The Formal Theory of Hopf Algebras

Part II: The case of Hopf algebras

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Abstract

The category $\mathbf{Hopf}_R$ of Hopf algebras over a commutative unital ring R is analyzed with respect to its categorical properties. The main results are: (1) For every ring R the category $\mathbf{Hopf}_R$ is locally presentable, it is coreflective in the category of bialgebras over R, over every R-algebra there exists a cofree Hopf algebra. (2) If, in addition, R is absoluty flat, then $\mathbf{Hopf}_R$ is reflective in the category of bialgebras as well, and there exists a free Hopf algebra over every R-coalgebra. Similar results are obtained for relevant subcategories of $\mathbf{Hopf}_R$. Moreover it is shown that, for every commutative unital ring R, the so-called "dual algebra functor" has a left adjoint and that, more generally, universal measuring coalgebras exist.

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Introduction

The first monograph on Hopf algebra theory (Sweedler 1969 [27]) paid considerable attention to categorical properties of those. Somewhat surprisingly more recent successors like e.g. [11] — though using categorical language throughout — hardly touch these questions. The question we here have in mind are, e.g., *Does the category of Hopf algebras have products* (or, more generally, all limits) and *how are they constructed? Does it have colimits?* Or *Do the naturally occurring functors* (e.g., the embedding of the category of Hopf algebras into that of bialgebras) *have adjoints*? Some scattered results exist, as for example

1. Takeuchi proved the existence of free Hopf algebras over coalgebras (claimed to exist but not proved by Sweedler) in [29].

2. He also proved in that paper that coproducts of Hopf algebras exist and are constructed as for bialgebras.

3. The Hopf envelope, i.e., the reflection from bialgebras into Hopf algebras has been constructed in the 1980s, though its relation to Takeuchi's construction still seems to appear somewhat mysterious, as a remark in [26] shows.

4. Special instances of (co)free Hopf algebras are better known as, e.g., (co)commutative (co)free Hopf algebras over (co)commutative (co)algebras; this might be due to the fact that the tensor product is the product for cocommutative coalgebras and the coproduct for commutative algebras.

A systematic approach to tackle these problems surfaced only recently (see Part I for details). This approach requires not only categorical language but quite a bit of category theory, too. These abstract considerations were the content of *The Formal Theory of Hopf Algebras - Part I* [22], where it became clear in particular that the formal theory of Hopf algebras, when applied to $\mathbf{Mod}_R$, the category of R-modules, requires the following categorical properties of $\mathbf{Mod}_R$:

1. $\mathbf{Mod}_R$ is a locally finitely presentable category.
2. The functors *tensor squaring* $\otimes^2$ and *tensor cubing* $\otimes^3$ preserve directed colimits.
3. Extremal epimorphisms in $\mathbf{Mod}_R$ are stable under tensor squaring, i.e., the functor *tensor squaring* $\otimes^2$ on $\mathbf{Mod}_R$ preserves extremal epimorphisms.

and, occasionally

4. Extremal monomorphisms in $\mathbf{Mod}_R$ are stable under tensor squaring, i.e., the functor tensor squaring $\otimes^2$ on $\mathbf{Mod}_R$ preserves extremal monomorphisms.

While the first three conditions are satisfied by every module category $\mathbf{Mod}_R$, the last one is satisfied if and only if the ring R is absolutely flat (see Appendix).

The paper is organized as follows.

Section 1 contains the explicit translation of the results of Part I to the case of Hopf algebras over a commutative ring R and makes clear in particular, which of those require the additional assumption of R being absolutely flat. It also relates our results to known constructions, in particular to Takeuchi's. Moreover, partly using results of [10], we briefly discuss the various kinds of mono- and epimorphisms in $\mathbf{Hopf}_R$.

Section 2 presents extensions of the results of Section 1 to relevant subcategories of the category of Hopf algebras.

In Section 3 we review the concept of convolution algebra from our abstract point of view. This enables us to discuss, for any commutative until ring R, a generalization of the so called *finite dual* or *Sweedler dual* of an R-algebra and to prove the existence of arbitrary universal measuring coalgebras.

In Section 4 the question is raised whether the approach of this paper might also work in more general situations. We show in particular, that one hardly loses anything when generalizing from symmetric to braided monoidal categories. We close this section by suggesting a way of how to deal with weak Hopf algebras.

The Appendix contains some technical arguments omitted in the course of the text in order to make it as readable as possible.

1 The case of Hopf algebras

In this section we formulate the results of Part I, Section "Main results" for the case of $\mathbb{C} = \mathbf{Mod}_R$, the monoidal category of R-modules, where R is a commutative unital ring.

Recall that $\mathbf{Mod}_R$, as any variety, is a locally finitely presentable category. The extremal epimorphisms in $\mathbf{Mod}_R$ are the surjective R-linear maps; the extremal monomorphisms are the injective R-linear maps. Concerning the additional assumptions used above we note

1. For each $n \in \mathbb{N}$ the functor $\otimes^n$ on $\mathbf{Mod}_R$ preserves directed colimits (see Section 5 for an elementary argument).
2. The functor $\otimes^2$ preserves extremal epimorphisms; in fact more is known to be true: for any pair (f, g) of surjective linear maps its tensor product $f \otimes g$ is surjective.

3. The functor $\otimes^2$ preserves extremal monomorphisms iff R is an absolutely flat ring (see Section 5 in the Appendix).

Thus, in particular the following hold by Proposition I.49.

1 Lemma *1. The category $\mathbf{Alg}_R$ is locally (finitely) presentable and the forgetful functor $U_a\colon \mathbf{Alg}_R \to \mathbf{Mod}_R$ is extremally monadic.*

2. The category $\mathbf{Coalg}_R$ is locally presentable and the forgetful functor $U_c\colon \mathbf{Coalg}_R \to \mathbf{Mod}_R$ is extremally comonadic, provided that R is absolutely flat.

Recall that the category ${}^{\star}\mathbf{Bialg}_R$ has as objects pairs (B, S), where B is an R-bialgebra and as morphisms $S\colon \mathsf{B} \to \mathsf{B}^{\mathsf{op},\mathsf{cop}}$ bialgebra homomorphisms commuting with the S's. By Proposition I.47 and Lemma I.51. we also know:

2 Lemma *The categories $\mathbf{Bialg}_R$ and ${}^{\star}\mathbf{Bialg}_R$ are locally presentable and the forgetful functor ${}^{\star}\mathbf{Bialg}_R \to \mathbf{Bialg}_R$ has a left and a right adjoint.*

As promised in Part I we make explicit the meaning of the Crucial Lemma I.38 for the cases $\mathbf{Mod}_R$ and $\mathbf{Mod}_R^{\mathsf{op}}$. These are the following familiar results (which, however, usually are not seen as dual to each other, as we can do here).

3 Fact Let $\mathsf{B} = (B, m, e, \mu, \epsilon)$ be an R-bialgebra and $S\colon B \to B$ a linear map.

1. If S is an algebra homomorphism $\mathsf{B} \to (\mathsf{B}^a)^{\mathsf{op}}$, the following hold
   (a) $(S * \mathsf{id}_B)(1) = e \circ \epsilon(1) = (\mathsf{id}_B * S)(1)$ and
   (b) $(S * \mathsf{id}_B)(x) = e \circ \epsilon(x) = (\mathsf{id}_B * S)(x)$ and $(S * \mathsf{id}_B)(y) = e \circ \epsilon(y) = (\mathsf{id}_B * S)(y)$ imply $(S * \mathsf{id}_B)(xy) = e \circ \epsilon(xy) = (\mathsf{id}_B * S)(xy)$.
2. If S is a coalgebra homomorphism $\mathsf{B}^c \to (\mathsf{B}^c)^{\mathsf{op}}$ then, with

$$I = \mathrm{im}(S * \mathsf{id} - e \circ \epsilon) \text{ and } J = \mathrm{im}(\mathsf{id} * S - e \circ \epsilon)$$

   $I + J$ is a coideal. If R is absolutely flat this is equivalent to
   (a) $\epsilon[I] = 0 = \epsilon[J]$
   (b) $\mu[I] \subset B \otimes I + I \otimes B$ and $\mu[J] \subset B \otimes J + J \otimes B$.

1. expresses the statement of the Crucial Lemma for the special case $\mathbb{C} = \mathbf{Mod}_R$, while 2. is nothing but the same statement for $\mathbf{Mod}_R^{\mathsf{op}}$: if $(Q, \rho\colon B \to Q)$ is the (multiple) coequalizer of $S * \mathsf{id}$, $\mathsf{id} * S$ and $e \circ \epsilon$ (note that the coequalizer of these maps is the quotient $\rho\colon B \to B/(I + J)$ with I and J as in statement 2 above), its kernel $I + J$ is a coideal. And 2. (a) and (b) express this fact in case R is absolutely flat (see e.g. [9, 40.12]).

We also mention the following application of Lemma I.56

4 Lemma *If R is an absolutely flat ring, then each of the categories $\mathbf{Bialg}_R$, ${}^{\star}\mathbf{Bialg}_R$ and $\mathbf{Hopf}_R$ has a factorization structure (E, M) for morphisms, where E is the class of all surjective homomorphisms and M is the class of all injective homomorphisms in the respective category.*

1.1 Specializing to $\mathbf{Hopf}_R$

By Propositions I.52 and I.53 in connection with Remark I.50 we get, using Lemma 1 above:

5 Proposition *1. The category $\mathbf{Hopf}_R$ is closed in ${}^\star\mathbf{Bialg}_R$ as well as in $\mathbf{Bialg}_R$ under colimits.*

2. The category $\mathbf{Hopf}_R$ is closed in ${}^\star\mathbf{Bialg}_R$ as well as in $\mathbf{Bialg}_R$ under limits, provided that R is absolutely flat.

In case of R being a field k statement 1 above has essentially been an observation of Takeuchi, who proved in [29] that $\mathbf{Hopf}_k$ is closed in $\mathbf{Bialg}_k$ under coproducts (the case of coequalizers being trivial — see page 20). The generalization to arbitrary rings as well as statement 2, however, has only recently been shown by using the methods of this paper (see [20], [4]).

Proposition I.54 then implies:

6 Theorem *For every commutative ring R the category $\mathbf{Hopf}_R$ is locally presentable.*

while

7 Theorem *1. For every ring R the category $\mathbf{Hopf}_R$ is coreflective in ${}^*\mathbf{Bialg}_R$ and, thus, in $\mathbf{Bialg}_R$.*

2. $\mathbf{Hopf}_R$ is reflective in reflective in ${}^*\mathbf{Bialg}_R$ and, thus, in $\mathbf{Bialg}_R$, provided that R is absolutely flat.

is a special instance of Proposition I.49 in connection with Remark I.52.

