Limits and Colimits of Hopf Algebras

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Abstract

It is shown that for any commutative unital ring R the category $\mathbf{Hopf}_R$ of R–Hopf algebras is locally presentable and a coreflective subcategory of the category $\mathbf{Bialg}_R$ of R–bialgebras, admitting cofree Hopf algebras over arbitrary R–algebras. The proofs are based on an explicit analysis of the construction of colimits of Hopf algebras, which generalizes an observation of Takeuchi. Essentially be a duality argument also the dual statement, namely that $\mathbf{Hopf}_R$ is closed in $\mathbf{Bialg}_R$ under limits, is shown to hold, provided that the ring R is von Neumann regular. It then follows that $\mathbf{Hopf}_R$ is reflective in $\mathbf{Bialg}_R$ and admits free Hopf algebras over arbitrary R–coalgebras, for any von Neumann regular ring R. Finally, Takeuchi's free Hopf algebra construction is analysed and shown to be simply a composition of standard categorical constructions. By simple dualization also a construction of the Hopf coreflection is provided.

Keywords: Hopf algebras, bialgebras, limits, colimits, left and right adjoints
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Introduction

In his seminal monograph on Hopf algebras [24] Sweedler already made the claims that (a) for any algebra A there exists a cofree Hopf algebra over A and (b) for any coalgebra C there exists a free Hopf algebra over C. He did not give any proofs and it took quite a couple of years until Tackeuchi [26] proved claim (b). A proof of (a) seems not to be known. More recent books on the topic like [8] do not mention these problems at all. Pareigis' lecture notes [15] recall a construction of a Hopf reflection

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of given bialgebra (also called *Hopf envelope*), attributed to Manin [14] by Škoda [23], which is very much in line with Takeuchi's construction. In fact, the existence of free Hopf algebras over coalgebras, that is, the existence of a left adjoint to the forgetful functor from $\mathbf{Hopf}_k$, the category of Hopf algebras, to the category $\mathbf{Coalg}_k$ of coalgebras (all relative to a fixed field k) and the existence of Hopf reflections of bialgebras are equivalent (see [21]). Street [25] also shows reflectivity of $\mathbf{Hopf}_k$ in $\mathbf{Bialg}_k$ in a quite different way. Both approaches seem to be limited to the field case.

In this note proofs of Sweedler's claims (a) and (b) will be provided, which are based on the crucial results, stated as Theorem 11 below, that for any commutative unital ring R the category $\mathbf{Hopf}_R$ of Hopf algebras over R is (1) closed in the category $\mathbf{Bialg}_R$ of bialgebras over R with respect to colimits and (2) closed with respect to limits, provided that the ring R is von Neumann regular. This generalizes substantially Takeuchi's observation, that $\mathbf{Hopf}_R$ is closed in $\mathbf{Bialg}_R$ with respect to coproducts, if R is a field (see [26]). Standard category theoretic arguments, namely the Special Adjoint Functor Theorem and the reflection theorem for locally presentable categories respectively, then provide the required left and right adjoints in a straightforward way.

The proof of Theorem 11 certainly requires descriptions of limits and colimits in $\mathbf{Bialg}_R$. This seems to be a difficult problem at first since $\mathbf{Bialg}_R$ emerges as a combination of algebraic constructions ($\mathbf{Bialg}_R \to \mathbf{Coalg}_R$, $\mathbf{Alg}_R \to \mathbf{Mod}_R$) and coalgebraic constructions ($\mathbf{Bialg}_R \to \mathbf{Alg}_R$, $\mathbf{Coalg}_R \to \mathbf{Mod}_R$), where the algebraic constructions behave nicely with respect to limits and badly with respect to colimits (and the other way round for the coalgebraic ones). And this is probably the reason that not much seems to be known about these limits and colimits in general yet (with the exception of coproducts, which are described in the field case in [26]); even their sheer existence has only been proved recently [20]. As it turns out, however, a standard categorical construction of colimits along a suitable right adjoint functor (see [1, 23.11]) in connection with the well known fact that monadic functors create limits is enough to describe colimits and limits in $\mathbf{Bialg}_R$ in a sufficiently explicit way. This construction of colimits is in fact carried out on a somewhat higher level of abstraction, namely that of monoids, comonoids and bimonoids over a symmetric monoidal category, since this way one gets the required description of limits by simple dualization, that is, without a separate proof, out of that of colimits.

