coherence conditions mentioned in the definitions of a symmetric monoidal category and a monoidal functor and apply them in a sketch of the proof of Theorem 13. Moreover we explain, what we call *monoidal lifts* of adjunctions between monoidal categories to their respective categories of monoids and comonoids and recall the standard construction of free monoids. In addition, we recall the *Eckmann-Hilton argument*, since it is heavily used in Part II.

Factorization structures

Factorization structures play a crucial role in proving completeness properties of Hopf algebras. It is useful to even consider factorizations of sources and sinks respectively. Also the behavior of functors with respect to these structures is important and therefore included. We explain all concepts in detail and sketch proofs of those results which are heavily used in the text. For details we refer to [1]. The characterization of extremally monadic functors (Proposition 78) is new.

Functor algebras and coalgebras

Since it is useful to consider many of structures in the text as categories of *functor algebras* and *functor coalgebras* respectively, we explain these concepts in detail and list their properties which are used in the text. Most of them are easy to prove.

Locally presentable categories

Some of the results about solutions of universal problems in the context of Hopf monoids (existence of colimits, of cofree Hopf algebras, and of coreflections) could have been obtained by standard categorical methods. There is with one notable exception however, which is crucial for solutions to the existence problems dual to those just mentioned: the existence of a right adjoint to the forgetful functor of the category of comonoids. The setting suitable for answering this question, which in fact is at the heart of Barr's solution for the module based case [5], is that of *locally presentable* (or, more generally of *accessible*) categories.

Accessible categories are not necessarily very well behaved categories. But the locally presentable ones (an important subcollection) are. However, the collection of all accessible categories has extremely useful closure properties. We will briefly recall the facts about these types of categories, relevant in our context.

For a full account we refer to [3]. Note that Propositions 89, 90, 92, and 93 are particularly hard.

First however we supply the proofs of a couple results, which would have belonged to the main text, but where we felt they would unduly interrupt the presentation due to their length and technicality.

4.1 The missing proofs

Proof of Proposition 36

It suffices to show that $S\colon \mathsf{H}^a \to (\mathsf{H}^a)^{\mathsf{op}}$ is a monoid homomorphism (the rest follows by duality), i.e., that the morphisms $S \circ m$ and $m \circ s \circ (S \otimes S)$ coincide. This will be the case if $S \circ m$ is a right invers of m and $m \circ s \circ (S \otimes S)$ is a left invers of m in the convolution monoid $\Phi(\mathsf{H}^c, \mathsf{H}^a)$, equivalently, if the following diagram commutes.

$$
\begin{tikzcd}
 & \otimes^4 H \arrow[r, "H\otimes s\otimes H"] & \otimes^4 H \arrow[r, "m\otimes(S\circ m)"] & H\otimes H \arrow[dr, "m"] & \\
H\otimes H \arrow[ur, "\mu\otimes\mu"] \arrow[rr, "\epsilon\otimes\epsilon"'] \arrow[dr, "\mu\otimes\mu"] & & I\otimes I \simeq I \arrow[rr, "e"'] & & H \\
 & \otimes^4 H \arrow[r, "H\otimes s\otimes H"'] & \otimes^4 H \arrow[r, "\big((m\circ S)\circ(S\otimes S)\big)\otimes m"'] & H\otimes H \arrow[ur, "m"] &
\end{tikzcd}
$$

For the upper cell one uses the decomposition below, where all cells commute since μ and ϵ are monoid homomorphisms and S is an antipode.

$$
\begin{tikzcd}
 & \otimes^4 H \arrow[r, "H\otimes s\otimes H"] & \otimes^4 H \arrow[d, "m\otimes m"'] \arrow[dr, "m\otimes(S\circ m)"] & \\
H\otimes H \arrow[ur, "\mu\otimes\mu"] \arrow[r, "m"] \arrow[dr, "\epsilon\otimes\epsilon"'] & H \arrow[r, "\mu"] \arrow[d, "\epsilon"] & H\otimes H \arrow[r, "H\otimes S"] & H\otimes H \arrow[d, "m"] \\
 & I\otimes I \simeq I \arrow[rr, "e"'] & & H
\end{tikzcd}
$$

The lower cell can be decomposed as follows, where all inner cells except for the central pentagon commute for obvious reasons.

$$
\begin{tikzcd}
 & & I\otimes I \arrow[r, "l_I"] \arrow[dr, "I\otimes e"] & I \arrow[dr, "e"] & \\
H\otimes H \arrow[urr, "\epsilon\otimes\epsilon"] \arrow[rr, "H\otimes(e\circ\epsilon)"] \arrow[d, "H\otimes\mu"] \arrow[dd, bend right, "\mu\otimes\mu"'] & & H\otimes H \arrow[r, "\epsilon\otimes H"] & I\otimes H \arrow[r, "l_H"] & H \\
\otimes^3 H \arrow[r, "H\otimes S\otimes H"] \arrow[d, "\mu\otimes H\otimes H"] & \otimes^3 H \arrow[r, "\epsilon\otimes H\otimes H"] \arrow[ur, "H\otimes m"] & I\otimes H\otimes H \arrow[r, "l_H\otimes H"] \arrow[ur, "I\otimes m"] & H\otimes H \arrow[ur, "m"] & \\
\otimes^4 H \arrow[d, "H\otimes s\otimes H"'] & & \otimes^4 H \arrow[dr, "S\otimes S\otimes H\otimes H"] & \otimes^3 H \arrow[u, "m\otimes H"'] & \\
\otimes^4 H \arrow[urr, "s\otimes H\otimes H"] \arrow[rr, "S\otimes S\otimes H\otimes H"] \arrow[rrrr, bend right, "((m\circ s)\circ(S\otimes S))\otimes m"'] & & \otimes^4 H \arrow[r, "s\otimes H\otimes H"] & \otimes^4 H \arrow[r, "m\otimes m"] \arrow[u, "H\otimes m\otimes H"'] & H\otimes H \arrow[uuu, "m"']
\end{tikzcd}
$$

Commutativity of that pentagon now can be read off the following decomposition, where the dotted cells commute by coherence.

$$\begin{tikzcd}
\otimes^3 H \arrow[r, "H\otimes S\otimes H"] \arrow[dr, "s\otimes H"] \arrow[dd, "\mu\otimes H\otimes H"'] & \otimes^3 H \arrow[r, "\epsilon\otimes H\otimes H"] \arrow[dr, "s\otimes H"] & I\otimes H\otimes H \arrow[r, dotted, "l_H\otimes H"] \arrow[dr, "s\otimes H"'] & H\otimes H \\
 & \otimes^3 H \arrow[r, "S\otimes H\otimes H"'] \arrow[dr, "H\otimes\mu\otimes H"'] & \otimes^3 H \arrow[r, "H\otimes\epsilon\otimes H"] & H\otimes I\otimes H \arrow[u, "r_H\otimes H"] \arrow[d, "H\otimes e\otimes H"'] \\
\otimes^4 H \arrow[rr, "s\otimes H\otimes H"'] \arrow[d, dotted, "H\otimes s\otimes H"'] & & \otimes^4 H \arrow[dr, "S\otimes S\otimes H\otimes H"'] & \otimes^3 H \arrow[uu, bend right, "m\otimes H"'] \\
\otimes^4 H \arrow[urr, dotted, "s\otimes H\otimes H"'] & & & \otimes^4 H \arrow[u, "H\otimes m\otimes H"']
\end{tikzcd}$$

Proof of the Crucial Lemma (Lemma 38)

Assume E carries a multiplication m' as required. To get a unit $e'\colon I \to E$ that is preserved by η we first observe in a rather straightforward way that the following equation holds

$$(m \circ (S \otimes \mathsf{id}) \circ \mu) \circ e = (e \circ \epsilon) \circ e \tag{4}$$

Then, by the equalizer property of (E, η), $e\colon I \to B$ will factor as

$$I \xrightarrow{e} B \;=\; I \xrightarrow{e'} E \xrightarrow{\eta} B$$

It then remains to prove that e' acts as a unit for m'. But that can be read of the following diagram and its analog for $E \otimes e'$ (recall, that η is a monomorphism).

$$\begin{tikzcd}
I\otimes B \arrow[rr, "e\otimes B"] \arrow[ddd, bend right=60, "\lambda_B"'] & & B\otimes B \arrow[ddd, bend left=60, "m"] \\
I\otimes E \arrow[u, "I\otimes\eta"] \arrow[rr, "e'\otimes E"] \arrow[dr, "\lambda_E"'] & & E\otimes E \arrow[u, "\eta\otimes\eta"'] \arrow[dl, "m'"] \\
 & E \arrow[d, "\eta"] & \\
 & B &
\end{tikzcd}$$

In order to prove that E carries a multiplication m' preserved by η, it suffices show that the equations

$$(S * \mathsf{id}_B) \circ \big(m \circ (\eta \otimes \eta)\big) = (e \circ \epsilon) \circ \big(m \circ (\eta \otimes \eta)\big) = (\mathsf{id}_B * S) \circ \big(m \circ (\eta \otimes \eta)\big) \tag{5}$$

hold, since then, by the equalizer property of η, $m \circ (\eta \otimes \eta)$ factors through η. Associativity of m' then follows trivially from that of m since η is a monomorphism.