8 Remark Existence of the Hopf reflection, i.e., of the Hopf envelope, and the Hopf coreflection of a bialgebra B respectively is here shown by using the Special Adjoint functor Theorem and the General Adjoint functor Theorem respectively in connection with Proposition 5. For amore explicit construction (which in case of the coreflection gives a much weaker result) see Section 1.2 below.

While the construction of the Hopf envelope, in case of R being a field, seems to be known at least since [15], coreflectivity has only been shown recently using the methods used here (see [20], [4], [10]).

9 Remark Following up the discussion of Remark I.59 let us note, that indeed the coreflection maps above may not be injective: Consider a Hopf algebra (H, S) and a bi-ideal B in H which is not a Hopf ideal. If $c\colon \widetilde{\mathsf{H}} \to \mathsf{H}/B$ denotes the Hopf coreflection of the bialgebra H/B, then the quotient map $q\colon \mathsf{H} \to \mathsf{H}/B$ factors over c; if now c were injective, it would be an isomorphism, forcing B to be a bi-ideal. That ideals of this kind exist has been shown in [16].

Dually, consider a Hopf algebra (H, S) and a sub-bialgebra B of H which is not a Hopf algebra. If $r\colon \mathsf{B} \to \widetilde{\mathsf{B}}$ denotes the Hopf coreflection of B, then the embedding $i\colon \mathsf{B} \to \mathsf{H}$ factors over r; if now r were surjective, it would be an isomorphism, forcing B to be a Hopf algebra. This situation exists: simply take a submonoid M of a group G and consider the semigroup algebras k^M and k^G with their canonical bialgebra structure; then k^M is a sub-bialgebra of k^G and not a Hopf algebra, while k^G is a Hopf algebra (see [11, Sect. 4.3]).

Finally, Theorem I.54 specializes to

10 Theorem *1. The forgetful functor $V_a\colon \mathbf{Hopf}_R \to \mathbf{Alg}_R$ has a right adjoint and, thus, is comonadic.*

2. The forgetful functor $V_c\colon \mathbf{Hopf}_R \to \mathbf{Coalg}_R$ has a left adjoint and, thus, is monadic, provided that R is absolutely flat.

Free and cofree Hopf monoids here are obtained by composition of adjunctions. This is in detail:

11 Fact 1. For every commutative ring R the forgetful functor $V_a\colon \mathbf{Hopf}_R \to \mathbf{Alg}_R$ has a right adjoint, and the cofree Hopf algebra over an algebra M can be constructed stepwise as follows:

(a) Form M^*, the cofree bialgebra over M, which is the monoidal lift of the cofree coalgebra over M;

(b) then form the cofree *-bialgebra $(\overline{\mathsf{M}^*}, S)$ over M^* according to Lemma 2, that is, adjoin cofreely a potential antipode;

(c) finally form the Hopf coreflection $Cov(\overline{\mathsf{M}^*}, S)$ of $(\overline{\mathsf{M}^*}, S)$ (see Theorem 7).

and $Cov(\overline{\mathsf{M}^*}, S)$ is the cofree Hopf algebra over M.

2. If R is absolutely flat, then the forgetful functor $V_c\colon \mathbf{Hopf}_R \to \mathbf{Coalg}_R$ has a left adjoint, and the free Hopf algebra over a coalgebra C can dually be constructed stepwise as follows:

(a) Form $\mathsf{C}^\sharp$, the free bialgebra over C, which is the monoidal lift of the free algebra over C;

(b) then form the free *-bialgebra $(\widetilde{\mathsf{C}^\sharp}, S)$ over $\mathsf{C}^\sharp$ according to Lemma 2, that is, adjoin freely a potential antipode;

(c) finally, form the Hopf reflection $Env(\widetilde{\mathsf{C}^\sharp}, S)$ of $(\widetilde{\mathsf{C}^\sharp}, S)$ (see Theorem 7).

Then $Env(\widetilde{\mathsf{C}^\sharp}, S)$ is the free Hopf algebra over C.

Again, existence of free Hopf algebras in case of R being a field is known from Takeuchi's paper [29]. Existence of cofree Hopf algebras again has only recently been shown using the methods used here (see [20], [4]).

1.2 The relation to known constructions

For relating the constructions of the Hopf reflection and coreflection of a bialgebra to others appearing in the literature, we must restrict ourselves to the case of an absolutely flat ring R, since we need the factorization structure on $^\star\mathbf{Bialg}_R$ according to Lemma 4.

In this case the Hopf coreflection of a bialgebra can be constructed stepwise by first constructing the cofree *-bialgebra $(\bar{\mathsf{B}}, S)$ (see Lemma 2) and then (see Remark I.61) forming the largest sub-coalgebra of $\bar{\mathsf{B}}^c$, which is a Hopf algebra, i.e., which is contained in the equalizer of the maps $S * \mathsf{id}, \mathsf{id} * S, e \circ \epsilon$. This is the Hopf coreflection of B.

This description of the Hopf coreflection of a bialgebra is given in [10]. We believe, however, that the proof of the existence of the Hopf (co)reflection as given in for Theorem 7 is much simpler (moreover, it does not need the restrictive condition on R of being absolutely flat!).

By duality one gets a corresponding description of the Hopf envelope (Hopf reflection) of a bialgebra B. Construct first the free *-bialgebra $(\tilde{\mathsf{B}}, S)$ (see Lemma 2) and then (see again Remark I.61) the largest algebra quotient of $\tilde{\mathsf{B}}^a$, which is a Hopf algebra, i.e., which is a module quotient of the (multiple) coequalizer of $S * \mathsf{id}$, $\mathsf{id} * S$ and $e \circ \epsilon$ in $\mathbf{Mod}_R$. As already mentioned in Fact 3 above, this coequalizer is the module quotient $B \to B/(I + J)$, where $I = \mathrm{im}(S * \mathsf{id} - e \circ \epsilon)$ and $J = \mathrm{im}(\mathsf{id} * S - e \circ \epsilon)$. Thus, the largest quotient of $\tilde{\mathsf{B}}^a$, which is a quotient of this coequalizer, is $\mathsf{H}^a/\Im$ where $\Im$ is the ideal generated by $I + J$.

This ideal has first been used by Takeuchi in his construction of free Hopf algebras [29], and later in the construction of the *Hopf envelope*, i.e., the Hopf reflection of a bialgebra (see e.g. [18]). We therefore define:

12 Definition Let B be a bialgebra and $S\colon B \to B$ a linear map. The ideal $\Im$ in B^a generated by the submodules

$$I = \mathrm{im}(S * \mathsf{id} - e \circ \epsilon) \text{ and } J = \mathrm{im}(\mathsf{id} * S - e \circ \epsilon)$$

of B will be called the *Takeuchi ideal* of the pair (B, S).

Obviously, it would suffice to take, as a generating set of the ideal $\Im$, the set of all elements $S * \mathsf{id}(x) - e \circ \epsilon(x)$ and $\mathsf{id} * S(x) - e \circ \epsilon(x)$ where x belongs to (a generating set of) B.

It is clear from the discussion so far, that the Takeuchi ideal $\Im$ of (B, S) is a bi-ideal in B and $\mathsf{H}/\Im$ is a Hopf algebra.

We thus get the familiar description of the Hopf envelope (see e.g. [18]) as follows.

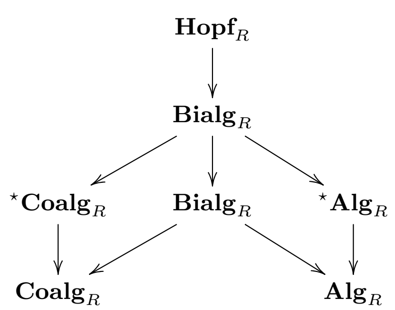
13 Fact Let R be an absolutely flat ring. Then the Hopf reflection of bialgebra B is obtained by first forming the free *-bialgebra $(\mathsf{B}^\sharp, S)$ over B and then forming the quotient $(\mathsf{B}^\sharp)^a/\Im$ of its algebra part $(\mathsf{B}^\sharp)^a$ modulo its Takeuchi ideal.

In order to relate the description of the free Hopf algebra as in Fact 11 to Takeuchi's construction, and also for avoiding the shortcoming of the above proof mentioned in Part I we introduce the following categories.

14 Definition A *-*algebra over* R is a pair (M, S), where M is an R-algebra and S is an algebra homomorphism $\mathsf{M} \to \mathsf{M}^{\mathsf{op}}$. A homomorphism $(\mathsf{M}, S) \to (\mathsf{M}', S')$ is an algebra homomorphism $\mathsf{M} \xrightarrow{f} \mathsf{M}'$ satisfying $S' \circ f = f \circ S$. *-algebras over R constitute the category $^\star\mathbf{Alg}_R$.

A *-*coalgebra over* R is a pair (C, S), where C is an R-algebra and S is an coalgebra homomorphism $\mathsf{C} \to \mathsf{C}^{\mathsf{cop}}$. A homomorphism $(\mathsf{C}, S) \to (\mathsf{C}', S')$ is a coalgebra homomorphism $\mathsf{C} \xrightarrow{f} \mathsf{C}'$ satisfying $S' \circ f = f \circ S$. *-coalgebra over R constitute the category $^\star\mathbf{Coalg}_R$.

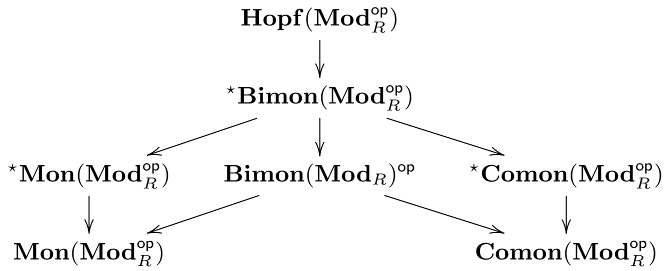
Obviously, $^\star\mathbf{Alg}_R = \mathbf{Alg}(-)^{\mathsf{op}}(= \mathbf{Coalg}(-)^{\mathsf{op}})$ for the functor $(-)^{\mathsf{op}}$ on $\mathbf{Alg}_R$, while $^\star\mathbf{Coalg}_R = \mathbf{Alg}(-)^{\mathsf{cop}}(= \mathbf{Coalg}(-)^{\mathsf{cop}})$ for the functor $(-)^{\mathsf{cop}}$ on $\mathbf{Coalg}_R$ (see Part I, Appendix, for the definitions of functor algebras and coalgebras). We display the categories just introduced in the following commutative diagram of categories and functors, all of which indicate the obvious forgetful ones.