The final step, showing that a colimit of Hopf algebras, when performed in $\mathbf{Bialg}_R$, is again Hopf algebra, then requires a property of symmetric monoidal categories, which so far (except for the trivial case of $\mathbf{Set}$) only was known to be satisfied by $\mathbf{Mod}_R$, the category of R-modules, and its dual—though with completely different and technically non trivial proofs. This fact is given in Lemma 7 in the first section.

The existence of colimits in $\mathbf{Hopf}_R$ for every commutative unital ring R obtained this way has a remarkable consequence: Since the category $\mathbf{Hopf}_R$ already was known to be accessible (see [21]) one now can conclude that it is even locally presentable, thus it not only has all colimits but in particular also all limits, is wellpowered and co-wellpowered, has (epi, extremal mono)- and (extremal epi, mono)-factorizations of morphisms and a generator.

Finally we try to provide a better understanding of Takeuchi's and Manin's constructions. We sketch how to show that they are nothing but special instances of standard categorical constructions.

In the final stage of preparing this paper I became aware of the recent preprints [5] and [7], which also deal with coreflectivity of the category of Hopf algebras in that of bialgebras (over a field only). The following comments concerning the overlap with these notes seem to be appropriate. [5] essentially reproves Takeuchi's result on coproducts of Hopf algebras mentioned above and then uses the Special Adjoint Functor Theorem as we do here; missing the categorical content of this coproduct construction the author cannot dualize her result. [7] essentially describes this coreflection explicitly. The author does not notice that this simply can be obtained by dualization of the construction of the Hopf envelope.

1. Notation and prerequisites

The results of this note are mainly obtained by using concepts and results from category theory. The reader not completely familiar with these is referred to the respective literature as follows: For general concepts use e.g. [1] or [13], for the theory of accessible and locally presentable categories use [4], concerning monoidal categories consult [13] or [12]. A suitable web reference is `http://ncatlab.org/nlab/show/HomePage`.

1.1. Categories of monoids

Throughout $\mathbb{C} = (\mathbf{C}, - \otimes -, I, \alpha, \lambda, \varrho, \tau)$ denotes a symmetric monoidal category with α the associativity and λ and ϱ the left and right unit constraints, respectively. τ denotes the symmetry. We assume in addition that $\mathbf{C}$ is a locally presentable category. A special instance of this situation is the category $\mathbf{Mod}_R$ of R–modules over a commutative unital ring.

Note that the dual $\mathbf{C}^{\mathsf{op}}$, equipped with the tensor product of $\mathbb{C}$ then also is symmetric monoidal category, $\mathbb{C}^{\mathsf{op}}$; $\mathbf{C}^{\mathsf{op}}$ however will fail to be locally presentable.

By $\mathbf{Mon}\mathbb{C}$ and $\mathbf{Comon}\mathbb{C}$ we denote the categories of monoids (C, M, e) in $\mathbb{C}$ and comonoids (C, Δ, ϵ) in $\mathbb{C}$, respectively. Obviously one has

$$\mathbf{Mon}(\mathbb{C}^{\mathsf{op}}) = (\mathbf{Comon}\mathbb{C})^{\mathsf{op}}. \tag{1}$$

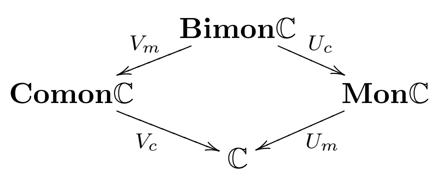
It is well known (see [12]) that $\mathbf{Mon}\mathbb{C}$ again is a symmetric monoidal category with tensor product

$$(C, M, e) \otimes (C', M', e') = \big(C \otimes C', (M \otimes M') \circ (C \otimes \tau \otimes C'), (e \otimes e') \circ \lambda_I^{-1}\big) \tag{2}$$

Consequently, $\mathbf{Mon}(\mathbf{Comod}\mathbb{C})$ and $\mathbf{Comon}(\mathbf{Mon}\mathbb{C})$ are defined. Both of these categories then coincide (more precisely: are isomorphic) and known as the category $\mathbf{Bimon}\mathbb{C}$ of bimonoids in $\mathbb{C}$ (see [20], [25]). It then is obvious that also

$$\mathbf{Bimon}(\mathbb{C}^{\mathsf{op}}) = (\mathbf{Bimon}\mathbb{C})^{\mathsf{op}} \tag{3}$$