We proceed as follows: Assume that the following two equations hold (with $m_3 := m \circ (B \otimes m) = (B \otimes m) \circ m$ and τ the respective symmetries)

$$m_3 \circ \big((\tau \otimes B) \circ (B \otimes S \otimes B) \circ ((S * \mathsf{id}_B) \otimes \mu)\big) = (S * \mathsf{id}_B) \circ m \tag{6}$$

$$m_3 \circ \big((\tau \otimes B) \circ (B \otimes S \otimes B) \circ ((e \circ \epsilon) \otimes \mu)\big) = (\epsilon \otimes B) \circ \big(B \otimes (S * B)\big) \tag{7}$$

Since, by the equalizing property of η, also

$$\big((S * B) \otimes \mu\big) \circ (\eta \otimes \eta) = \big((e \circ \epsilon) \otimes \mu\big) \circ (\eta \otimes \eta)$$

equations (6) and (7) imply (omitting the canonical isomorphism $I \otimes I \simeq I$)

$$\big((\epsilon \otimes B) \circ (B \otimes (S * \mathsf{id}_B))\big) \circ (\eta \otimes \eta) = \big((S * \mathsf{id}_B) \circ m\big) \circ (\eta \otimes \eta)$$

Since ϵ is a monoid homomorphism, one has $e \circ \epsilon \circ m = e \circ (\epsilon \otimes \epsilon) = \epsilon \otimes (e \circ \epsilon)$ which, together with the last equation, implies the first of the required equalities (5). It thus remains to prove the equalities (6) and (7) above.

Equation (6) means commutativity of the outer frame of the diagram

$$\begin{tikzcd}
B \otimes B \arrow[r, "\mu\otimes\mu"] \arrow[dd, "m"'] & \otimes^4 B \arrow[r, "S\otimes B\otimes B\otimes B"] \arrow[d, "B\otimes\tau\otimes B"'] & \otimes^4 B \arrow[r, "m\otimes B\otimes B"] & \otimes^3 B \arrow[r, "B\otimes S\otimes B"] & \otimes^3 B \arrow[d, "\tau\otimes B"] \\
 & \otimes^4 B \arrow[r, "S\otimes S\otimes B\otimes B"] \arrow[d, "m\otimes m"'] & \otimes^4 B \arrow[r, "\tau\otimes B\otimes B"] & \otimes^4 B \arrow[r, "B\otimes m\otimes B"] \arrow[d, "m\otimes m"'] & \otimes^3 B \arrow[d, "m_3"] \\
B \arrow[r, "\mu"'] & B \otimes B \arrow[rr, "S\otimes B"'] & & B \otimes B \arrow[r, "m"'] & B
\end{tikzcd}$$

Here the left hand rectangle commutes, since μ is a homomorphism of monoids; the lower middle rectangle commutes, since S is an anti-homomorphism of monoids; the lower right hand rectangle commutes by associativity of m. Commutativity of the upper right hand rectangle is a consequence of naturality of τ and τ's coherence property.

Equation (7) is equivalent to the commutativity of the outer frame of the following diagram, which follows from naturality of τ, associativity of m and the axioms for the unit e.

$$\begin{tikzcd}
B \otimes B \arrow[r, "B\otimes\mu"] \arrow[d, "B\otimes\mu"'] & \otimes^3 B \arrow[r, "\epsilon\otimes B\otimes B"] \arrow[d, "\tau\otimes B"] & I \otimes B \otimes B \arrow[r, "e\otimes B\otimes B"] \arrow[d, "\tau\otimes B"] & \otimes^3 B \arrow[r, "B\otimes S\otimes B"] \arrow[d, "\tau\otimes B"] & \otimes^3 B \arrow[d, "\tau\otimes B"] \\
\otimes^3 B \arrow[r, "\tau\otimes B"] \arrow[d, "B\otimes S\otimes B"'] & \otimes^3 B \arrow[r, "B\otimes\epsilon\otimes B"] \arrow[d, "S\otimes B\otimes B"] & B \otimes I \otimes B \arrow[r, "B\otimes e\otimes B"] \arrow[d, "S\otimes B\otimes B"] & \otimes^3 B \arrow[r, "S\otimes B\otimes B"] \arrow[d, "S\otimes B\otimes B"] & \otimes^3 B \arrow[d, "m\otimes B"] \\
\otimes^3 B \arrow[r, "\tau\otimes B"] \arrow[rrd, "\epsilon\otimes B\otimes B"'] \arrow[ddd, "B\otimes m"'] & \otimes^3 B \arrow[r, "B\otimes\epsilon\otimes B"] & B \otimes I \otimes B \arrow[r, "B\otimes e\otimes B"] \arrow[d, "\tau\otimes B"] & \otimes^3 B \arrow[r, "m\otimes B"] & B \otimes B \arrow[ddd, "m"] \\
 & & I \otimes B \otimes B \arrow[r, "e\otimes B\otimes B"] \arrow[dd, "I\otimes m"'] & \otimes^3 B \arrow[ru, "m\otimes B"'] \arrow[d, "B\otimes m"] & \\
 & & & B \otimes B \arrow[rd, "m"] & \\
B \otimes B \arrow[rr, "\epsilon\otimes B"] & & I \otimes B \arrow[ru, "e\otimes B"] \arrow[rr, equal] & & B
\end{tikzcd}$$

The second of the required equalities (5) follows analogously.

Proof of Lemma 60

The proof will be obtained by applying the following simple lemma.

63 Lemma *Let $\mathbb{C}$ have the coextremal factorization structure for morphisms and let $\otimes^2$ preserve extremal monomorphisms. Let H be a comonoid, $E \xrightarrow{\eta} H$ an extremal monomorphism and C the largest subcomonoid of H contained in E (i.e. there exists an extremal monomorphism $\mathsf{C} \xrightarrow{i} \mathsf{H}$ in* **Comon**$\mathbb{C}$ *and an extremal monomorphism $C \xrightarrow{j} E$ in* $\mathbf{C}$ *with $i = \eta \circ j$).*

Then every homomorphism of comonoids $f\colon \mathsf{X} \to \mathsf{H}$ which factors (in $\mathbf{C}$*) over E as $f = X \xrightarrow{f'} E \xrightarrow{\eta} H$, factors in* **Comon**$\mathbb{C}$ *over C, that is, it induces a unique homomorphism of comonoids $\tilde{f}\colon \mathsf{X} \to \mathsf{C}$ with $f = i \circ \tilde{f}$*

Proof By assumption **Comon**$\mathbb{C}$ has co-extremal factorizations and these are preserved by its forgetful functor. If then $\mathsf{X} \xrightarrow{e} \mathsf{D} \xrightarrow{m} \mathsf{H}$ is the factorization of f with an extremal monomorphism m and an epimorphism e, the diagram

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[d, "f'"'] & D \arrow[d, "m"] \arrow[dl, dotted, "d"] \\
E \arrow[r, "\eta"'] & H
\end{tikzcd}$$

has a diagonal d, which is an extremal monomorphism since m is. Thus, D is an extremal subcomonoid of H, contained in E. Since C is the largest such subcomonoid, there exists an extremal monomorphism $n\colon \mathsf{D} \to \mathsf{C}$ with $i \circ n = m$. Put $\tilde{f} := n \circ e$. $\square$

Proof (of Lemma 60) Let $\mathsf{C} \xrightarrow{i} \mathsf{H}^c$ the comonoid embedding. For showing that C carries the structure of submonoid of H^a with monoid embedding i we apply Lemma 63.

Since $f = \mathsf{C} \otimes \mathsf{C} \xrightarrow{i \otimes i} \mathsf{H}^c \otimes \mathsf{H}^c \xrightarrow{a} \mathsf{H}^c$ is a comonoid homomorphism, which factors over E (recall that by the Crucial Lemma 38: $a \circ (i \otimes i) = a \circ ((\eta \circ j) \otimes (\eta \circ j)) = \eta \circ a' \circ (j \otimes j)$, where a' denotes the multiplication on E) we can apply Lemma 63 to it. The resulting homomorphism $\tilde{a} := \tilde{f}$ makes the diagram

$$\begin{tikzcd}
C \otimes C \arrow[r, "j \otimes j"] \arrow[d, "\tilde{a} = n \circ e"'] \arrow[rr, bend left, "i \otimes i"] & E \otimes E \arrow[r, "\eta \circ \eta"] \arrow[d, "a'"] & H \otimes H \arrow[d, "a"] \\
C \arrow[r, "j"'] \arrow[rr, bend right, "i"'] & E \arrow[r, "\eta"'] & H
\end{tikzcd} \tag{8}$$

commute, such that $i = \eta \circ j$ is compatible with the multiplications $\tilde{a} = n \circ e$ on C and a of H^a.

Similarly one defines a potential unit $I \xrightarrow{\tilde{u}} C$ on C: Applying the lemma to the unit $\mathsf{I} \xrightarrow{u} \mathsf{H}^c$ of H^a, which also is a comonoid homomorphism factoring over E as $u = \eta \circ u'$ by the Crucial Lemma, produces $\tilde{u} = n \circ e$ with $i \circ \tilde{u} = u$.

Thus, $(C, \tilde{a}, \tilde{u})$ is a T_+-subalgebra, hence a submonoid, of H^a. By Lemma 28 $(\mathsf{C}, \tilde{a}, \tilde{u})$ is a subbimonoid of H.