15 Remark It is obvious that one can define categories ${}^{\star}\mathbf{Comon}(\mathbb{C}^{\mathsf{op}})$ and $({}^{\star}\mathbf{Mon}\mathbb{C})^{\mathsf{op}}$ the same way for an arbitrary symmetric monoidal category $\mathbb{C}$ instead of $\mathbf{Mod}_R$ and then gets, with $\mathbb{C} = \mathbf{Mod}_R$

$$ {}^{\star}\mathbf{Comon}(\mathbf{Mod}_R^{\mathsf{op}}) = ({}^{\star}\mathbf{MonMod}_R)^{\mathsf{op}} = ({}^{\star}\mathbf{Alg}_R)^{\mathsf{op}} $$

and that the following digram is the dualization of the above.



16 Proposition *For every commutative ring R the following hold.*

1. (a) ${}^{\star}\mathbf{Alg}_R$ *and* ${}^{\star}\mathbf{Coalg}_R$ *are locally presentable categories.*
   (b) *Their forgetful functors into* $\mathbf{Alg}_R$ *and* $\mathbf{Coalg}_R$ *respectively have left and right adjoints and, thus, are monadic and comonadic.*
2. (a) *The forgetful functor* ${}^{\star}\mathbf{Bialg}_R \xrightarrow{|-|} {}^{\star}\mathbf{Coalg}_R$ *has a left adjoint.*
   (b) *The forgetful functor* ${}^{\star}\mathbf{Bialg}_R \xrightarrow{|-|} {}^{\star}\mathbf{Alg}_R$ *has a right adjoint.*

Proof

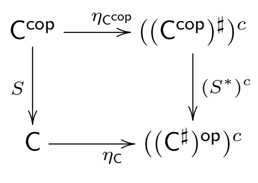
1. (a) follows from Part I, Corollary 94, since the functors $(-)^{\mathsf{op}}$ and $(-)^{\mathsf{cop}}$ are isomorphisms, hence accessible in particular.

1. (b) Since the functors $(-)^{\mathsf{op}}$ and $(-)^{\mathsf{cop}}$ preserve coproducts and products, this follows by the *free algebra construction* of Lemma I.85 and its dual.

2. (a) Denote the unit adjunction for $(-)^c$ by $\eta_{\mathsf{C}}\colon \mathsf{C} \to (\mathsf{C}^\sharp)^c$. Note that[1], since $(-)^{\mathsf{cop}} \circ (-)^c = (-)^c \circ (-)^{\mathsf{cop}}$ and since both functors $(-)^{\mathsf{cop}}$ are isomorphisms on $\mathbf{Coalg}_R$ and $\mathbf{Bialg}_R$ respectively, one has $(-)^\sharp \circ (-)^{\mathsf{cop}} = (-)^{\mathsf{cop}} \circ (-)^\sharp$ and, for each comonoid C,

$$ \eta_{\mathsf{C}^{\mathsf{cop}}} = (\eta_{\mathsf{C}})^{\mathsf{cop}}\colon \mathsf{C}^{\mathsf{cop}} \to ((C^{\mathsf{cop}})^\sharp)^c = ((C^\sharp)^{\mathsf{cop}})^c = ((C^\sharp)^{\mathsf{op},\mathsf{cop}})^c $$

Let (C, S) be a $(-)^{\mathsf{cop}}$-algebra and $S^*\colon (\mathsf{C}^\sharp)^{\mathsf{op},\mathsf{cop}} \to \mathsf{C}^\sharp$ the $\mathbf{Bialg}_R$-morphism (equivalently: the $\mathbf{Bialg}_R$-morphism $(\mathsf{C}^\sharp)^{\mathsf{cop}} = (\mathsf{C}^{\mathsf{cop}})^\sharp \to (\mathsf{C}^\sharp)^{\mathsf{op}}$) defined by commutativity of the diagram



Then $\eta_{\mathsf{C}}\colon (\mathsf{C}, S) \to |(\mathsf{C}^\sharp, S^*)|$ is a $|-|$-universal morphism for (C, S) and $F(\mathsf{C}, S) = (\mathsf{C}^\sharp, S^*)$, as is easy to see.

2. (b) is dual to the above. □

We now get immediately the following alternative construction of free and cofree Hopf algebras:

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[1] We do not distinguish notationally here between the functor $(-)^{\mathsf{cop}}$ on $\mathbf{Coalg}_R$ and the functor on $\mathbf{Bialg}_R$, which sends a bimonoid (B^a, B^c) to $(B^a, (B^c)^{\mathsf{cop}})$.

17 Corollary *In Fact 11 above steps 1. (a) and 1. (b) can be replaced by*

(a′) *form the cofree* **-algebra* $(\tilde{\mathsf{M}}, S)$ *over* M *as in Proposition 16 (1.b),*

(b′) *form the cofree* **-bialagebra* $(\tilde{\mathsf{M}}^\sharp, S^*)$ *over* $(\tilde{\mathsf{M}}, S)$ *as in Proposition 16 (2).*

2. (a) and (b) can be replaced dually.

This enables us to relate Takeuchi's construction of free Hopf algebras in [29] to the approach presented above and so correct a claim in [20].

18 Fact Takeuchi's construction of the free R-Hopf algebra over an arbitrary R-coalgebra C (R an absolutely flat ring) is done stepwise as follows:

1. He first constructs the free *bialgebra $(\tilde{\mathsf{C}}^\sharp, S^*)$ over C according to Corollary 17 (not as in Fact 11), and then
2. reflects this into $\mathbf{Hopf}_R$ according to Fact 13 by forming the quotient $\tilde{\mathsf{C}}^\sharp/\Im$, where $\Im$ is the Takeuchi ideal of $(\tilde{\mathsf{C}}^\sharp, S^*)$.

19 Remark The formal analogy between the constructions of the Hopf envelope and free Hopf algebras (the same final step, namely factoring out the Takeuchi ideal, and the similarity in freely adjoining a morphism) has drawn some attention before. There is a recent attempt of an explanation for example in [26]. However, the author's conclusive statement "Notice that the algebra structure [in case of the Hopf envelope construction] is also opposite between even and odd cases (superfluous/unvisible condition in the case of [the free Hopf algebra construction])" suggests that he didn't completely understand Takeuchi's construction. Specifically, that this is *not* done as in Theorem 11, but rather as in Corollary 17. Indeed, as becomes clear from his description of that construction, he believes that, in the notation of the proof of Proposition 16 (2). S is extended to a morphism $(\mathsf{C}^{\mathsf{cop}})^\sharp \to (\mathsf{C}^\sharp)$ and not, as we showed, to a morphism $(\mathsf{C}^{\mathsf{cop}})^\sharp \to (\mathsf{C}^\sharp)^{\mathsf{op}}$. Takeuchi's notation, which does not distinguish between the various dualizing operators supports this misunderstanding.

1.3 Some additional results

Monomorphisms and epimorphisms in $\mathbf{Hopf}_R$

As stated in Lemma 4 the category $\mathbf{Hopf}_R$ has a factorization structure for morphisms (E, M) where E is the class of all surjective Hopf homorphisms and M the class of all injective Hopf homomorphisms, provided that the ring R is absolutely flat. This factorization structure has been a crucial ingredient in Section 1.2. It therefore is interesting to ask whether these classes E and M can be characterized categorically, e.g. as those of all extremal epimorphisms and monomorphisms respectively in $\mathbf{Hopf}_R$.

Clearly, each surjective Hopf homomorphism is an epimorphism in $\mathbf{Hopf}_R$, and each injective Hopf homomorphism is a monomorphism in $\mathbf{Hopf}_R$.

The converses of these statement, however, do not hold as has been shown in [10]. The arguments are obvious, since the antipode of a Hopf algebra is a monomorphism as well an an epimorphism in $\mathbf{Hopf}_R$ (see Corollary I.37), taking into account the following classical results:

20 Fact
1. There exists a Hopf algebra, whose antipode is injective, but not surjective (see [16]).
2. There exists a Hopf algebra, whose antipode is surjective, but not injective (see [30]).

Every extremal epimorphism in $\mathbf{Hopf}_R$, however, is surjective. This is immediate, if we assume that extremal epimorphisms have (surjective, injective)-factorizations (i.e., if R is absolutely flat). Dually, every extremal monomorphism is injective.

The converse to this observation, again, does not hold: Consider a Hopf algebra (H, S) whose antipode is injective, but not surjective. Consider now S as Hopf homomorphism $(\mathsf{H}, S) \to (\mathsf{H}, S)^{\mathsf{op},\mathsf{cop}}$ and its (surjective, injective)-factorization $S = m \circ e$. If S were an extremal monomorphism, e would be an isomorphism and, thus, S would be injective. Hence, not every injective Hopf homomorphism is an extremal monomorphism.

Arguing dually with a Hopf algebra (H, S) whose antipode is surjective, but not injective one sees that not every surjective Hopf homomorphism is an extremal epimorphism. This improves the result on epimorphisms stated above.

Since the notion of extremal epimorphism is the weakest categorical strengthening of the notion of epimorphism, the results

$$ExtrEpi(\mathbf{Hopf}_R) \subsetneq SurjHom(\mathbf{Hopf}_R) \subsetneq Epi(\mathbf{Hopf}_R)$$

$$ExtrMono(\mathbf{Hopf}_R) \subsetneq InjHom(\mathbf{Hopf}_R) \subsetneq Mono(\mathbf{Hopf}_R)$$

show, that the (surjective,injective)-factorization structure on $\mathbf{Hopf}_R$, as useful as it is, can probably not be characterized categorically.

The presentability degree of $\mathbf{Hopf}_R$

Concerning the presentablity degree of $\mathbf{Hopf}_R$ we can say more provided that R is absolutely flat. Since, in this case, $\mathbf{Hopf}_R$ is closed in $\mathbf{Bialg}_R$ under limits and colimits and, moreover, $\mathbf{Bialg}_R$ is finitary monadic over $\mathbf{Coalg}_R$, $\mathbf{Hopf}_R$ is locally λ-presentable provided that $\mathbf{Coalg}_R$ is so (use [3, 2.48]). Now $\mathbf{Coalg}_R$ is locally $\aleph_1$ presentable by Proposition I.95.