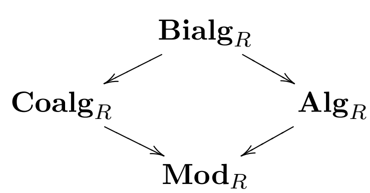
There then are natural underlying functors as follows



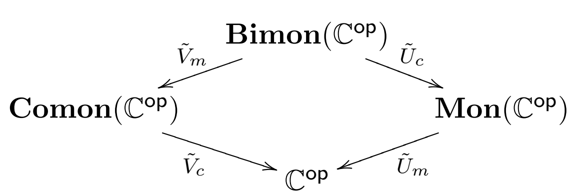
(4)

With $\mathbb{C} = \mathbf{Mod}_R$ this is

(4')



Note that by equations (1) and (3) the following digrams coincide, where (5) is simply (4), with $\mathbb{C}$ replaced by its dual, and (6) is (4^{op}).



(5)

$$\begin{array}{ccccc} & & (\mathbf{Bimon}\mathbb{C})^{\mathsf{op}} & & \\ & \swarrow^{U_c^{\mathsf{op}}} & & \searrow^{V_m^{\mathsf{op}}} & \\ (\mathbf{Mon}\mathbb{C})^{\mathsf{op}} & & & & (\mathbf{Comon}\mathbb{C})^{\mathsf{op}} \\ & \searrow_{U_m^{\mathsf{op}}} & & \swarrow_{V_c^{\mathsf{op}}} & \\ & & \mathbb{C}^{\mathsf{op}} & & \end{array} \tag{6}$$

By $\mathbf{Hopf}\mathbb{C}$ we denote the full subcategory of $\mathbf{Bimon}\mathbb{C}$ formed by the Hopf monoids over $\mathbb{C}$, that is, those bimonoids $(B, M, e, \Delta, \epsilon)$ which carry an antipode. An antipode here is a $\mathbf{C}$–morphism $S\colon B \to B$ satisfying the equations

$$\big(B \xrightarrow{\Delta} B \otimes B \underset{\mathsf{id}\otimes S}{\overset{S\otimes\mathsf{id}}{\rightrightarrows}} B \otimes B \xrightarrow{M} B\big) = \big(B \xrightarrow{\epsilon} I \xrightarrow{e} B\big). \tag{7}$$

Occasionally the morphisms $M \circ (S \otimes \mathsf{id}) \circ \Delta$ and $M \circ (\mathsf{id} \otimes S) \circ \Delta$ are abbreviated by $S \star \mathsf{id}$ and $\mathsf{id} \star S$ respectively.

As in the special case of $\mathbb{C} = \mathbf{Mod}_R$ an antipode is both, a monoid–morphism $(B, M, e) \to (B, M \circ \tau, e) =: (B, M, e)^{\mathsf{op}}$ and a comonoid morphism $(B, \Delta, \epsilon) \to (B, \tau \circ \Delta, \epsilon) =: (B, \Delta, \epsilon)^{\mathsf{cop}}$ (in fact, the latter property comes out by simple dualization of the former due to the dualization principle expressed by equality of diagrams (5) and (6) above) thus, S is bimonoid morphism from $(B, M, e, \Delta, \epsilon)$ into $(B, M, e, \Delta, \epsilon)^{\mathsf{op,cop}}$. Also, any bimonoid morphism between Hopf monoids commutes with the antipodes.

One clearly has

$$\mathbf{Hopf}(\mathbb{C}^{\mathsf{op}}) = (\mathbf{Hopf}\mathbb{C})^{\mathsf{op}} \tag{8}$$

The full embedding $\mathbf{Hopf}\mathbb{C} \to \mathbf{Bimon}\mathbb{C}$ will be called E. It has been shown in [21] that $\mathbf{Hopf}\mathbb{C}$ is reflective in $\mathbf{Bimon}\mathbb{C}$, i.e., that E has a left adjoint, iff $V_m \circ E$ has a left adjoint, i.e., if there is a free Hopf monoid over each comonoid over $\mathbb{C}$. Also, $\mathbf{Hopf}\mathbb{C}$ is coreflective in $\mathbf{Bimon}\mathbb{C}$ iff $U_c \circ E$ has a right adjoint.