It thus remains to find a comonoid morphism $\tilde{S}\colon \mathsf{C}^{\mathsf{cop}} \to \mathsf{C}$ with $i \circ \tilde{S} = S \circ i$. For this apply Lemma 63 to the comonoid morphism $f := \mathsf{C}^{\mathsf{cop}} \xrightarrow{i} (\mathsf{H}^c)^{\mathsf{cop}} \xrightarrow{S} (\mathsf{H}^c)$. By Lemma 46 $S \circ \eta$ equalizes the morphisms $S * \mathsf{id}_H, \mathsf{id}_H * S, \epsilon \circ e$ and so induces a morphism $E \xrightarrow{S'} E$ with $\eta \circ S' = S \circ \eta$. Thus, f factors over E and we get a morphism $\tilde{S} := n \circ e\colon \mathsf{C}^{\mathsf{cop}} \to \mathsf{C}$ with $i \circ \tilde{S} = S \circ i$ as required. Since C is contained in E, $(\mathsf{C}, \tilde{S})$ is a Hopf monoid, if E is the equalizer under consideration. $\square$

4.2 Monoidal categories

Symmetric monoidal categories

In the definition of a monoidal category we referred to so-called coherence conditions. These are, in detail, the requirements that the following diagrams commute.

$$\begin{tikzcd}
& (A \otimes B) \otimes (C \otimes D) \arrow[dr, "a"] & \\
\big((A \otimes B) \otimes C\big) \otimes D \arrow[ur, "a"] \arrow[d, "a \otimes D"'] & & A \otimes \big(B \otimes (C \otimes D)\big) \\
\big(A \otimes (B \otimes C)\big) \otimes D \arrow[rr, "a"'] & & A \otimes \big((B \otimes C) \otimes D\big) \arrow[u, "A \otimes a"']
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes I) \otimes C \arrow[rr, "a"] \arrow[dr, "r \otimes C"'] & & A \otimes (I \otimes C) \arrow[dl, "A \otimes l"] \\
& A \otimes C &
\end{tikzcd}$$

These conditions guarantee that one can, in general, deal with the constraints as if they were identities (see [12] for details). We made use of this already in the definition of monoids, where the associativity constraint was suppressed. Without this possibility we should have written the respective diagram as

$$\begin{tikzcd}
& (M \otimes M) \otimes M \arrow[r, "m \otimes M"] \arrow[dl, "a_{MMM}"'] & M \otimes M \arrow[dd, "m"] \\
M \otimes (M \otimes M) \arrow[d, "M \otimes m"'] & & \\
M \otimes M \arrow[rr, "m"'] & & M
\end{tikzcd}$$

The symmetry in a symmetric monoidal category is subject to the coherence condition that the following diagram commutes.

$$\begin{tikzcd}
(B \otimes A) \otimes C \arrow[d, "a"] & (A \otimes B) \otimes C \arrow[l, "s \otimes C"'] \arrow[r, "a"] & A \otimes (B \otimes C) \arrow[d, "s"] \\
B \otimes (A \otimes C) \arrow[r, "B \otimes s"] & B \otimes (C \otimes A) & (B \otimes C) \otimes A \arrow[l, "a"']
\end{tikzcd}$$

Monoidal functors

The coherence conditions referred to in the definition of a monoidal functor are that the following diagrams commute (where we suppress the constraints mentioned above).

$$\begin{tikzcd}
FA \otimes FB \otimes FC \arrow[r, "\Phi_{A,B} \otimes FC"] \arrow[d, "FA \otimes \Phi_{B,C}"'] & F(A \otimes B) \otimes FC \arrow[d, "\Phi_{A \otimes B, C}"] \\
FA \otimes F(B \otimes C) \arrow[r, "\Phi_{A, B \otimes C}"'] & F(A \otimes B \otimes C)
\end{tikzcd}$$

$$\begin{tikzcd}
I' \otimes FC = FC = FC \otimes I' \arrow[rr, "\mathsf{id}\otimes\phi"] \arrow[d, "\phi\otimes\mathsf{id}"'] \arrow[drr, "\mathsf{id}"] & & FC \otimes FI \arrow[d, "\Phi_{C,I}"] \\
FI \otimes FC \arrow[rr, "\Phi_{I,C}"'] & & F(I \otimes C) = FC = F(C \otimes I)
\end{tikzcd}$$

Given a functor F between monoidal categories $\mathbb{C}$ and $\mathbb{C}'$, a triple (F, Φ, ϕ) is called an *opmonoidal functor from* $\mathbb{C}$ *to* $\mathbb{C}'$, if $(F^{\mathsf{op}}, \Phi, \phi)$ is a monoidal functor from $\mathbb{C}^{\mathsf{op}}$ to $\mathbb{C}'^{\mathsf{op}}$. Obviously, if (F, Φ, ϕ) is a strong monoidal functor, then $(F, \Phi^{-1}, \phi^{-1})$ is opmonoidal. Moreover the following holds.

64 Lemma *If* $(F, \Phi, \phi)\colon \mathbb{C} \to \mathbb{C}'$ *is an opmonoidal functor, where* F *admits a right adjoint* G *with counit* ϵ*, then there exist* Ψ *and* ψ *such that* (G, Ψ, ψ) *is a monoidal functor.*

Given C' *and* D' *in* $\mathbf{C}'$*,* $\Psi_{C',D'}$ *is the unique morphism making the following diagram commute,*

$$\begin{tikzcd}
FG(C' \otimes D') \arrow[r, "\epsilon_{C'\otimes D'}"] & C' \otimes D' \\
F(GC' \otimes GD') \arrow[r, "F\Phi_{GC',GD'}"'] \arrow[u, "F\Psi_{C',D'}"] & FGC' \otimes FGD' \arrow[u, "\epsilon_{C'}\otimes\epsilon_{D'}"']
\end{tikzcd}$$

while ψ *is the morphism corresponding to* ϕ *by adjunction.*

From this one easily gets the following.

65 Proposition *Let* $F\colon \mathbb{C} \to \mathbb{C}'$ *be a strong monoidal functor. If* F *has a right adjoint* G *with counit* $\varepsilon\colon FG \to 1_{\mathbf{C}'}$*, then the monoidal lift* $G^\sharp$ *of* G *is right adjoint to* $F^\sharp$ *with counit* $\tilde{\varepsilon}$ *such that* $U'_a\tilde{\varepsilon}_{(C,m,e)} = \varepsilon_C$ *for each* (C, m, e) *in* $\mathbf{Mon}\mathbb{C}'$*.*

Proof It essentially suffices to observe that for any monoid (C, m, e) in $\mathbb{C}'$ one has $G^\sharp(C, m, e) = (GC, \tilde{m}, \tilde{e})$, where $\tilde{m}$ and $\tilde{e}$ are the unique morphisms making the following diagrams commute.

$$\begin{tikzcd}
FGC \arrow[rr, "\varepsilon_C"] & & C \\
F(GC \otimes GC) \arrow[r, "\Phi^{-1}_{GC,GC}"'] \arrow[u, "F\tilde{m}"] & FGC \otimes FGC \arrow[r, "\varepsilon_C\otimes\varepsilon_C"'] & C \otimes C \arrow[u, "m"']
\end{tikzcd}
\qquad
\begin{tikzcd}
FI \arrow[r, "F\tilde{e}"] \arrow[d, "\phi^{-1}"'] & FGC \arrow[d, "\varepsilon_C"] \\
I' \arrow[r, "e"] & C
\end{tikzcd}$$

Then $\varepsilon_C\colon F^\sharp G^\sharp(C, m, e) \to (C, m, e)$ is a monoid homomorphism which is $F^\sharp$-couniversal for (C, m, e). $\square$

66 Remark By duality, if a strong monoidal functor $\mathbb{C} \xrightarrow{F} \mathbb{C}'$ has left adjoint G, then its monoidal lift $\mathbf{Comon}\mathbb{C} \xrightarrow{F^*} \mathbf{Comon}\mathbb{C}'$ has the monoidal lift G^* of G as a left adjoint.

The monoidal structure of Mon$\mathbb{C}$

We here give a hint towards the proof of Theorem 13. That the associativity constraints of $\mathbb{C}$ lift to monoid homomorphisms means that the outer frame of the diagram below commutes; and this is a consequence of commutativity of the inner cells, where one uses, except for naturality and the coherence conditions for a, the coherence conditions for the symmetry s twice and, for the bottom cell, the fact that the monoids are commutative.

$$\begin{tikzcd}
{[(M \otimes N) \otimes K] \otimes [(M \otimes N) \otimes K]} \arrow[r, "a_{MNK} \otimes a_{MNK}"] \arrow[d, "a"'] & {[M \otimes (N \otimes K)] \otimes [M \otimes (N \otimes K)]} \arrow[d, "a"] \\
{(M \otimes N) \otimes [K \otimes (M \otimes N)] \otimes K} \arrow[r, "a"] \arrow[d, "\mathsf{id} \otimes [s_{K, M \otimes N} \otimes K]"'] & {(M \otimes N) \otimes [K \otimes \big((M \otimes N) \otimes K\big)]} \arrow[d, "\mathsf{id} \otimes [s_{K, (M \otimes N) \otimes K} \otimes K]"] \\
{(M \otimes N) \otimes [(M \otimes N) \otimes K] \otimes K} \arrow[d, "a"'] & {(M \otimes N) \otimes [\big((M \otimes N) \otimes K\big) \otimes K)]} \arrow[d, "a"] \\
{(M \otimes N) \otimes [(M \otimes N) \otimes (K \otimes K)]} \arrow[r, "\mathsf{id} \otimes [\mathsf{id} \otimes s_{K,K}]"] \arrow[d, "a"'] & {(M \otimes N) \otimes [(M \otimes N) \otimes (K \otimes K)]} \arrow[d, "a"] \\
{M \otimes [(N \otimes M) \otimes N)] \otimes (K \otimes K)} \arrow[r, "M \otimes a \otimes s_{K,K}"] \arrow[d, "M \otimes [s_{N,M} \otimes N] \otimes \mathsf{id}"'] & {M \otimes [N \otimes (M \otimes N)] \otimes (K \otimes K)} \arrow[d, "M \otimes s_{N, M \otimes N} \otimes \mathsf{id}"] \\
{M \otimes [(M \otimes N) \otimes N] \otimes (K \otimes K)} \arrow[d, "a"'] & {M \otimes [(M \otimes N) \otimes N] \otimes (K \otimes K)} \arrow[d, "a"] \\
{M \otimes [M \otimes (N \otimes N)] \otimes (K \otimes K)} \arrow[r, "M \otimes [M \otimes s_{N,N}] \otimes s_{K,K}"'] \arrow[d, "a"'] & {M \otimes [M \otimes (N \otimes N)] \otimes (K \otimes K)} \arrow[d, "a"] \\
{[(M \otimes M) \otimes (N \otimes N)] \otimes (K \otimes K)} \arrow[d, "{[m \otimes n] \otimes k}"'] & {(M \otimes M) \otimes [(N \otimes N) \otimes (K \otimes K)]} \arrow[d, "m \otimes [(n \circ s_{N,N}) \otimes (k \circ s_{K,K})]"] \\
{(M \otimes N) \otimes K} \arrow[r, "a_{MNK}"'] & {M \otimes (N \otimes K)}
\end{tikzcd}$$