The category of Hopf algebras over a field k is even locally finitely presentable: by the so-called *Fundamental Theorem of Coalgebras* (see e.g. [11, 1.4.7]) every coalgebra is a directed colimit of finitely dimensional vector spaces, which form a set of finitely presentable objects in the category of coalgebras (use [2]). This proves that both $\mathbf{Coalg}_k$ and $\mathbf{Hopf}_k$ are locally finitely presentable.

We thus have got the following result, which analogously holds for all the reflective subcategories discussed in Section 2 below.

21 Proposition *1. The category $\mathbf{Hopf}_k$ is locally finitely presentable, for any field k.*

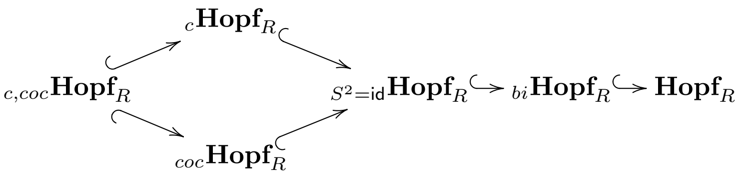
2. The category $\mathbf{Hopf}_R$ is locally $\aleph_1$-presentable, for any absolutely flat ring R.

2 Subcategories of $\mathbf{Hopf}_R$

This section complements the results obtained in the main part of this paper by investigating the categories

- ${}_c\mathbf{Hopf}_R$, the category of commutative Hopf algebras over R,
- ${}_{coc}\mathbf{Hopf}_R$, the category of cocommutative Hopf algebras over R,
- ${}_{c,coc}\mathbf{Hopf}_R$, the category of commutative and cocommutative Hopf algebras over R,
- ${}_{S^2=\mathsf{id}}\mathbf{Hopf}_R$, the category of Hopf algebras over R with antipode satisfying $S^2 = \mathsf{id}$,
- ${}_{bi}\mathbf{Hopf}_R$, the category of Hopf algebras over R with bijective antipode.

The following diagram below illustrates the inclusions between these categories:



Indeed, if (H, S) is commutative, its antipode S is an algebra homomorphism $\mathsf{H}^a \to \mathsf{H}^a$. Applying the (convolution monoid) homomorphism $\phi := \Phi(\mathsf{id}_H, S)\colon \hom(H,H) \to \hom(H,H)$ of Proposition I.22 to S one obtains $e \circ \epsilon = \phi(\mathsf{id}_H * S) = \phi(\mathsf{id}_H) * \phi(S) = S * (S \circ S)$. Since S is the inverse of id_H in the convolution monoid one concludes $S \circ S = \mathsf{id}_H$. Dually, ${}_{coc}\mathbf{Hopf}_R \subset {}_{S^2=\mathsf{id}}\mathbf{Hopf}_R$.

Moreover, each of the categories ${}_{c,coc}\mathbf{Hopf}_R$, ${}_{coc}\mathbf{Hopf}_R$, ${}_{c}\mathbf{Hopf}_R$, and $\mathbf{Hopf}_R$ is a full subcategory of the respective subcategory of $\mathbf{Bialg}_R$, denoted analogously. An analysis of these categories will be simplified by observing that, by the Eckmann-Hilton Principle and the fact that $\mathbf{Alg}_R$, $\mathbf{Coalg}_R$, $\mathbf{Bialg}_R$ and $\mathbf{Hopf}_R$ again are symmetric monoidal categories (see e.g. Part I), one has

$$
\begin{aligned}
{}_{c}\mathbf{Alg}_R &= {}_{c}\mathbf{Mon}(\mathbf{Alg}_R) && (1)\\
{}_{coc}\mathbf{Coalg}_R &= {}_{coc}\mathbf{Comon}(\mathbf{Alg}_R) && (2)\\
{}_{c}\mathbf{Bialg}_R &= {}_{c}\mathbf{Mon}(\mathbf{Coalg}_R) = \mathbf{Mon}(\mathbf{Bialg}_R) && (3)\\
{}_{coc}\mathbf{Bialg}_R &= {}_{coc}\mathbf{Comon}(\mathbf{Alg}_R) = \mathbf{Comon}(\mathbf{Bialg}_R) && (4)\\
{}_{c,coc}\mathbf{Bialg}_R &= {}_{coc}\mathbf{Comon}({}_{c}\mathbf{Alg}_R) = {}_{c}\mathbf{Mon}({}_{coc}\mathbf{Coalg}_R) && (5)\\
{}_{c}\mathbf{Hopf}_R &= \mathbf{Mon}(\mathbf{Hopf}_R) && (6)\\
{}_{coc}\mathbf{Hopf}_R &= \mathbf{Comon}(\mathbf{Hopf}_R) && (7)\\
{}_{c,coc}\mathbf{Hopf}_R &= \mathbf{Comon}({}_{c}\mathbf{Hopf}_R) = \mathbf{Mon}({}_{coc}\mathbf{Hopf}_R) && (8)
\end{aligned}
$$

Limits and Colimits

22 Proposition *1. In the chains of subcategories*

$${}_{coc}\mathbf{Hopf}_R \subset {}_{S^2=\mathsf{id}}\mathbf{Hopf}_R \subset {}_{bi}\mathbf{Hopf}_R \subset \mathbf{Hopf}_R$$

$${}_{coc}\mathbf{Hopf}_R \subset {}_{coc}\mathbf{Bialg}_R$$

each subcategory is closed under all colimits in each of its successors. In particular, each of these categories is cocomplete.

2. *If R is absolutely flat, then in the chains of subcategories*

$${}_{c}\mathbf{Hopf}_R \subset {}_{S^2=\mathsf{id}}\mathbf{Hopf}_R \subset {}_{bi}\mathbf{Hopf}_R \subset \mathbf{Hopf}_R$$

$${}_{c}\mathbf{Hopf}_R \subset {}_{c}\mathbf{Bialg}_R$$

each subcategory is closed under all limits in each of its successors. In particular, each of these categories is complete.

3. *For every ring R ${}_{c}\mathbf{Hopf}_R$ is closed under limits in $\mathbf{Hopf}_R$ and under directed colimits in each of its successors.*

4. *All the categories above are locally presentable, as are the categories ${}_{c,coc}\mathbf{Hopf}_R$ and ${}_{c,coc}\mathbf{Bialg}_R$.*

Proof 1. To show that ${}_{bi}\mathbf{Hopf}_R \subset \mathbf{Hopf}_R$ is closed under colimits consider a colimit $(\mathsf{H}_i, S_i) \xrightarrow{\lambda_i} (\mathsf{H}, S)$ in $\mathbf{Hopf}_R$. By the construction of colimits in $\mathbf{Hopf}_R$ (see Section I.3.1) one, thus, has to prove that the morphism S in diagram below is bijective if all the S_i are bijective.

$$\begin{array}{ccccc} H^{\mathsf{op,cop}} & \overset{T}{\dashrightarrow} & H & \overset{S}{\longrightarrow} & H^{\mathsf{op,cop}} \\ \uparrow{\scriptstyle \lambda_i} & & {\scriptstyle \lambda_i}\uparrow & & \uparrow{\scriptstyle \lambda_i} \\ H_i^{\mathsf{op,cop}} & \underset{S_i^{-1}}{\longrightarrow} & H_i & \underset{S_i}{\longrightarrow} & H_i^{\mathsf{op,cop}} \end{array}$$

This is a commutative diagram in $\mathbf{Bialg}_R$, when T is the bialgebra homomorphism induced by the family $(S_i^{-1})_i$ (note that the left hand column is a colimit as well). The equations $S_i \circ S_i^{-1} = \mathsf{id}$ for each i then induce $S \circ T = \mathsf{id}$ by the colimit property. $T \circ S = \mathsf{id}$ is proved analogously.

A similar argument shows that ${}_{S^2=\mathsf{id}}\mathbf{Hopf}_R$ is closed under colimits in $\mathbf{Hopf}_R$ and, thus, in ${}_{bi}\mathbf{Hopf}_R$.

Since ${}_{coc}\mathbf{Bialg}_R = {}_{coc}\mathbf{Comon}(\mathbf{Alg}_R)$ closure of colimits in $\mathbf{Bialg}_R$ is a trivial consequence of the dual of Fact I.10. But then ${}_{coc}\mathbf{Hopf}_R$ is closed under colimits in ${}_{coc}\mathbf{Bialg}_R$, since ${}_{coc}\mathbf{Hopf}_R$ is closed under colimits in $\mathbf{Bialg}_R$ by the above in connection with Proposition 5.

Since ${}_{coc}\mathbf{Hopf}_R = \mathbf{Comon}(\mathbf{Hopf}_R)$ by (7), ${}_{coc}\mathbf{Hopf}_R$ is closed in $\mathbf{Hopf}_R$ under colimits, too, again by the dual of Fact I.10.

Cocompleteness of the categories under consideration now follows from cocompleteness of $\mathbf{Bialg}_R$ and $\mathbf{Hopf}_R$ respectively.

2. Closure under limits is dual (in case of directed colimits analogous) to 1.

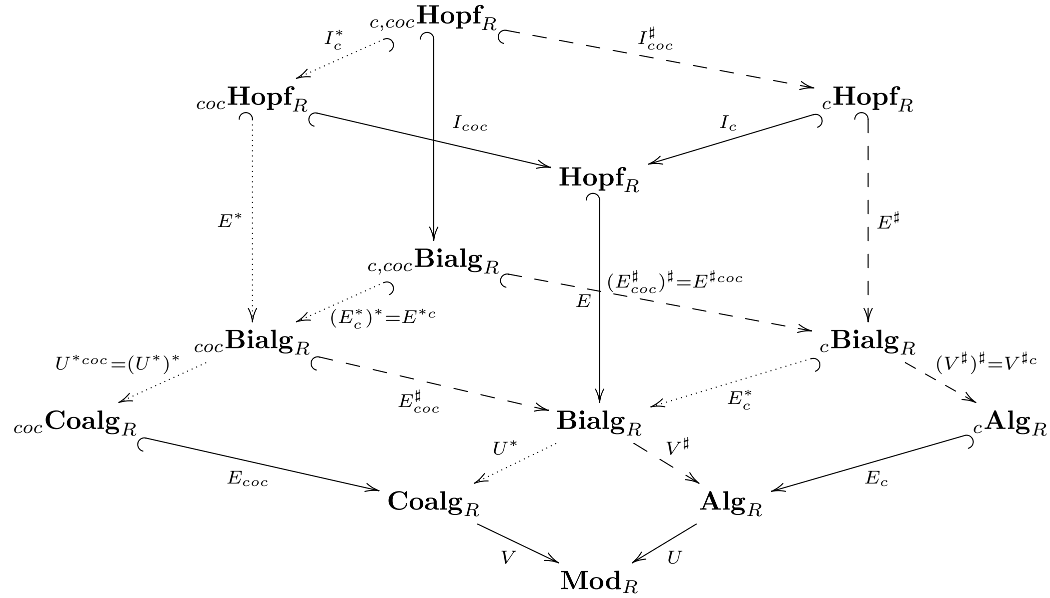
3. This follows from Proposition I.47 in view of (6).