We recall from [20]

1 Facts *For any symmetric monoidal category $\mathbb{C}$ with $\mathbf{C}$ locally presentable*

1. *the categories $\mathbf{Mon}\mathbb{C}$, $\mathbf{Comon}\mathbb{C}$ and $\mathbf{Bimon}\mathbb{C}$ are locally presentable,*
2. *the category $\mathbf{Hopf}\mathbb{C}$ is accessible,*
3. *the functors U_m and V_m are monadic,*
4. *the functors U_c and V_c are comonadic.*

2 Remark The left adjoints of U_m and V_m are given by MacLane's standard construction of free monoids (see [13]). In particular, the free R-algebra over an R-module M is the tensor algebra TM over M and the free R-bialgebra $T^{\star}C$ over a coalgebra C is the tensor algebra TV_cC over the underlying module of C endowed with the unique coalgebra structure (Δ, ϵ) making the embedding of V_cC into TV_cC (the unit of the adjunction for T) a coalgebra morphism.

The following has essentially been shown in [19] or follows from standard arguments concerning factorization structures:

3 Fact *Let $\mathbb{C}$ be a symmetric monoidal category, where $\mathbf{C}$ carries an (E, M)-factorization system for morphisms with $e \otimes e \in E$ for each $e \in E$. Assume further that the underlying functor $U\colon \mathbf{Mon}\mathbb{C} \to \mathbf{C}$ has a left adjoint. Then the following hold:*

1. $(U^{-1}[E], U^{-1}[M])$ *is a factorization system for morphisms in* $\mathbf{Mon}\mathbb{C}$ *and this is created by U.*
2. *If (E, M) is the (extremal epi, mono)-factorization, then so is $(U^{-1}[E], U^{-1}[M])$. In particular, U then preserves and reflects extremal epimorphism.*
3. *If (E, M) is the (extremal epi, mono)-factorization and extremal and regular epimorphisms coincide in $\mathbf{C}$, then they also coincide in $\mathbf{Mon}\mathbb{C}$.*

4 Remark In $\mathbf{Mod}_R$ epimorphisms, extremal epimorphisms and regular epimorphisms coincide (they are the surjective linear maps); but also monomorphisms, extremal monomorphisms and regular monomorphisms coincide (they are the injective linear maps). As a consequence, the image factorization of homomorphisms in $\mathbf{Mod}_R$ lifts to a factorization system not only always in $\mathbf{Alg}_R$, but also in $\mathbf{Coalg}_R$, provided that R is von Neumann regular (recall that a commutative unital ring R is called *von Neumann regular* iff R is a subring of a product of fields closed under taking “weak inverses” of elements $x \in R$—the unique element y such that $xyx = x$ and $yxy = y$—and that this is equivalent to the fact that, for each injective R-linear map f and each R-module M, the map $f \otimes \mathsf{id}_M$ is injective). While the lifted factorization in $\mathbf{Alg}_R$ is the (regular epi, mono)-factorization, it is the (epi, regular mono)-factorization in $\mathbf{Coalg}_R$. Consequently, then the surjections are precisely the epimorphisms in $\mathbf{Coalg}_R$, while the injections are the regular monomorphisms.

Note that the category $\mathbf{Coalg}_R$, being locally presentable, in addition carries the (extremal epi, mono)-factorization system, different from the above. It should be as difficult to describe this explicitely as it is difficult to describe the (epi, extremal mono)-factorizations in $\mathbf{Alg}_R$.

The category $\mathbf{Bialg}_R$—again by local presentablity—has the (epi, extremal mono)- as well as the (extremal epi, mono)-factorization structure. In case of a von Neumann regular ring R it follows from the lemma above that the (coinciding) liftings of the image-factorization in $\mathbf{Mod}_R$ along $V_c \circ V_m$ and $U_m \circ U_c$ also yield a (surjective, injective)-factorization structure. It seems unclear whether this coincides with one of the others. One certainly has, for a morphism f in $\mathbf{Bialg}_R$, the implications

1. f is an extremal epic $\implies$ f is surjective $\implies$ f is an epimorphism.
2. f is an extremal mono $\implies$ f is injective $\implies$ f is a monomorphism.

It is, moreover, easy to see that the category $\mathbf{Hopf}_R$ is closed in $\mathbf{Bialg}_R$ under image factorizations, if R is von Neumann regular.

It is easy to see that the statements of Fact 3 above generalize to factorization systems of cones in the sense of [1]. In particular, if $\mathbf{C}$ has regular factorizations of cones (see [1, 14.14]) then so has $\mathbf{Mon}\mathbb{C}$, provided that the tensor product of two regular epimorphisms in $\mathbb{C}$ again is a regular epimorphism. Clearly, $\mathbf{Mod}_R$ and $\mathbf{Alg}_R$ are instances of this situation, but also $\mathbf{Mod}_R^{\mathsf{op}}$, provided that the ring R is von Neumann regular.