Free monoids

The standard construction of a free monoid $X^\star$ over a set X (word monoid with concatenation) is well known to generalize to a free monoid construction over a symmetric monoidal category $\mathbb{C}$ as follows:

67 Proposition ([12]) *Let $\mathbb{C}$ be a symmetric monoidal category with countable coproducts which are preserved by each functor $C \otimes - \colon \mathbf{C} \to \mathbf{C}$. Then, for any C in $\mathbf{C}$, there exists a free monoid $C^\star$ over C. The underlying object of $C^\star$ is $\coprod_{n \in \mathbb{N}} \otimes^n C$ and the unit of this adjunction is given by the first coproduct injection.*

An example of this construction in our context is the construction of the free R-algebra over an R-module M as the tensor algebra $\bigoplus_n (\otimes^n M)$ of M. Since the forgetful functor $\mathbf{Coalg}_R \to \mathbf{Mod}_R$ is strict monoidal and creates colimits, the category $\mathbf{Coalg}_R$ satisfies the assumptions of the above proposition as well, and the free bimonoid over a comonoid also can be constructed this way.

The Eckmann-Hilton argument

As one checks easily, the multiplication m of a monoid $\mathsf{M} = (M, m, e)$ is a monoid homomorphism $(M, m, e) \otimes (M, m, e) \to (M, m, e)$, provided that the monoid is commutative. Since the unit e always is a homomorphism $\mathsf{I} \to \mathsf{M}$, every commutative monoid belongs to $\mathbf{Mon}(\mathbf{Mon}\mathbb{C})$.

It thus suffices to show: If (M, m, e) and $(M, \bar{m}, \bar{e})$ are monoids where $\mathsf{I} \xrightarrow{\bar{e}} (M, m, e)$ and $(M, m, e) \otimes (M, m, e) \xrightarrow{\bar{m}} (M, m, e)$ are monoid homomorphisms, i.e., where the following diagrams commute

$$\begin{tikzcd}
I \arrow[r, "\mathsf{id}_I"] \arrow[dr, "e"'] & I \arrow[d, "\bar{e}"] \\
& M
\end{tikzcd}
\qquad
\begin{tikzcd}
\otimes^4 M \arrow[r, "M\otimes s\otimes M"] \arrow[d, "\bar{m}\otimes\bar{m}"] & \otimes^4 M \arrow[r, "m\otimes m"] & M \otimes M \arrow[d, "\bar{m}"] \\
M \otimes M \arrow[rr, "m"'] & & M
\end{tikzcd}$$

then $e = \bar{e}$ (which is clear by the first diagram), $m = \bar{m}$ and (M, m, e) is commutative.

Now, consider the diagrams below which commute, since s is natural and $\bar{m}$ is a homomorphism (use also $s_{II} = \mathsf{id}_{I \otimes I}$) .

Since $e = \bar{e}$ is a unit for m as well as for $\bar{m}$, all rows are identities. Thus, Diagram 10 shows $m = \bar{m}$ and Diagram 9 commutativity of m.

$$\begin{tikzcd}
\otimes^2 M \arrow[r] \arrow[dd, "s"] & I \otimes M \otimes M \otimes I \arrow[r, "e\otimes M\otimes M\otimes e"] \arrow[dd, "I\otimes s\otimes I"] & \otimes^4 M \arrow[r, "\bar{m}\otimes\bar{m}"] \arrow[dd, "M\otimes s\otimes M"] & \otimes^2 M \arrow[dr, "m"] & \\
& & & & M \\
\otimes^2 M \arrow[r] & I \otimes M \otimes M \otimes I \arrow[r, "e\otimes M\otimes M\otimes e"] & \otimes^4 M \arrow[r, "m\otimes m"] & \otimes^2 M \arrow[ur, "\bar{m}"'] &
\end{tikzcd} \tag{9}$$

$$\begin{tikzcd}
\otimes^2 M \arrow[r] \arrow[dd, "\mathsf{id}"] & M \otimes I \otimes I \otimes M \arrow[r, "M\otimes e\otimes e\otimes M"] \arrow[dd, "M\otimes s\otimes M"] & \otimes^4 M \arrow[r, "\bar{m}\otimes\bar{m}"] \arrow[dd, "M\otimes s\otimes M"] & \otimes^2 M \arrow[dr, "m"] & \\
& & & & M \\
\otimes^2 M \arrow[r] & M \otimes I \otimes I \otimes M \arrow[r, "M\otimes e\otimes e\otimes M"] & \otimes^4 M \arrow[r, "m\otimes m"] & \otimes^2 M \arrow[ur, "\bar{m}"'] &
\end{tikzcd} \tag{10}$$

4.3 Factorization structures and extremally monadic functors

Some special types of morphisms, sources and sinks

A *source* in a category $\mathbf{A}$ is a pair $(A, (A \xrightarrow{f_i} A_i)_{i \in I})$, where I may be a proper class. A source in $\mathbf{A}^{\mathsf{op}}$ is called a *sink* in $\mathbf{A}$.

A *monosource* in $\mathbf{A}$ is a source $(A, (A \xrightarrow{m_i} A)_{i \in I})$ where the family $(m_i)_i$ is jointly monomorphic, that is, for each pair $r, s \colon B \to A$ one has $r = s$, provided that $m_i \circ r = m_i \circ s$ for all $i \in I$. A monosource in $\mathbf{A}^{\mathsf{op}}$ is called an *episink in* $\mathbf{A}$.

The following collections of morphisms, sources and sinks in a category $\mathbf{A}$ will be of particular use:

1. $Epi(\mathbf{A})$ and $Mon(\mathbf{A}) := Epi(\mathbf{A}^{\mathsf{op}})$, the classes of all epimorphisms and monomorphims in $\mathbf{A}$ respectively.

2. $ExtrEpi(\mathbf{A})$ and $ExtrMon(\mathbf{A}) := ExtrEpi(\mathbf{A}^{\mathsf{op}})$, the classes of all extremal epimorphisms and extremal monomorphims in $\mathbf{A}$ respectively.

   Recall that a morphism e is called an *extremal epimorphism*, provided that e is an epimorphism and, for any factorization $e = m \circ g$ with m a monomorphism, m is an isomorphism. In categories such as those of groups, rings or modules the extremal epimorphisms are precisely the surjective homomorphisms.

3. $MonoSource(\mathbf{A})$ is the collection of all monosources in $\mathbf{A}$; $Episink(\mathbf{A})$ is the collection of all episinks in $\mathbf{A}$.

4. A morphism $A \xrightarrow{m} B$ in $\mathbf{A}$ is called *U-initial* (with respect to a functor $U \colon \mathbf{A} \to \mathbf{X}$), if the following holds. Whenever

   $$(UC \xrightarrow{h} UA \xrightarrow{Um} UB) = (UC \xrightarrow{Ug} UB),$$

   then $h = Uh'$ for some uniquely determined $C \xrightarrow{h'} B$.

5. More generally, a source $(A, (A \xrightarrow{m_i} A_i)_{i \in I})$ in $\mathbf{A}$ is called *U-initial*, if the following holds. Whenever, for each $i \in I$,

   $$(UC \xrightarrow{h} UA \xrightarrow{Um_i} UA_i) = (UC \xrightarrow{Ug_i} UA_i),$$

   then $h = Uh'$ for some uniquely determined $B \xrightarrow{h'} A$.

   Dually, a sink $(A, (A_i \xrightarrow{e_i} A)_{i \in I}$ in $\mathbf{A}$ is called *U-final* if it is U^{op}-initial in $\mathbf{A}^{\mathsf{op}}$, that is, whenever, for each $i \in I$,

   $$(UA_i \xrightarrow{Ue_i} UA \xrightarrow{h} UC) = (UA_i \xrightarrow{Ug_i} UC),$$

   then $h = Uh'$ for some uniquely determined $A \xrightarrow{h'} C$.

If $\mathcal{M}$ is a collection of sources in $\mathbf{A}$, then a subcategory $\mathbf{B}$ of $\mathbf{A}$ is said to be *closed under $\mathcal{M}$-sources*, if for any source $(A, (A \xrightarrow{m_i} B_i)_{i \in I}) \in \mathcal{M}$ the object A belongs to $\mathbf{B}$, provided that all B_i do.