4. Concerning ${}_{S^2=\mathsf{id}}\mathbf{Hopf}_R$ observe that, if E is the embedding of $\mathbf{Hopf}_R$ into $\mathbf{Bialg}_R$, then $\lambda_{(\mathsf{H},S)} = S \circ S$ defines a natural transformation $E \Rightarrow E$ and ${}_{S^2=\mathsf{id}}\mathbf{Hopf}_R = \mathbf{Eq}(\lambda, \mathsf{id}_E)$. Thus, ${}_{S^2=\mathsf{id}}\mathbf{Hopf}_R$ is accessible by Proposition I.92 and, thus, locally presentable by Proposition I.89 due to cocompleteness.

Concerning ${}_{bi}\mathbf{Hopf}_R$ consider the functor $K\colon \mathbf{Hopf}_R \to \mathbf{Hopf}_R$ with $K(\mathsf{H},S) = (\mathsf{H},S)^{\mathsf{op,cop}}$. A K-algebra thus is a pair $((\mathsf{H},S),T)$, where $T\colon (\mathsf{H},S)^{\mathsf{op,cop}} \to (\mathsf{H},S)$ is a Hopf algebra homomorphism. Denoting the forgetful functor $\mathbf{Alg}K \to \mathbf{Hopf}_R$ by $|-|$, one has natural transformations $\lambda, \mu\colon |-| \Rightarrow |-|$ with $\lambda_{((\mathsf{H},S),T)} = S \circ T$ and $\mu_{((\mathsf{H},S),T)} = T \circ S$. ${}_{bi}\mathbf{Hopf}_R$ then is the equifier $\mathbf{Eq}((\lambda, \mathsf{id}_{|-|}), (\mu, \mathsf{id}_{|-|}))$ and, thus again, a cocomplete accessible and therefore locally presentable category. For the other cases use Proposition I.47. □

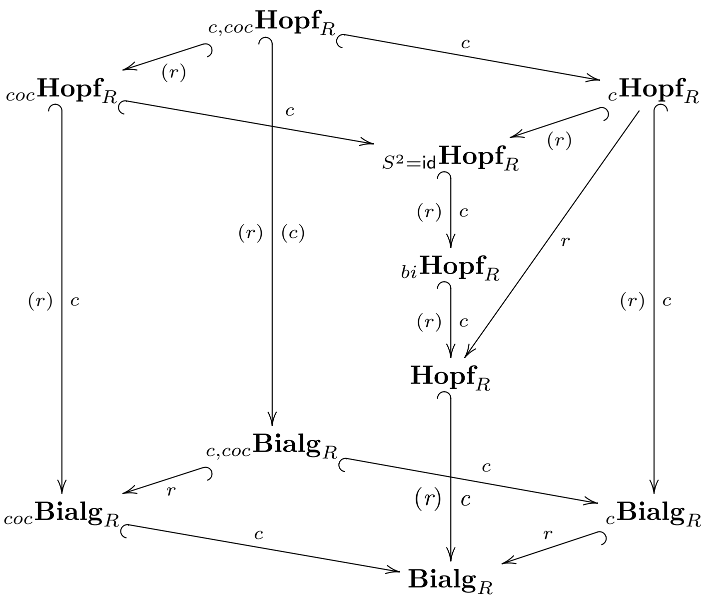
Reflectivity and Coreflectivity

The following diagram summarizes our results where each label c marks a coreflective embedding and each label r a reflective one. Labels in brackets indicate that the respective result only holds over absolutely flat rings.



(10)

(9)



Most of these results we get out of Diagram (10), where we use notation as in Section I.4.2 and indicate the monoidal lift $F^\sharp$ of a monoidal functor F to the respective categories of monoids by a dashed arrow $\dashrightarrow$ and the monoidal lift F^* of a strong monoidal functor F to the respective categories of comonoids by a dotted arrow ⋯> . In case of a monoidal lift to the categories of (co)commutative (co)monoids we denote the respective functors by $F^{\sharp_{coc}}$ and F^{*_c} respectively. Note that, with these notations, we have in particular $(V^\sharp)^\sharp = V^{\sharp_c}$ and $(U^*)^* = U^{*_{coc}}$, as well as $(E_c^*)^* = E^{*_c}$ and $(E_{coc}^\sharp)^\sharp = E^{\sharp_{coc}}$.

Thus, using the list of equations above, Proposition I.65 and its dual imply that for each functor F in the diagram

- $F^\sharp$ and $F^{\sharp_{coc}}$ will have a right adjoint, provided that F has one,
- F^* and F^{*_c} will have a left adjoint, provided that F has one,

and we will only have to check existence of adjunctions for the functors indicated by a solid arrow $\to$ in Diagram 10. Here we already know that U has a left adjoint and V has a right adjoint by 1, and that E has a right adjoint for each ring R, while E also has a left adjoint if R is absolutely flat (see Theorem 7).

Now all the reflections and coreflections indicated in Diagram (9) above are consequences of the next two Propositions and their corollary.

23 Proposition *For any ring R, in the chains of subcategories*

$$_{coc}\mathbf{Hopf}_R \subset {}_{S^2=\mathsf{id}}\mathbf{Hopf}_R \subset {}_{bi}\mathbf{Hopf} \subset \mathbf{Hopf}_R$$

$$_{coc}\mathbf{Hopf}_R \subset {}_{coc}\mathbf{Bialg}_R$$

every category is coreflective in each of its successors.

Proof All embeddings have locally presentable domains and preserve colimits by Proposition 22. Coreflectivity then follows by the Special Adjoint Functor Theorem. □

24 Proposition *For every absolutely flat ring R, in the chains of subcategories*

$$_{c}\mathbf{Hopf}_R \subset {}_{S^2=\mathsf{id}}\mathbf{Hopf}_R \subset {}_{bi}\mathbf{Hopf}_R \subset \mathbf{Hopf}_R$$

$$_{c}\mathbf{Hopf}_R \subset {}_{c}\mathbf{Bialg}_R \subset \mathbf{Bialg}_R$$

every category is reflective in each of its successors.

$_c\mathbf{Hopf}_R$ is reflective in $\mathbf{Hopf}_R$ even for every commutative ring R.

Proof All embeddings are accessible and preserve limits by Proposition 22. Now use the General Adjoint Functor Theorem in connection with Proposition I.93.

For the final statement one uses the same argument after observing that, for every ring R, the embedding of $_c\mathbf{Hopf}_R = \mathbf{Mon}(\mathbf{Hopf}_R)$ into $\mathbf{Hopf}_R$ preserves limits and directed colimits by Proposition I.47. □

25 Corollary *For every absolutely flat ring R the category $_{c,coc}\mathbf{Hopf}_R$ is reflective and coreflective in $_{c,coc}\mathbf{Bialg}_R$.*

Proof Referring to Diagram (10), one has for the embedding $I\colon {}_{c,coc}\mathbf{Hopf}_R \hookrightarrow {}_{c,coc}\mathbf{Bialg}_R$ $(E^*)^\sharp = I = (E^\sharp)^*$ by Eqns. (5) and (8).

By Propositions 23 and 24 $E^*\colon {}_{coc}\mathbf{Hopf}_R \hookrightarrow {}_{coc}\mathbf{Bialg}_R$ and $E^\sharp\colon {}_c\mathbf{Hopf}_R \hookrightarrow {}_c\mathbf{Bialg}_R$ both have a left as well as a right adjoint. Now apply Proposition I.65. □

26 Remark Some of the results above can, in the restricted case of an absolutely flat regular ring, also be obtained by using the explicit construction of the Hopf (co)reflection as in Section 1.2. Recall that, given a bialgebra B, one constructs its (co) reflection as follows: Define a family of bialgebras $(B_n)_{n\in\mathbb{N}}$ by $B_0 := B$ and $B_{n+1} := B_n^{\mathsf{op},\mathsf{cop}}$. Then the Hopf reflection RB of B is a (suitable) homomorphic image of $\coprod B_n$ while the Hopf coreflection of B is a (suitable) subbialgebra of $\prod B_n$.

Now, obviously, if B is commutative (cocommutative, commutative and cocommutative) so is each B_n and then $\coprod B_n$ and $\prod B_n$ respectively (since the functors on $\mathbf{Bialg}_R$ sending B to B^{op} or B^{cop} are isomorphisms and therefore preserve (co)products). It is, moreover, easy to see that images and subbialgebras of a commutative (cocommutative, commutative and cocommutative) bialgebra have the respective property again. Thus the Hopf (co)reflection of a commutative (cocommutative, commutative and cocommutative) bialgebra is a commutative (cocommutative, commutative and cocommutative) Hopf algebra.

An explicit construction of the reflection of $\mathbf{Hopf}_R$ into ${}_{bi}\mathbf{Hopf}_R$ has been given in [25].

Monadicity

The following results now follow by composition of adjunctions, where the second statement generalizes a result of [29].

27 Proposition 1. *Let R be an arbitrary commutative ring. Then there exists a cofree cocommutative R-Hopf algebra over any R-algebra.*

2. *Let R be an absolutely flat ring. Then there exists a free commutative R-Hopf algebra over any R-coalgebra.*

Extending the results from Theorem 10 that $\mathbf{Hopf}_R$ is comonadic over $\mathbf{Alg}_R$ (always) and monadic over $\mathbf{Coalg}_R$, provided that R is absolutely flat, we also get

28 Proposition 1. *For every ring R the following hold:*

(a) *The cofree Hopf algebra on a commutative algebra A is commutative and, thus, the cofree commutative Hopf algebra on A.*

(b) *${}_c\mathbf{Hopf}_R$ is comonadic over ${}_c\mathbf{Alg}_R$.*

2. *For every absolutely flat ring R, the following hold:*

(a) *The free Hopf algebra on a cocommutative coalgebra C is cocommutative and, thus, the free cocommutative Hopf algebra on C.*

(b) *${}_{coc}\mathbf{Hopf}_R$ is monadic over ${}_{coc}\mathbf{Coalg}_R$.*

Proof Since the forgetful functor $W\colon \mathbf{Hopf}_R \to \mathbf{Alg}_R$ has a right adjoint C by Theorem 10, the forgetful functor $\mathbf{Mon}W\colon {}_c\mathbf{Hopf}_R \to {}_c\mathbf{Alg}_R$ has, in view of Eqn. (1), a right adjoint $\tilde{C}$ by Proposition I.65, and this makes the following diagram commutative.

$$\begin{array}{ccc} {}_c\mathbf{Alg}_R & \xrightarrow{\tilde{C}} & {}_c\mathbf{Hopf}_R \\ {\scriptstyle |-|}\downarrow & & \downarrow{\scriptstyle |-|} \\ \mathbf{Alg}_R & \xrightarrow[C]{} & \mathbf{Hopf}_R \end{array}$$

This proves 1. Statement 2 is dual. □

3 Some remarks on convolution monoids

Recall from Section I.1.3 that, given $\mathsf{C} = (C, \mu, \epsilon)$, a comonoid in $\mathbb{C}$ and $\mathsf{M} = (M, m, e)$ a monoid in $\mathbb{C}$, the hom-set $\hom_{\mathbf{C}}(C, M)$ becomes an (ordinary) monoid $\Phi_{\mathbb{C}}(\mathsf{C}, \mathsf{M})$ — called *convolution monoid of* (C, M) — as follows.

- Given $f, g \colon C \longrightarrow M$, define their product (called *convolution product*) $f * g = C \xrightarrow{\mu} C \otimes C \xrightarrow{f \otimes g} M \otimes M \xrightarrow{m} M$,
- chose as unit $C \xrightarrow{\epsilon} I \xrightarrow{e} M$.

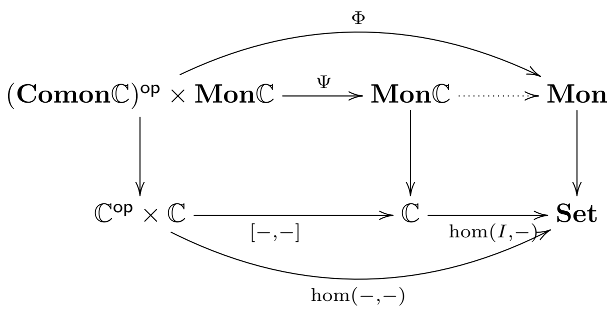
and that this construction is functorial.

In the case of a symmetric monoidal closed category $\mathbb{C}$ such as $\mathbf{Mod}_R$ we even get a functor $\Psi \colon (\mathbf{Comon}\mathbb{C})^{\mathsf{op}} \times \mathbf{Mon}\mathbb{C} \to \mathbf{Mon}\mathbb{C}$ as follows. By Proposition 37 below the internal hom-functor $[-,-]$ is monoidal and, hence, induces a functor $\Psi \colon \mathbf{Mon}(\mathbb{C}^{\mathsf{op}} \times \mathbb{C}) \to \mathbb{C}$ (see Part I). Observing that $\mathbf{Mon}(\mathbb{C}^{\mathsf{op}} \times \mathbb{C}) = \mathbf{Mon}\mathbb{C}^{\mathsf{op}} \times \mathbf{Mon}\mathbb{C} = (\mathbf{Comon}\mathbb{C})^{\mathsf{op}} \times \mathbf{Mon}\mathbb{C}$ one has $\Psi\big((C, \mu, \epsilon), (M, m, e)\big) = ([C, M], a, \varepsilon)$ with

- $a = [C, M] \otimes [C, M] \xrightarrow{n} [C \otimes C, M \otimes M] \xrightarrow{[\mu, m]} [C, M]$ and
- $\varepsilon = I \xrightarrow{u_I} [I, I] \xrightarrow{[\epsilon, e]} [C, M]$

We call this monoid in $\mathbb{C}$ the $\mathbb{C}$-*convolution monoid* of the pair $\big((C, \mu, \epsilon), (M, m, e)\big)$.

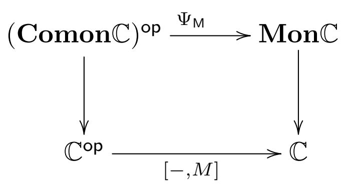
The following commutative diagram of functors illustrates the situation, where the dotted arrow only exists in the case where $\hom(I, -)$ is monoidal. This is the case for example for $\mathbb{C} = \mathbf{Mod}_R$ (see Example I.3 (4)), where $\Psi\big((C, \mu, \epsilon), (M, m, e)\big)$ then is the convolution algebra of the pair $\big((C, \mu, \epsilon), (M, m, e)\big)$.



Fixing a monoid M in the construction of $\mathbb{C}$-convolution monoids, one obviously gets a functor $\Psi_{\mathsf{M}} \colon (\mathbf{Comon}\mathbb{C})^{\mathsf{op}} \to \mathbf{Mon}\mathbb{C}$.

29 Theorem *Let $\mathbb{C}$ be a symmetric monoidal closed locally finitely presentable category, where the hom-functor* $\hom(I, -)$ *create limits. Then, for each $\mathbb{C}$-monoid M, the functor $\Psi_{\mathsf{M}} \colon (\mathbf{Comon}\mathbb{C})^{\mathsf{op}} \to \mathbf{Mon}\mathbb{C}$ has left adjoint.*

Proof Consider the commutative diagram of functors, where the vertical arrows are the forgetful functors.



Since the forgetful functors create limits and the internal hom functor (due to our assumptions) preserves those, it is clear that Φ_M preserves limits. Now **Comon**$\mathbb{C}$, being locally presentable (Proposition I.47 1(a) applies: the functors $\otimes^2$ and $\otimes^3$ preserve directed colimits, since all functors $C \otimes -$ and $-\otimes$ do so by monoidal closedness — see e.g. [19]), has a generator, and is cocomplete and co-wellpowered (see Proposition I.90). Thus, Φ_M is a limit preserving functor on a well powered complete category with a cogenerator. It therefore has a left adjoint by the Special Adjoint Functor Theorem. □

The dual algebra functor

Specializing to $\mathbb{C} = \mathbf{Mod}_R$ and $\mathsf{M} = R$ in the previous section, the functor $\mathbf{Coalg}_R^{\mathsf{op}} \xrightarrow{\Psi_R} \mathbf{Alg}_R$ is the so-called *dual algebra functor* $(-)^\star \colon \mathbf{Coalg}_R^{\mathsf{op}} \to \mathbf{Alg}_R$.

It is known to have a left adjoint $(-)^o$, called the *finite dual* or *Sweedler dual* (see [27]), in case $R = k$, a field. A generalization to arbitrary commutative rings seems to be unknown in the standard literature. However, by Theorem 29 we immediately have

30 Theorem *The dual algebra functor $(-)^\star \colon \mathbf{Coalg}_R^{\mathsf{op}} \to \mathbf{Alg}_R$ has a left adjoint $(-)^\circ$, for every commutative ring R.*

The universal measuring coalgebra

Given R-algebras A and B, the *universal measuring coalgebra* $\mu(\mathsf{A},\mathsf{B})$ is an R-coalgebra, such that there are natural isomorphisms $\mathbf{Alg}_R(\mathsf{A}, \Psi(\mathsf{C},\mathsf{B})) \simeq \mathbf{Coalg}_R(\mathsf{C}, \mu(\mathsf{A},\mathsf{B}))$, for all R-coalgebras C (see [27]). Sweedler proves that, for $R = k$ a field, these coalgebras always exist. By generalizing the above application of Theorem 29 we can easily prove and at the same time generalize Sweedler's result to arbitrary rings as follows.

31 Theorem *For every ring R and any pair (A,B) of R-algebras there exists a universal measuring coalgebra $\mu(\mathsf{A},\mathsf{B})$.*

Proof By Theorem 29 the functor $\Psi_\mathsf{B} \colon (\mathbf{Coalg}_R)^{\mathsf{op}} \to \mathbf{Alg}_R$ has a left adjoint μ_B, for each R-algebra B. That is, there exist isomorphisms $\mathbf{Alg}_R(\mathsf{A}, \hom_R(\mathsf{C},\mathsf{B})) \simeq \mathbf{Coalg}_R(\mathsf{C}, \mu_\mathsf{B}(\mathsf{A}))$, which are natural. In other words, for each B and A the coalgebra $\mu_\mathsf{B}(\mathsf{A})$ is the universal measuring coalgebra for (A,B). □

Set-like elements in a comonoid

An element c of an R-coalgebra (C, μ, ϵ) is called *set-like* or *group-like* (see [28] or [11]) provided that

$$\mu(c) = c \otimes c \quad (11)$$
$$\epsilon(c) = 1 \quad (12)$$

Identifying elements in $c \in C$ with R-linear maps $R \xrightarrow{c} C$ these conditions are equivalent to

$$R \xrightarrow{c} C \xrightarrow{\mu} C \otimes C = R \simeq R \otimes R \xrightarrow{c \otimes c} C \otimes C \quad (13)$$
$$R \xrightarrow{c} C \xrightarrow{\epsilon} R = \mathsf{id}_R \quad (14)$$

Observing that the canonical isomorphism $R \simeq R \otimes R$ is the comultiplication and id_R is the counit of the canonical comonoid structure of R, Equations (13) and (14) are equivalent to saying that a is a comonoid homomorphism $R \to C$. We, thus, define with respect to an arbitrary symmetric monoidal category $\mathbb{C}$

32 Definition Given a $\mathbb{C}$-monoid C, a *(generalized) set-like element of* C is a comonoid homomorphism $\mathsf{I} \xrightarrow{c} \mathsf{C}$.[2] $S(\mathsf{C})$ denotes the set of set-like elements of C.

We immediately get the following generalization of [11, 1.4.15].

33 Proposition *For every R-algebra A the set-like elements of the A° are (in bijection to) the algebra homomorphisms $\mathsf{A} \to R$.*

Proof By the adjunction of Theorem 30 above, one has $\mathbf{Alg}(\mathsf{A}, R) = \mathbf{Alg}(\mathsf{A}, R^*) \simeq \mathbf{Coalg}(R, \mathsf{A}^\circ)$. □

The following is generalizing a well known result about Hopf algebras, namely that set-like elements of a Hopf algebra form a group (see e.g. [11, 4.2.9]).