Factorization structures

68 Definition A pair $(E, \mathcal{M})$ where E is a class of morphisms in $\mathbf{A}$ and $\mathcal{M}$ a collection[9] of sources, both closed under composition with isomorphisms, is called a *factorization structure (for sources)*, provided that

1. for each source $(A, (A \xrightarrow{f_i} A_i)_I)$ there exists some $A \xrightarrow{e} B \in E$ and a source $(B, (B \xrightarrow{m_i} A)_I) \in \mathcal{M}$ such that $f_i = m_i \circ e$ for all $i \in I$,

2. each commutative diagram

$$\begin{tikzcd} A_i \arrow[r, "e"] \arrow[d, "f"'] & B \arrow[d, "g_i"] \arrow[dl, dotted, "d"] \\ C \arrow[r, "m_i"'] & D \end{tikzcd}$$

with $e \in E$ and $(C, (m_i)_I) \in \mathcal{M}$ admits a (unique) *diagonal fill in* d, i.e., a morphism $B \xrightarrow{d} C$ with $d \circ e = f$ and $m_i \circ d = g_i$ for all $i \in I$.

A category $\mathbf{A}$ is called an *$(E, \mathcal{M})$-category*, if $\mathbf{A}$ has a factorization structure $(E, \mathcal{M})$ for sources. The dual of a factorization structure for sources is a *factorization structure for sinks*.

In the case that E consists of epimorphisms only and for each $(C, (m_i)_{i \in I}) \in \mathcal{M}$ the class I is a singleton set, thus $\mathcal{M}$ is a proper class, we write M instead of $\mathcal{M}$ and call (E, M) a *factorization structure for morphisms* and $\mathbf{A}$ an *(E, M)-structured category.*[10]

69 Examples

1. The image factorization of a (linear) map provides a factorization structure for morphisms on $\mathbf{Set}$ and $\mathbf{Mod}_R$ respectively.

2. Of particular importance in our context are the *extremal factorization structure for morphisms* $(ExtrEpi(\mathbf{A}), Mono(\mathbf{A}))$ (the first example provides instances of this) and the *extremal factorization structure for sources* $(ExtrEpi(\mathbf{A}), MonoSource(\mathbf{A}))$. A category with this factorization structure will be called an *$(ExtrEpi, MonoSource)$-category.*

3. If $\mathbf{A}$ has the extremal factorization structure, then $(Mono(\mathbf{A}),\ ExtrEpi(\mathbf{A}))$ is a factorization structure on $\mathbf{A}^{\mathsf{op}}$, which is the the *co-extremal factorization structure* $(Epi(\mathbf{A}^{\mathsf{op}}), ExtrMono(\mathbf{A}^{\mathsf{op}}))$ on that category.

70 Remark As usual we call, for a class of M of monomorphisms in $\mathbf{A}$, a pair $(B, B \xrightarrow{m} A)$ with $m \in M$ an *M-subobject of A*. For subobjects (B, m) and (B', m') of A we say $(B, m) \geqslant (B', m')$ iff there exists some $B' \xrightarrow{h} B$ with $m \circ h = m'$, and we call them *isomorphic*, if $(B, m) \leq (B', m')$ and $(B', m') \leq (B, m)$ (equivalently: if there exists an isomorphism $B' \xrightarrow{h} B$ with $m \circ h = m'$). The class $\mathrm{Sub}_M(\mathrm{A})$ of M-subobjects of A then is preordered and the class of isomorphism classes is ordered. Recall also that a category $\mathbf{A}$ is called *M-well powered*, if all isomorphism classes of M-subobjects are sets.

Dually we use, for a class of E of epimorphisms in $\mathbf{A}$, the terms E-quotient of an object and *E-co-wellpowered* category.

The following are good to know.

[9] $\mathcal{M}$ might fail to be a class.

[10] Note that this is more restrictive than e.g. in [1].

71 Lemma ([1, 15.8]) *1. For every (E, M)-structured category $\mathbf{A}$ the following hold:*

(a) $E \cap M$ is the class of all isomorphisms,

(b) E and M are closed under composition.

2. For every $(E, \mathcal{M})$-category $\mathbf{A}$ the following hold:

(a) $E \subset Epi(\mathbf{A})$.

(b) If $\mathcal{M} = MonoSource(\mathbf{A})$, then $E = ExtrEpi(\mathbf{A})$.

(c) If $E = ExtrEpi(\mathbf{A})$, the following are equivalent

i. $\mathcal{M} = MonoSource(\mathbf{A})$.

ii. $\mathbf{A}$ has coequalizers.

(d) $\mathbf{A}$ is (E, M)-structured for the subclass M of $\mathcal{M}$ consisting of those sources whose indexing class is 1.

(e) For each $\mathbf{A}$-object A the isomorphism classes of E-quotients of A form a (large) complete lattice, where $\sup\{(Q_i, e_i) \mid i \in I\} = (Q, e)$ iff there is an $(E, \mathcal{M})$-factorization $A \xrightarrow{e_i} Q_i = A \xrightarrow{e} Q \xrightarrow{m_i} Q_i$.

72 Remark In the presence of products in $\mathbf{A}$ any factorization structure for morphisms (E, M) can be extended to a factorization structure $(E, \mathcal{M})$ of sources, provided that $\mathbf{A}$ is E-cowellpowered. For details see [1, 15.19, 15.20].

73 Remark Let $\mathbf{A}$ be an $(E, \mathcal{M})$-category. A subcategory $\mathbf{B}$ of $\mathbf{A}$ is called *closed under $(E, \mathcal{M})$-factorizations*, iff for each $(E, \mathcal{M})$-factorization $B \xrightarrow{e} A \xrightarrow{m_i} B_i$ of some source $(B, (B \xrightarrow{f_i} B_i)_{i \in I})$ with B and all B_i in $\mathbf{B}$, the object A belongs to $\mathbf{B}$. Every full isomorphism closed subcategory $\mathbf{B}$ of an $(E, \mathcal{M})$-category $\mathbf{A}$, which is closed under $(E, \mathcal{M})$-factorizations, is an $(E_{\mathbf{B}}, \mathcal{M}_{\mathbf{B}})$-category for $E_{\mathbf{B}}$ the class of all morphisms in E belonging to $\mathbf{B}$ and $\mathcal{M}_{\mathbf{B}}$ the collection of all sources $\mathcal{M}$ belonging to $\mathbf{B}$.

Recall that a reflective subcategory $\mathbf{B}$ of an (E, M)-structured category $\mathbf{A}$ is called *E-reflective* if all reflection morphisms belong to E. In what follows we always assume that $\mathbf{B}$ is a full and isomorphism closed subcategory.

The following *characterizations of E-reflective subcategories* are frequently used:

74 Proposition ([1, 16.17]) *Let $\mathbf{A}$ be an $(E, \mathcal{M})$-category. Then the following are equivalent for any full subcategory $\mathbf{B}$ of $\mathbf{A}$.*

1. $\mathbf{B}$ is E-reflective in $\mathbf{A}$, that is, $\mathbf{B}$ is reflective in $\mathbf{A}$ and every reflection map belongs to E.

2. $\mathbf{B}$ is closed in $\mathbf{A}$ under $\mathcal{M}$-sources.

3. $\mathbf{B}$ is closed in $\mathbf{A}$ under M-subobjects and, if A is an $\mathbf{A}$-object and if (Q, e) is the supremum of all E-quotients $(A, A \xrightarrow{q} B)$ of A with B in $\mathbf{B}$, then Q belongs to $\mathbf{B}$.

In case that $\mathbf{A}$ has products, but only a factorization structure (E, M) for morphisms such that $\mathbf{A}$ is E-co-wellpowered, condition 2. above can be replaced by

2'. $\mathbf{B}$ is closed in $\mathbf{A}$ under products and M-subobjects.

In more detail, the reflection $A \xrightarrow{r} RA$ of an **A**-object A is, in the situation of 2., given by the $(E, \mathcal{M})$-factorization $A \xrightarrow{r} RA \xrightarrow{m_f} B_f$ of the source of all morphisms $A \xrightarrow{f} B_f$ with B_f in **B**. In the situation of 3 it is $A \xrightarrow{e} Q$.

75 Remark A more intuitive description of the above result is obtained easily: under the assumptions of the Proposition above, $r\colon A \to RA$ is the **B**-reflection of A iff $r\colon A \to RA$ is the largest E-quotient of A which belongs to the subcategory **B**. Or; dually: An M-coreflection of A is given by the largest M-subobject of A which belongs to **B**. We note, in view of Lemma 60 however, that the existence of such a largest subobject with respect to some class M of monomorphisms does not necessarily imply coreflectivity: M needs to be part of a factorization structure for morphisms.

76 Proposition ([1, 16B]) *If* **A** *is an extremally epireflective subcategory of an* $(ExtrEpi, MonoSource)$*-category* **B***, then* **A** *is an* $(ExtrEpi, MonoSource)$*-category. In particular its embedding* E *preserves and reflects extremal epimorphisms.*

Extremally monadic functors

77 Definition A monad (T, η, μ) on an $(ExtrEpi, MonoSource)$-category **X** is called an *extremal monad*, if T preserves extremal epimorphisms.

A monadic functor $U\colon \mathbf{A} \to \mathbf{X}$, where **X** is an $(ExtrEpi, MonoSource)$-category, is called *extremally monadic*, provided that its associated monad is extremal.

78 Proposition *Let* **X** *be an* $(ExtrEpi, MonoSource)$*-category and* $U\colon \mathbf{A} \to \mathbf{X}$ *a monadic functor. Then the following conditions are equivalent:*

1. *U is extremally monadic.*
2. *U preserves extremal epimorphisms.*

Proof Let $\mathbb{T} = (T, \mu, \eta)$ the monad associated to U, hence $T = UF$, where F is left adjoint to U.