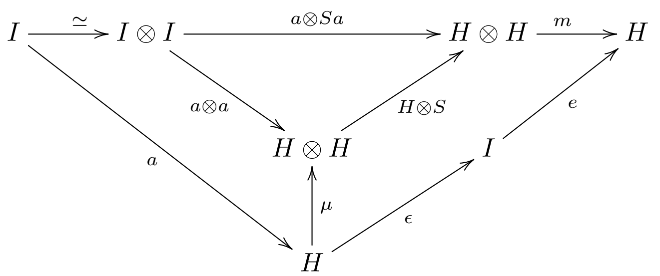
34 Proposition *1. If H is a bimonoid in $\mathbb{C}$, then $S(\mathsf{H}^c)$, the set of set-like elements of (the coalgebra underlying) H, form a monoid.*

2. If (H, S) is a Hopf monoid in $\mathbb{C}$, thenc, the set of set-like elements of (the coalgebra underlying) (H, S), form a group.

Proof Recall first that I carries the structure of a Hopf monoid (I, S) with the canonical isomorphism $I \otimes I \xrightarrow{r_I} I$ its multiplication, r_I^{-1} its comultiplication, and the identity id_I serving as unit, counit and antipode S. In particular, all these maps are monoid and comonoid homomorphisms.

To prove 1 observe that $S(\mathsf{H}^c)$ is a subset of the convolution monoid $\Phi_{\mathbb{C}}(\mathsf{I}^c, \mathsf{H}^a)$. It, thus, suffices to show that $S(\mathsf{H}^c)$ is closed under convolution and contains the monoid unit $I \xrightarrow{\mathsf{id}_I} I \xrightarrow{e} C$, where the latter is clear, since C is a bimonoid. Also, for set-like elements, i.e, comonoid homomorphisms $a, b\colon \mathsf{I}^c \to \mathsf{H}^c$ their convolution product $a * b = m \circ (a \otimes b) \circ r_I^{-1}$ is a composition of comonoid homomorphisms and, thus, a set-like element.

To prove 2 we show that $a \in S(\mathsf{H}^c)$ implies $S \circ a \in S(\mathsf{H}^c)$ and $a * (S \circ a) = e \circ \mathsf{id}_I$. The first of these statements is trivial, since $S \circ a$ is comonoid homomorphism $\mathsf{I}^c \to (\mathsf{H}^c)^{\mathsf{cop}}$ iff it is a comonoid homomorphism $\mathsf{I}^c = (\mathsf{I}^c)^{\mathsf{cop}} \to \mathsf{H}^c$. The required equation then follows by commutativity of the diagram, where the left hand cell commutes since a is set-like and the right hand one by the antipode equation for S.



□

4 Limitations to this approach

One may now ask the obvious question whether our approach works in more general contexts as well, which arise naturally. We will briefly discuss two of them.

[2]Here I is the canonical comonoid on I, the unit object of the monoidal structure of $\mathbb{C}$ — see Proposition I.21.

4.1 When $\mathbb{C}$ is braided only

Recall that a *braided monoidal category* is the generalization of the concept of symmetric monoidal category, obtained by dropping the condition of the symmetry s that $s_{BC} \circ s_{CB} = \mathsf{id}_{B\otimes C}$ for each pair of **C**-objects (B, C) (see [14]). The question thus is, which of our results can be obtained when using braided monoidal categories, instead of symmetric ones.

The effect of generalizing to a braided monoidal category $\mathbb{C}$ is first, that the category **Mon**$\mathbb{C}$, though still a monoidal category by the same construction as in the symmetric case, will in general no longer be braided monoidal. As a consequence the category **Bimon**$\mathbb{C}$ (and then **Hopf**$\mathbb{C}$) cannot be supplied with a monoidal structure at all in general. Note, however, that the Eckmann-Hilton principle also works in the braided case.

Since the monoidal structure on **Bimon**$\mathbb{C}$ has only been used from Section 2 onwards, everything in the earlier sections holds in the braided case as well. When generalizing the results of Section 2 to arbitrary symmetric monoidal categories, which can be done by replacing the phrases "for every ring" and "for every absolutely flat ring" by the assumptions on $\mathbb{C}$ as in Theorem I.54, only the arguments for the embeddings

$$_{coc}\mathbf{Hopf}\mathbb{C} \xrightarrow{E^*} {}_{coc}\mathbf{Bialg}\mathbb{C} \quad \text{and} \quad {}_{c}\mathbf{Hopf}\mathbb{C} \xrightarrow{E^\sharp} {}_{c}\mathbf{Bialg}\mathbb{C}$$

$$_{c,coc}\mathbf{Hopf}\mathbb{C} \xrightarrow{I_c^*} {}_{coc}\mathbf{Hopf}\mathbb{C} \quad \text{and} \quad {}_{c,coc}\mathbf{Hopf}_{\mathbb{C}} \xrightarrow{I_{coc}^\sharp} {}_{c}\mathbf{Hopf}\mathbb{C}$$

and

$$_{c,coc}\mathbf{Hopf}\mathbb{C} \xrightarrow{(E^*)^\sharp = (E^\sharp)^*} {}_{c,coc}\mathbf{Bialg}\mathbb{C}$$

are falling short.

Concerning E^* and $E^\sharp$ we observe that, by the results of Section 2, E^* always preserves colimits, while $E^\sharp$ preserves limits if $\otimes^2$ preserves extremal monomorphisms. Thus, E^* always has a right adjoint by the Special Adjoint Functor Theorem and $E^\sharp$ has a left adjoint by Proposition I.93 if $\otimes^2$ preserves extremal monomorphisms.

We do not see a way, however, to maintain the additional results on these embeddings and the (co)reflectivity results about the other embeddings in the braided situation.

4.2 When R is not commutative

Considering a non commutative ring A one is tempted to use the category $_A\mathbf{Mod}_A$ as a base category $\mathbb{C}$. Indeed, $_A\mathbf{Mod}_A$ is a monoidal category by means of the usual tensor product; however, this monoidal structure not only fails to be symmetric, it also fails to be braided in general (see [5] for detailed study). Thus, though we can define monoids and comonoids in $_A\mathbf{Mod}_A$ (the latter being called *A-corings* see [7]). The missing braiding has the effect that one cannot define a tensor product on $\mathbf{Mon}(_A\mathbf{Mod}_A)$ and $\mathbf{Comon}(_A\mathbf{Mod}_A)$. Thus, we fail in the constructions of bimonoids in $\mathbf{Mon}(_A\mathbf{Mod}_A)$. For an alternative see e.g. [7] or [24].

The particular properties of $\mathbf{Mod}_R$ we have used above concerning properties of the tensor product and extremal morphisms are shared by $\mathbf{Mon}(_A\mathbf{Mod}_A)$. Thus, everything said above concerning monoids and comonoids in $\mathbf{Mod}_R$ also holds w.r.t. $\mathbf{Mon}(_A\mathbf{Mod}_A)$.

But if A is such that $_A\mathbf{Mod}_A$ is braided (e.g., if A is a finite dimensional algebra over a field k, which is a central simple algebra – see [5]), then everything works as in Section 4.1 above.

4.3 Weak Hopf algebras

The various concepts of *weak Hopf algebras* (see e.g. [8], [12]) have in common, that, in the definition of a weak Hopf algebra, the conditions on the (co)multiplication or (co)unit, to be homomorphisms of monoids and comonoids respectively are weakened. Thus, categories **wHopf**$\mathbb{C}$ of weak Hopf algebras are no longer defined to be a subcategories of **Bimon**$\mathbb{C}$, but rather to be a subcategories of the following pullback **MC** in **CAT**.

$$\begin{array}{ccc} \mathbf{MC} & \longrightarrow & \mathbf{Mon}\mathbb{C} \\ \downarrow & & \downarrow{\scriptstyle U_a} \\ \mathbf{Comon}\mathbb{C} & \xrightarrow[U_c]{} & \mathbf{C} \end{array}$$

Thus, the literal application of the methods used above, is not possible. However, a first important observation is that, given the standing assumptions on $\mathbb{C}$, the pullback **MC** is once more a locally presentable category (see Proposition I.80). One would have to investigate, whether the forgetful functors from **MC** into **Mon**$\mathbb{C}$ and **Comon**$\mathbb{C}$ respectively share some crucial properties with the respective functors from **Bimon**$\mathbb{C}$ and, whether the categories of weak Hopf algebras relate to **MC** in a similar way as **Hopf**$\mathbb{C}$ does to **Bimon**$\mathbb{C}$. But this is beyond the scope of this paper.

5 Appendix

Absolutely flat rings

A standard definition of an absolutely flat ring R is that every R-module M is flat, i.e., that for every injective R-linear map $A \xrightarrow{m} B$ the map $M \otimes A \xrightarrow{\mathsf{id}_M \otimes m} M \otimes B$ is injective. This, obviously, implies that for every injective R-linear map $A \xrightarrow{m} B$ the map $A \otimes A \xrightarrow{m \otimes m} B \otimes B$ is injective. Thus, for every absolutely flat ring R the category $\mathbf{Mod}_R$ satisfies our crucial condition that the functor $\otimes^2$ preserves extremal monomorphisms.

Whether the converse holds, was posed as a problem in [20]. That this indeed is the case has been shown by George Janelidze [13]: If $\otimes^2$ preserves injective homomorphisms, then R is absolutely flat.

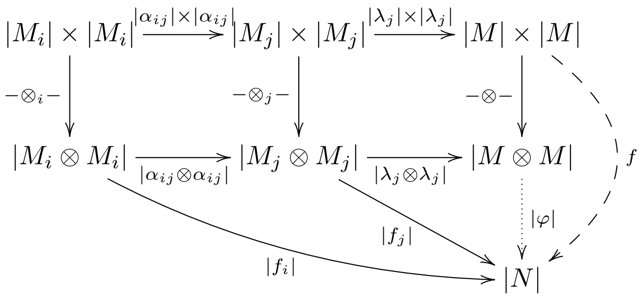
Indeed, if $A \xrightarrow{m} B$ is an R-linear injective map and M an arbitrary R-module, then also $\mathsf{id}_M \oplus m \colon M \oplus A \to M \oplus B$ is injective: By the (obvious) argument above now $(\mathsf{id}_M \oplus \mathsf{id}_A) \otimes (\mathsf{id}_M \oplus m) \colon (M \oplus A) \otimes (M \oplus A) \to (M \oplus A) \otimes (M \oplus B)$ is injective as well. Since $\otimes$ distributes over $\oplus$ one concludes that this map is $(\mathsf{id}_M \otimes \mathsf{id}_M) \oplus (\mathsf{id}_M \otimes m) \oplus (\mathsf{id}_A \otimes \mathsf{id}_M) \oplus (\mathsf{id}_A \otimes m)$. Thus, $\mathsf{id}_M \otimes m$ is injective as required.

$\otimes^n$ preserves directed colimits

On every category $\mathbf{Mod}_R$ the functor $\otimes^n$, raising to the n^{th} tensor power, is finitely accessible for every $n \in \mathbb{N}$. An elementary argument (only for $n = 2$) is the following[3].

Let $M_i \xrightarrow{\alpha_{ij}} M_j$ be a directed diagram in $\mathbf{Mod}_R$ and $M_j \xrightarrow{\lambda_j} M$ its colimit. Since the forgetful functor $|-| \colon \mathbf{Mod} \to \mathbf{Set}$ preserves directed colimits, as does $(-)^2$ on **Set**, the top row of the following commutative diagram is a directed colimit.

[3] A more elegant and general (categorical) argument is possible (see [19]). Since the present paper is written in the attempt to be comprehensible for readers not too familiar with categorical reasoning we refrain from using that argument here.



If now $f_i\colon M_i \otimes M_i \to N$ is a compatible family in $\mathbf{Mod}_R$, the family $|f_j| \circ - \otimes_j -$ is compatible with the top row diagram, hence yields unique map f making the right hand 'square' commute.

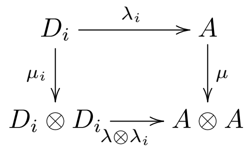
Since all maps $|f_j|\circ-\otimes_j-$ are bilinear and the maps $|\lambda_j|\times|\lambda_j|$, being collimating maps in **Set**, form a covering family, f is bilinear, hence induces a linear map $\varphi\colon M\otimes M \longrightarrow N$ such that the right hand triangle commutes. It now is the unique linear map with $\varphi\circ(\lambda_i\otimes\lambda_i) = f_i$ for all i.

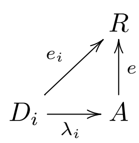
Colimits in algebras and bialgebras

This is an application of (the proof of) Proposition I.80 (5).

Colimits: In order to construct a colimit of a diagram $D\colon \mathbf{I} \to \mathbf{Alg}_R$ one first forms a colimit $UD_i \xrightarrow{\mu_i} C$ of UD in $\mathbf{Mod}_R$, where $U\colon \mathbf{Alg}_R \to \mathbf{Mod}_R$ is the forgetful functor; then one builds the tensor algebra TC of C (this is the application of a left adjoint of U) and finally factors TC modulo an appropriate ideal I (since the regular epimorphisms in $\mathbf{Alg}_R$ are the surjective homomorphisms) — see [17] for an explicit description of I. This gives the colimit $(A, (\lambda_i)_{i\in I})$ in $\mathbf{Alg}_R$.

Since the forgetful functor $V\colon \mathbf{Bialg}_R \to \mathbf{Alg}_R$ is comonadic, V creates colimits. Therefore, a colimit of a diagram $D\colon \mathbf{I} \to \mathbf{Bialg}_R$ can be constructed as follows: First form a colimit $(A, (VD_i \xrightarrow{\lambda_i} A)_{i\in I})$ of VD in $\mathbf{Alg}_R$ as above. Then the algebra A can be equipped with a unique pair of homomorphisms $(A \xrightarrow{\mu} A\otimes A, A \xrightarrow{\epsilon} R)$ such that it becomes a bialgebra $\bar{A}$ and each $\lambda_i\colon D_i \to \bar{A}$ is a bialgebra homomorphism. $(\bar{A}, (\lambda_i)_{i\in I})$ then is a colimit of D. In particular, μ and ϵ are determined by commutativity of the diagrams





Concerning coequalizers in $\mathbf{Bialg}_R$ this simply means that a coequalizer of a pair $f, g\colon B \to A$ — when performed in $\mathbf{Alg}_R$ as A/I with the ideal I generated by $\{f(b) - g(b) \mid b \in B\}$ — carries a unique bialgebra structure such that the quotient map also is a coalgebra homomorphism (in other words, I is a coideal), and that this is a coequalizer of f and g in $\mathbf{Bialg}_R$.

Limits: For constructing limits in $\mathbf{Coalg}_R$ one can make use of the dual of the above construction provided that R is an absolutely flat ring. Thus, a limit of the diagram $D\colon \mathbf{I} \to \mathbf{Coalg}_R$ is obtained from a limit $\big(A, (\pi_i\colon A \to VD_i)_i\big)$ of VD in $\mathbf{Mod}_R$ (with $V\colon \mathbf{Coalg}_R \to \mathbf{Mod}_R$ the forgetful functor) by first forming the cofree coalgebra $VA^* \xrightarrow{\varrho} A$ on A. A limit L of D then is obtained by performing the (epi-sink, injective)–factorization of the family of all coalgebra homomorphisms $f_j\colon A_j \to A^*$ such that, for all $i \in \mathsf{ob}\mathbf{I}$, $\pi_i \circ \varrho \circ f_j$ is a coalgebra homomorphism.

Somewhat more explicitely, L is given by forming the sum of all subcoalgebras S_k of A^* such that the restriction of $\pi_i \circ \varrho$ to S_k is a coalgebra homomorphism.

Concerning equalizers it would be simpler to proceed as follows. Since $\mathbf{Coalg}_R$ has (episink, extremal mono)-factorizations and extremal mono morphisms are regular in $\mathbf{Mod}_R$ and $\mathbf{Coalg}_R$, an equalizer $E \xrightarrow{\eta} B$ of a pair $f, g \colon B \to A$ of homomorphisms is obtained by forming this factorization

$$C_h \xrightarrow{e_h} E \xrightarrow{\eta} B$$

of the sink of all homomorphims $h \colon C_h \to B$ with $f \circ h = g \circ h$ (see [1, 15.7]). E thus is, as a module, the sum of all subcoalgebras of B contained in the kernel of $f - g$.

Since the forgetful functor $V \colon \mathbf{Bialg}_R \to \mathbf{Coalg}_R$ is monadic it creates limits. Therefore, a limit of a diagram $D \colon \mathbf{I} \to \mathbf{Bialg}_R$ can be constructed as follows: First form a limit $A \xrightarrow{\pi_i} VD_i$ in $\mathbf{Coalg}_R$ as above. Then the coalgebra A can be equipped with a unique pair of homomorphisms $(A \otimes A \xrightarrow{M} A,\ R \xrightarrow{e} A)$ such that it becomes a bialgebra $\tilde{A}$ and each $\pi_i \colon \tilde{A} \to D_i$ a bialgebra homomorphism. $(\tilde{A}, (\pi_i))$ then is a limit of D. In particular, M and e are determined by commutativity of the diagrams

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\pi_i \otimes \pi_i} & D_i \otimes D_i \\ {\scriptstyle M}\downarrow & & \downarrow{\scriptstyle M_i \otimes M_i} \\ A & \xrightarrow[\pi_i]{} & D_i \end{array} \qquad\qquad \begin{array}{ccc} A & \xrightarrow{\pi_i} & D_i \\ & {\scriptstyle e}\nwarrow & \uparrow{\scriptstyle e_i} \\ & & R \end{array}$$

Note that the condition on R to be absolutely flat is only needed to construct limits *this way*. Their sheer existence is given for any ring, for every ring R, by Proposition 6.

Symmetric monoidal closed categories

35 Definition A symmetric monoidal category $\mathbb{C}$ is called *closed* provided that each functor $- \otimes C \colon \mathbf{C} \to \mathbf{C}$ has a right adjoint, which will be denoted by $[C, -]$.

The counits $[C, D] \otimes C \to D$ are called *evaluation morphisms* and will be denoted by $ev_{C,D}$.

The morphism $I \to [C, C]$ corresponding to id_C by the isomorphism $\hom_{\mathbf{C}}(C, C) \simeq \hom_{\mathbf{C}}(I \otimes C, C) \simeq \hom_{\mathbf{C}}(I, [C, C])$ is called *unit* and will be denoted by u_C.

36 Fact In every symmetric monoidal closed category there are, for each triple (A, B, C) of $\mathbf{C}$-objects, *composition morphisms*

$$c_{A,B,C} \colon [A, B] \otimes [B, C] \to [A, C]$$

defined as those morphisms which correspond by adjunction to

$$[A, B] \otimes [B, C] \otimes A \simeq [A, B] \otimes A \otimes [B, C] \xrightarrow{ev_{A,B} \otimes [B,C]} B \otimes [B \otimes C] \xrightarrow{ev_{B,C}} C.$$

Similarly there are, for each quadruple (A, B, A', B'), *natural morphisms*

$$n_{A,B,A',B'} \colon [A, B] \otimes [A', B'] \to [(A \otimes A'), (B \otimes B')]$$

corresponding by adjunction to

$$[A, B] \otimes [A', B'] \otimes (A \otimes A') \xrightarrow{\mathsf{id} \otimes s \otimes \mathsf{id}} [A, B] \otimes A \otimes [A', B'] \otimes A' \xrightarrow{ev_{A,B} \otimes ev_{A',B'}} B \otimes B'.$$

Note that, by definition, the composition and the natural morphisms arise as adjoints of (components of) natural transformations, hence they form natural transformations c and n as well. Moreover, one has

37 Proposition *Let $\mathbb{C}$ be a symmetric monoidal closed category. Then $(C,D) \mapsto [C,D]$ defines a bifunctor (called the* internal hom-functor*) $[-,-]\colon \mathbf{C}^{\mathsf{op}} \times \mathbf{C} \to \mathbf{C}$ and the following hold:*

1. $\hom_{\mathbf{C}}(I,-) \circ [-,-] = \hom_{\mathbf{C}}(-,-)$,
2. $([-,-], n, u_I)$ *is a monoidal functor* $\mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}$.

38 Example

1. The cartesian category **Set** is closed with $[X,Y]$ being the set of all maps from X to Y. The evaluation morphisms are the usual evaluation maps, and the units are the maps 'picking out' the identities. The composition morphisms map pairs of composable maps (f,g) to $f \circ g$. The natural morphisms send pairs of maps $(A \xrightarrow{f} B, A' \xrightarrow{g} B')$ to their product $A \times A' \xrightarrow{f\times g} B \times B'$.
2. The monoidal category $\mathbf{Mod}_R$ is closed with $[X,Y]$ being the set of all R-linear maps from X to Y. The evaluation morphisms are the maps with $f \otimes x \mapsto f(x)$ and the units are, again, the (linear) maps 'picking out' the identities. The composition morphisms map pairs of composable R-linear maps (f,g) to $f \circ g$. The natural morphisms send elements $f \otimes g \in \hom_R(A,B) \otimes \hom_R(A',B')$ to the linear map $A \otimes A' \xrightarrow{f\otimes g} B \otimes B'$.

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