1. implies 2.: Let $(X, \xi) \xrightarrow{e} (Y, \upsilon)$ be an extremal epimorphism in $X^{\mathbb{T}}$ and $x \xrightarrow{r} Z \xrightarrow{m} Y$ the (extremal epi, mono)-factorization of Ue. We then have the following commutative diagram in **X**

$$\begin{tikzcd}
TX \arrow[r, "Tr"] \arrow[d, "\xi"'] & TZ \arrow[r, "Tm"] \arrow[d, dotted, "\zeta"] & TY\upsilon \arrow[d] \\
X \arrow[r, "r"'] & Z \arrow[r, "m"'] & Y
\end{tikzcd}$$

Here the dotted 'diagonal' ζ exists, since Tr is an extremal epimorphism by assumption 1. This makes $(X, \xi) \xrightarrow{r} (T, \zeta) \xrightarrow{m} (Y, \upsilon)$ a factorization of e with m a monomorphism in $\mathbf{X}^{\mathbb{T}}$. Since e is extremal, m is an isomorphism, firstly in $\mathbf{X}^{\mathbb{T}}$, but then in **X**, too. Thus, e is an extremal epimorphism in **X**.

2. implies 1.: Let $X \xrightarrow{e} Y$ be an extremal epimorphism in **X**. It suffices to show that $Fe = (TX, \mu_X) \xrightarrow{Te} (TY, \mu_Y)$ satisfies the extremality condition (Fe is an epimorphism, since left adjoints preserve these). Indeed, if $(TX, \mu_X) \xrightarrow{g} (Z, \zeta) \xrightarrow{m} (TY, \mu_Y)$ is a factorization of Fe

with a monomorphism m in $\mathbf{X}^{\mathbb{T}}$, we have the following commutative diagram in $\mathbf{X}$

$$\begin{tikzcd} X \arrow[rr, "e"] \arrow[d, "\eta_X"'] & & Y \arrow[d, "\eta_Y"] \arrow[dl, dotted, "d"] \\ TX \arrow[r, "Ug"'] \arrow[rr, bend right, "Te"'] & Z \arrow[r, "Um"'] & TY \end{tikzcd}$$

where the dotted diagonal d exists since e is an extremal epimorphism in $\mathbf{X}$ by assumption and the right adjoint U sends the monomorphism m in $\mathbf{X}^{\mathbb{T}}$ to a monomorphism in $\mathbf{X}$. The homomorphic extension $d^\sharp\colon (TY, \mu) \to (Z, \zeta)$ the satisfies the equality

$$Um \circ Ud^\sharp \circ \eta_Y = Um \circ d = \eta_Y = U\mathsf{id}_{(TY,\mu)} \circ \eta_Y$$

from which $m \circ d^\sharp = \mathsf{id}_{(TY,\mu)}$ follows. Since m is a monomorphism by assumption, it thus is an isomorphism, as was to be shown. □

79 Example Let $\mathbf{X}$ be an $(ExtrEpi, MonoSource)$-category and $T\colon \mathbf{X} \to \mathbf{X}$ a functor which preserves extremal epimorphisms. Then the forgetful functor $U\colon \mathbf{Alg}T \to \mathbf{X}$ is extremally monadic, provided that U has a left adjoint. Indeed, since U is monadic (see Proposition 81) we only need to show by the above that U preserves extremal epimorphisms.

Now, by Proposition 81 (4.), $\mathbf{Alg}T$ is an $(U^{-1}[ExtrEpi(\mathbf{A})], U^{-1}[MonoSource(\mathbf{A})])$-category. Since U is a faithful right adjoint, U preserves and reflects monosources, thus, $U^{-1}[MonoSource(\mathbf{A})] = MonoSource(\mathbf{Alg}T)$. By Lemma 71 $U^{-1}[ExtrEpi(\mathbf{A})]$ then is the class of all extremal epimorphisms in $\mathbf{Alg}T$. In particular, U preserves these[11].

Besides the usual properties of monadic functors, like creation of limits, extremally monadic functors have the following ones important for our context.

80 Proposition *Let $U\colon \mathbf{A} \to \mathbf{X}$ be an extremally monadic functor. Then the following hold:*

1. *U preserves and reflects extremal epimorphisms.*
2. *$\mathbf{A}$ is an $(ExtrEpi, MonoSource)$-category and its respective factorization structure is created by U, i.e.,*

$$(U^{-1}[ExtrEpi(\mathbf{X})], U^{-1}[MonoSource(\mathbf{X})]) = (ExtrEpi(\mathbf{A}), MonoSource(\mathbf{A}))$$

   and this is a factorization structure on $\mathbf{A}$.

3. *Every extremal epimorphism $A \xrightarrow{e} B$ in $\mathbf{A}$ is U-final.*
4. *Every monosource $(A, (A \xrightarrow{m_i} A_i)_{i \in I})$ is U-initial.*
5. *If $\mathbf{X}$ has colimits, so has $\mathbf{A}$.*

[11]For the categorical specialist: What we have proved here is the statement: *If T is an endofunctor on an $(ExtrEpi, MonoSource)$-category which preserves extremal epimorphisms, then the monad freely generated by T is an extremal monad.*

We only prove the last statement here, because it is of crucial importance in Section 3.

This construction of colimits in $\mathbf{A}$, due to [9], is nothing but a categorical abstraction of the familiar colimit construction in Birkhoff varieties (see e.g. [10, Thm. 2.11]): Let $U\colon \mathbf{A} \to \mathbf{C}$ be an extremally monadic functor. Then a colimit of a diagram $D\colon \mathbf{I} \to \mathbf{A}$ (with $D_i := D(i)$ for $i \in \mathsf{ob}\mathbf{I}$) can be constructed as follows:

1. Chose a colimit $\big(C, (UD_i \xrightarrow{\mu_i} C)_{i\in\mathsf{ob}\mathbf{I}}\big)$ of UD in $\mathbf{C}$.
2. Choose a U–universal morphism $u_C\colon C \to UC^\sharp$.
3. Form the collection of all $\mathbf{A}$–morphisms $f_j\colon C^\sharp \to A_j$ $(j \in J)$ such that, for each $i \in \mathsf{ob}\mathbf{I}$,
$$UD_i \xrightarrow{\mu_i} C \xrightarrow{u_C} UC^\sharp \xrightarrow{Uf_j} UA_j$$
is the U-image of some $\mathbf{A}$-morphism $h_{ij}\colon D_i \to A_j$ (note, that J may be a proper class).
4. Factorize the source $\big(C^\sharp, (f_j)_{j\in J}\big)$ as
$$C^\sharp \xrightarrow{q} A \xrightarrow{m_j} A_j$$
with an extremal epimorphism q and a monosource $\big(A, (m_j)_{j\in J}\big)$. This is possible by our assumptions.

Then, again by the assumptions on U, for each $i \in \mathsf{ob}\mathbf{I}$, the morphism
$$UD_i \xrightarrow{\mu_i} C \xrightarrow{u_C} UC^\sharp \xrightarrow{Uq} UA \tag{11}$$
is the U–image of a (unique) $\mathbf{A}$–morphism $D_i \xrightarrow{\lambda_i} A$. The sink $\big(A, (D_i \xrightarrow{\lambda_i} A)_{i\in I}\big)$ then is a colimit of D.

4.4 Functor (co)algebras and equifiers: Properties

Functor algebras and coalgebras as well as equifers have been introduced in Section 2.1 (see Definitions 41 and 42). We here collect some of their properties, which are important in our context.

81 Proposition *The underying functor $U\colon \mathbf{Alg}T \to \mathbf{A}$*

1. *reflects isomorphisms,*
2. *creates limits,*
3. *creates those colimits which are preserved by T,*[12]
4. *lifts a factorization structure $(E, \mathcal{M})$ on $\mathbf{A}$ to a factorization structure $(U^{-1}[E], U^{-1}[\mathcal{M}])$ on $\mathbf{Alg}T$, provided that $T[E] \subset E$,*
5. *is monadic, provided that it has a left adjoint,*
6. *is extremally monadic, provided that U has a left adjoint, and that, moreover, $\mathbf{A}$ is an $(ExtrEpi, MonoSource)$-category and T preserves extremal epimorphisms.*[13]

[12]More precisely, by this is meant the following: If D is diagram in $\mathbf{Alg}T$ and $(L, (D_i \xrightarrow{\lambda_i} L)_i)$ is a colimit of UD in $\mathbf{A}$, then T preserves this colimit.

[13]See Section 4.3 for the respective definitions.

Because of the extensive use of these results in Section 2.3 we add the following arguments:

To 3.: If D is a diagram in $\mathbf{Alg}T$ with $D(i) = (D_i, \delta_i)$ and $(L, (D_i \xrightarrow{\lambda_i} L)_i)$ is a colimit of UD in $\mathbf{A}$, then — since $(TL, T\lambda_i)$ is a colimit of UD by assumption — the family $(\lambda_\circ \delta_i)_i$ induces a unique morphism $TL \xrightarrow{\delta} L$ such that the diagram

$$\begin{tikzcd} TL \arrow[r, dotted, "\delta"] & L \\ L \arrow[u, "T\lambda_i"] \arrow[r, "\delta_i"] & D_i \arrow[u, "\lambda_i"'] \end{tikzcd}$$

commutes. Thus, (L, δ) is the unique T-algebra such that all $\lambda_i \colon D(i) \to (L, \delta)$ are homomorphisms. One now easily checks that $((L, \delta), (\lambda_i)_i)$ is a colimit of D.

To 4: In order to construct a factorization of a family of T-algebra homomorphisms $f_i \colon (A, \alpha) \to (B_i, \beta_i)$ out of an $(E, \mathcal{M})$-factorization $A \xrightarrow{e} Z \xrightarrow{m_i} B_i$ of $(f_i)_i$ one needs to find a T-algebra structure $\zeta \colon TZ \to Z$ such that the following diagram commutes.

$$\begin{tikzcd} TA \arrow[r, "Te"] \arrow[d, "\alpha"'] & TZ \arrow[r, "Tm_i"] \arrow[d, dotted, "\zeta"] & TB_i \arrow[d, "\beta_i"] \\ A \arrow[r, "e"'] & Z \arrow[r, "m_i"'] & B_i \end{tikzcd}$$

This can be obtained by a diagonal fill in (see Definition 68), since $T[E] \subset E$. To check the diagonal condition for $(U^{-1}[E], U^{-1}[\mathcal{M}])$, more precisely to see, that the obvious diagonal one obtains in $\mathbf{A}$ is a a homomorphism, requires the following fact, which is useful otherwise, too.

To 5.: This follows as in the proof of Proposition 47. For the proof of 6. see Example 79.

82 Fact A morphism $f \colon (A, \alpha) \to (B, \beta)$ in $\mathbf{Alg}T$ is

1. initial, if f is a monomorphism in $\mathbf{A}$,
2. final, if Tf is an epimorphism in $\mathbf{A}$.

83 Remark Since the category $\mathbf{Coalg}T$ of functor coalgebras coincides with $(\mathbf{Alg}T^{\mathsf{op}})^{\mathsf{op}}$ by definition, Propostion 81 and Fact 82 can be dualized and so yield results on categories coalgebras.

Concerning equifers the following hold.

84 Lemma *An equifier* $\mathbf{Eq}\big((\phi_\kappa, \psi_\kappa)_{\kappa \in K}\big)$ *is closed in* $\mathbf{A}$ *under*

1. *those subobjects* $S \xrightarrow{m} A$ *where, for all* $\kappa \in K$, $G_\kappa m$ *is a monomorphism in* $\mathbf{B}_\kappa$*; more generally, under sources* $(S, (S \xrightarrow{m_i} A_i)_{i \in I})$ *for which each source* $(G_\kappa S, (G_\kappa S \xrightarrow{G_\kappa m_i} G_\kappa A_i)_{i \in I})$ *is a monosource.*[14]
2. *those limits which are preserved by all* G_κ*,*
3. *those colimits which are preserved by all* F_κ*.*

[14]See Section 4.3 for a definition of this kind of closure.

Moreover, if $\mathbf{A}$ *is an* $(E, \mathcal{M})$*-category, then* $\mathbf{Eq}\big((\phi_\kappa, \psi_\kappa)_{\kappa \in K}\big)$ *is closed in* $\mathbf{A}$ *under* $(E, \mathcal{M})$*-factorizations, provided that* $F_\kappa[E] \subset Epi(\mathbf{B}_\kappa)$ *or* $G_\kappa[\mathcal{M}] \subset Monosource(\mathbf{B}_\kappa)$*, for all* $\kappa \in K$.

Again, because of the extensive use of these facts in Section 2.3 we add some arguments:

To 1.: Let $S \xrightarrow{m} A$ be a monomorphism in $\mathbf{A}$ where A belongs to $\mathbf{Eq}\big((\phi_\kappa, \psi_\kappa)_{\kappa \in K}\big)$. In the commutative diagram

$$\begin{tikzcd}
F_\kappa S \arrow[r, "F_\kappa m"] \arrow[d, "\phi_{\kappa S}"', shift right] \arrow[d, "\psi_{\kappa S}", shift left] & F_\kappa A \arrow[d, "\phi_{\kappa A} = \psi_{\kappa A}"] \\
G_\kappa S \arrow[r, "G_\kappa m"] & G_\kappa A
\end{tikzcd}$$

we have $\phi_{\kappa A} = \psi_{\kappa A}$ for all κ. Since each $G_\kappa m$ is a monomorphism, the claim follows.

To 2.: We sketch the case of products; the general case is similar. Thus, let $(A \xrightarrow{\pi_i} A_i)_i$ be a product in $\mathbf{A}$ with each A_i belonging to $\mathbf{Eq}\big((\phi_\kappa, \psi_\kappa)_{\kappa \in K}\big)$. By assumption $(G_\kappa A, (G_\kappa A \xrightarrow{G_\kappa \pi_i} G_\kappa A_i)_i)$ is a product in $\mathbf{B}_\kappa$ for all κ. Now argue as in 1. with $S \xrightarrow{m} A$ replaced by $A \xrightarrow{\pi_i} A_i$.

Extensive use of the above results is made in the proof of Proposition 47.

Concerning free functor algebras we need — besides the crucial existence results of Corollary 94 below — the following simple *free algebra construction.*

85 Lemma *Let* $\mathbf{A}$ *have finite and countable coproducts and* $T \colon \mathbf{A} \to \mathbf{A}$ *preserve these. Then the free* T*-algebra* $(X^\sharp, \alpha_X)$ *over an object* X *in* $\mathbf{A}$ *is given by* $X^\sharp = \coprod_{n \in \mathbb{N}} T^n(X)$ *with action* $\alpha_X \colon T(X^\sharp) \to X^\sharp$ *determined by commutativity of the following diagram (for all* $n > 0$*), where we write* $T^0 = \mathsf{id}_{\mathbf{A}}$*,* $T^{n+1} = T \circ T^n$ *and where* ι_n *is the* n^{th} *coproduct injection.*

$$\begin{tikzcd}
T^n X \arrow[r, "\mathsf{id}_{T^n X}"] \arrow[d, "T\iota_{n-1}"'] & T^n X \arrow[d, "\iota_n"] \\
TX^\sharp \arrow[r, "\alpha_X"] & X^\sharp
\end{tikzcd}$$

The assignment $X \mapsto (X^\sharp, \alpha_X)$ *thus defines a left adjoint of the forgetful functor* $\mathbf{Alg}T \to \mathbf{A}$ *with units of this adjunction being the* 0^{th} *coproduct injections* ι_0.

In particular, the forgetful functor $\mathbf{Alg}T \to \mathbf{A}$ *is monadic.*

Proof Since T preserves coproducts the left column of the diagram is a coproduct, and this implies existence of α_X.

If now $f \colon X \to H$ is a $\mathbf{A}$-morphism where (H, α_H) is a T-algebra, define a family $(f_n)_{n \in \mathbb{N}}$ of $\mathbf{A}$-morphisms by

$$\begin{aligned}
f_0 &:= T^0 X = X \xrightarrow{f} H \\
f_{n+1} &:= T^{n+1} X = T(T^n X) \xrightarrow{T f_n} TH \xrightarrow{\alpha_H} H
\end{aligned}$$

The coproduct property yields a unique $\mathbf{A}$-morphism $f^\sharp \colon \coprod_{n \in \mathbb{N}} T^n X \to H$ with $f^\sharp \circ \iota_n = f_n$ for all n.

It is now easy to see that $f^\sharp$ is the (unique) homomorphic extension of f. $\square$

4.5 Accessible and locally presentable categories

Because of its importance in this section we recall the definition of a *directed colimit* (also called *direct* or *inductive limit* as follows. Here λ always denotes a regular cardinal.

A (λ-)*directed set* is a poset in which every finite subset (every subset of cardinality $< \lambda$) has an upper bound. A (λ-)*directed diagram* in a category $\mathbf{A}$ then is a functor $\mathbf{I} \xrightarrow{D} \mathbf{A}$, where $\mathbf{I}$ is a (λ-) directed set considered as a category. In other words, D is a family of $\mathbf{A}$-morphisms $(D_i \xrightarrow{d_{ij}} D_j)_{i,j\in I, i\leq j}$ with $d_{jk} \circ d_{ij} = d_{ik}$ for all $i \leq j \leq k$ and $d_{ii} = \mathsf{id}_{D_i}$ for each i.

A (λ-)*directed colimit* is colimit of a (λ-) directed diagram. More in detail, if $\mathbf{I} \xrightarrow{D} \mathbf{A}$ is a directed diagram, its colimit is a source $(C, (D_i \xrightarrow{\mu_i} D_i)_{i\in I})$ with the following properties

1. for all $i, j \in I,\ i \leq j$ one has $\mu_j \circ d_{i,j} = \mu_i$;

2. if $(A, (D_i \xrightarrow{f_i} A)_i)$ is a source satisfyiying $f_j \circ d_{i,j} = f_i$ for all $i, j \in I,\ i \leq j$, then there exists a unique $\mathbf{A}$-morphism $C \xrightarrow{f} A$ such that the following diagram commutes.

$$\begin{tikzcd} D_i \arrow[r, "\mu_i"] \arrow[dr, "f_i"'] & C \arrow[d, dotted, "f"] \\ & A \end{tikzcd}$$

86 Definition An object A in a category $\mathbf{A}$ is called λ*-presentable*, provided that the hom-functor $\hom(A, -)\colon \mathbf{A} \to \mathbf{Set}$ preserves λ-directed colimits.

A category $\mathbf{A}$ is called *locally* λ*-presentable*, provided that

1. $\mathbf{A}$ is cocomplete and

2. $\mathbf{A}$ has a small subcategory of λ-presentable objects, such that every object in $\mathbf{A}$ is a λ-directed colimit of those.

A category $\mathbf{A}$ is called *locally presentable*, provided that it is *locally* λ*-presentable*, for some λ.

The notion of *accessible category* is obtained from the above, when one requires, instead of cocompleteness, only existence of λ-directed colimits, for some λ.

A λ*-accessible functor* is a functor between λ-accessible categories which preserves λ-directed colimits. A functor is called accessible, if it is λ-accessible for some λ.

Instead of $\aleph_0$-presentable one rather uses the term *finitely presentable*; instead of calling a functor $\aleph_0$-accessible it is called *finitary.* A finitary functor is λ-accessible for each λ.

87 Examples A group P is finitely presentable in the sense of the definition above, i.e., the hom-functor $\hom(P, -)$ on the category of groups preserves directed colimits, if and only if P is a finitely presentable group in the algebraic sense (which motivates the terminology used here). The functor $(-)^n$, raising to the n^{th} power (n any cardinal), on the category of sets is finitely accessible if n is finite; in other words: a set is finitely presentable in the sense of the definition above, if and only if it is finite.

Every variety and every quasi variety in the sense of Universal Algebra is a locally finitely presentable category, as are the categories of partially ordered sets and of small categories. A partially ordered set, considered as a category, is locally presentable iff it is a complete lattice; it is locally finitely presentable iff it is a complete and algebraic lattice.

The category of fields is finitely accessible, but it is far from being locally presentable (it not even has products!). Also, the category of rings with injective homomorphisms only is accessible, but not locally presentable.

Because of its frequent use in the main part of this paper we state explicitely

88 Fact 1. On a locally finitely presentable category the functor $(-)^n$, raising to the n^{th} power, is finitely accessible for every $n \in \mathbb{N}$. This is a special instance of property 5 of Proposition 90 below.

2. On every category $\mathbf{Mod}_R$ the functor $\otimes^n$, raising to the n^{th} tensor power, is finitely accessible for every $n \in \mathbb{N}$. An elementary argument (only for $n = 2$) is contained in the appendix of Part II. For a more elegant argument see [15].

The following propositions are highly non-trivial. See e.g. [3] for proofs.

89 Proposition *For a λ-accessible category $\mathbf{A}$ the following are equivalent:*

1. *$\mathbf{A}$ is cocomplete.*
2. *$\mathbf{A}$ is complete.*
3. *$\mathbf{A}$ is locally λ-presentable.*

90 Proposition *Any locally λ-presentable category $\mathbf{A}$ has the following properties:*

1. *$\mathbf{A}$ is complete and cocomplete.*
2. *$\mathbf{A}$ has a strong generator consisting of λ-presentable objects.*
3. *$\mathbf{A}$ is wellpowered and cowellpowered.*
4. *$\mathbf{A}$ has the extremal as well as the co-extremal factorization structure for morphisms.*
5. *In $\mathbf{A}$ λ-small limits*[15] *and λ-directed colimits commute.*

The following is an immediate consequence (use Remark 72 and its dual as well as Lemma 71):

91 Corollary *Every locally presentable category $\mathbf{A}$ is an (ExtrEpi,MonoSource)- as well an an (EpiSink,ExtrMono)-category.*

Concerning the closure properties of accessible categories mentioned above we state those relevant to our context.

92 Proposition *1. Let $T\colon \mathbf{A} \to \mathbf{A}$ be an accessible functor. Then the categories $\mathbf{Alg}T$ and $\mathbf{Coalg}T$ are accessible.*

2. Let $\phi_\kappa, \psi_\kappa\colon F_\kappa \Longrightarrow G_\kappa$ $(\kappa \in K)$ be a family of natural transformations between accessible functors $F_\kappa, G_\kappa\colon \mathbf{A} \to \mathbf{B}_t$ $(\kappa \in K)$. Then $\mathbf{Eq}\big((\phi_\kappa, \psi_\kappa)\big)_{\kappa \in K}$ is an accessible category and its embedding into $\mathbf{A}$ is accessible.

3. If $F\colon \mathbf{A} \to \mathbf{C}$ and $G\colon \mathbf{B} \to \mathbf{C}$ are accessible functors with $\mathbf{A}, \mathbf{B}, \mathbf{C}$ locally presentable, then their pullback in $\mathbf{CAT}$ is locally presentable, provided one of the functors F, G is monadic or comonadic[16]*.*

One moreover has

[15] That is, limits of diagrams with less than λ morphisms.

[16] In fact it suffices to ask F or G to be an *isofibration*, i.e. a functor lifting isomorphisms.

93 Proposition *1. Every composition of accessible functors is accessible.*

2. Every accessible functor satisfies the solution set condition.

Note that, by Propositions 89 and 92, one even has

94 Corollary *Let* $T\colon \mathbf{A} \to \mathbf{A}$ *be an accessible functor where* $\mathbf{A}$ *is locally presentable. Then*

1. *the categories* $\mathbf{Alg}T$ *and* $\mathbf{Coalg}T$ *are locally presentable,*
2. *the forgetful functor* $\mathbf{Alg}T \to \mathbf{A}$ *has a left adjoint, and*
3. *the forgetful functor* $\mathbf{Coalg}T \to \mathbf{A}$ *has a right adjoint.*

Proof 1. follows from the facts that $\mathbf{Alg}T$ is complete and $\mathbf{Coalg}T$ is cocomplete (see 89 and its dual) in connection with Proposition 92.

2. follows from Proposition 93, since the forgetful functor of $\mathbf{Alg}T$ preserves limits.

3. follows by means of the Special Adjoint Functor Theorem: $\mathbf{Coalg}T$ has a generator by Proposition 90 and its forgetful functor preserves colimits. □

Statement 1. of this corollary can be improved as follows.

95 Proposition ([2]) *Let* $T\colon \mathbf{X} \to \mathbf{X}$ *be a* λ*-accessible functor with* $\lambda > \aleph_0$ *, where* $\mathbf{X}$ *is locally* λ*-presentable. Then*

1. $\mathbf{Coalg}T$ *is a locally* λ*-presentable category.*
2. *If* T *preserves extremal monomorphisms, then a (full and isomorphism-closed) subcategory* $\mathbf{A}$ *of* $\mathbf{Coalg}T$ *is locally* λ*-presentable, provided that it is closed under coproducts, extremal subobjects and epimorphic quotients.*

Acknowledgment

I am grateful to the anonymous referee, whose scrutiny and constructive comments lead to a noticeable improvement of the presentation of this paper.

References

[1] Adámek, J., Herrlich H., and Strecker, G.E., *Abstract and Concrete Categories*, John Wiley, New York (1990).

[2] Adámek, J. and Porst, H.-E., On Tree Coalgebras and Coalgebra Presentations, *Theor. Comp. Science*, **311**, 257 - 283, (2004).

[3] Adámek, J. and Rosický, J., *Locally Presentable and Accessible Categories.* Cambridge University Press, Cambridge (1994).

[4] Aguiar, M. and Mahajan, S., *Monoidal Functors, Species and Hopf Algebras.* CRM Monograph Series **29**. American Mathematical Society, Providence, RI, 2010.

[5] Barr, M., Coalgebras over a Commutative Ring, *J. Algebra* **32**, 600–610 (1974).

[6] Brzezinski, T. and Wisbauer, R., *Corings and Comodules*, Cambridge University Press, Cambridge (2003).

[7] Chirvasitu, A., On epimorphisms and monomorphisms of Hopf algebras, *J. Algebra* **323**, 1593–1606 (2010)

[8] Dăscălescu, S., et al., *Hopf Algebras – An Introduction*, Marcel Dekker, New York – Basel (2001).

[9] Herrlich, H., Regular Categories and Regular Functors, *Can. J. Math.* **XXVI**, 709 – 720 (1974).

[10] Jacobson, N., *Basic Algebra II*, W.H. Freeman and Company, San Francisco (1980).

[11] Joyal, A. and Street, R., Braided tensor categories, *Advances in Math.* **102**, 20 – 78 (1993).

[12] Mac Lane, S., *Categories for the Working Mathematician*, 2nd ed., Springer, New York (1998).

[13] Manin, Yu. I., *Quantum groups and noncommutative geometry.* Université de Montréal, Centre de Recherches Mathmatiques, Montreal, QC (1988).

[14] Pareigis, B., *Advanced Algebra*, Lecture Notes TU Munich, http://www.mathematik.uni-muenchen.de/∼pareigis/Vorlesungen/01WS/ advalg.pdf.

[15] Porst, H.–E., On Corings and Comodules, *Arch. Math. (Brno)* **42**, 419–425 (2006).

[16] Porst, H.–E., On Categories of Monoids, Comonoids, and Bimonoids, *Quaest. Math.* **31**, 127–139 (2008).

[17] Porst, H.–E., Fundamental Constructions for Coalgebras, Corings, and Comodules, *Appl. Categorical Struct.*, **16**, 223 – 238 (2008).

[18] Porst, H.–E., Dual Adjunctions Between Algebras and Coalgebras, *Arab. J. Sci. Eng.* **33 - 2 C**, 407 – 411 (2008).

[19] Porst, H.–E., Universal constructions for Hopf algebras, *J. Pure Appl. Algebra* **212**, 2547 – 2554 (2008).

[20] Porst, H.–E., On Subcategories of the Category of Hopf Algebras, *Arab. J. Sci. Eng.* **36**, 1023 – 1029 (2011).

[21] Porst, H.–E., Limits and Colimits of Hopf Algebras, *J. Algebra* **328**, 254–267 (2011).

[22] Porst, H.–E., Takeuchi's Free Hopf Algebra Construction Revisited, *J. Pure Appl. Algebra* **216**, 1768-1774 (2012)

[23] Sweedler, M.E., *Hopf Algebras*, Benjamin, New York (1969).

[24] Street, R., *Quantum Groups*, Cambridge University Press, Cambridge (2007).

[25] Takeuchi, M., Free Hopf Algebras Generated by Coalgebras, *J. Math. Soc. Japan* **23**, 561 – 582 (1971).

Department of Mathematical Sciences, University of Stellenbosch, Stellenbosch, South Africa
porst@math.uni-bremen.de