We recall for further use how the regular factorizations in these cases are performed (see [19]): If $((M_i \xrightarrow{f_i} M)_{i\in I}, M)$ is a cocone of homomorphisms, where I is a non-empty class, chose a representative set $S = \{\mathrm{im} f_j \mid j \in J\}$ of the class of all images $\mathrm{im} f_i,\ i \in I$ (which is possible since M only has a set of subobjects). Denote by $m\colon N \to M$ the embedding of the submodule and subalgebra respectively $N := < \cup_J Imf_j >$ generated by $\cup_J Imf_j$ (in the module case this is simply $\sum_J \mathrm{im} f_j$) into M and by $\tilde{f}_i\colon M_i \to N$ the obvious homomorphism induces by f_i. Then

$$f_i = M_i \xrightarrow{\tilde{f}_i} N \xrightarrow{m} M$$

is the desired factorization. If $I = \emptyset$ the factorization is simply given by the embedding of the trivial submodule into M.

Calling a monadic functor $U\colon \mathbf{A} \to \mathbf{C}$ *regularly monadic* (see [1]), whenever $\mathbf{C}$ has regular factorizations and U preserves regular epimorphisms, we thus obtain:

5 Lemma *The underlying functor* $\mathbf{Alg}_R \to \mathbf{Mod}_R$ *is regularly monadic. The underlying functor* $\mathbf{Coalg}_R \to \mathbf{Mod}_R$ *is* coregularly comonadic*, that is, the dual of regularly monadic functor, provided that* R *is a von Neumann regular ring.*

1.2. Testing on antipodes

There is a familiar test for antipodes (see e.g. [15, 2.1.3] or [8, 4.3.3]) based on the first of the following facts:

6 Fact *Let B be an R-bialgebra and $S\colon B \to B^{\mathsf{op},\mathsf{cop}}$ a bialgebra homomorphism.*

1. $S \star \mathsf{id}(x) = e \circ \epsilon(x) = \mathsf{id} \star S(x)$ *and* $S \star \mathsf{id}(y) = e \circ \epsilon(y) = \mathsf{id} \star S(y)$ *implies* $S \star \mathsf{id}(xy) = e \circ \epsilon(xy) = \mathsf{id} \star S(xy)$.
2. *For* $I = \mathrm{im}(S \star \mathsf{id} - e \circ \epsilon)$ *and* $J = \mathrm{im}(\mathsf{id} \star S - e \circ \epsilon)$ *one has* $\Delta[I] \subset B \otimes I + I \otimes B$ *and* $\Delta[J] \subset B \otimes J + J \otimes B$.

These facts are special instances of the following result.

7 Lemma *Let $(B, M, \Delta, e, \epsilon)$ be a bimonoid and $S\colon (B, M, e) \to (B, M, e)^{\mathsf{op}}$ a homomorphism of bimonoids. Denote by $(E, \eta\colon E \to B)$ the (multiple) equalizer of $S \star \mathsf{id}$, $\mathsf{id} \star S$ and $e \circ \epsilon$ in $\mathbf{C}$. Then E carries a (unique) monoid structure such that η becomes the embedding of a submonoid of (B, M, e).*

Proof: In order to prove that E carries a multiplication M' preserved by η, it suffices show that the equations

$$(S\star\mathsf{id})\circ\big((M\circ(\eta\otimes\eta)\big) = (e\circ\epsilon)\circ\big((M\circ(\eta\otimes\eta)\big) = (\mathsf{id}\star S)\circ\big((M\circ(\eta\otimes\eta)\big) \quad (9)$$

hold, since then, by the equalizer property of η, $M \circ (\eta \otimes \eta)$ factors through η. Associativity of M' then follows trivially from that of M since η is a monomorphism.

We proceed as follows: Assume that the following two equations hold (with $M_3 := M \otimes (\mathsf{id} \otimes M) = (\mathsf{id} \otimes M) \circ M$)

$$M_3 \circ \big((\tau \otimes \mathsf{id}) \circ (\mathsf{id} \otimes S \otimes \mathsf{id}) \circ (S \star \mathsf{id}) \otimes \Delta\big) = (S \star \mathsf{id}) \circ M \qquad (10)$$

$$M_3 \circ \big((\tau \otimes \mathsf{id}) \circ (\mathsf{id} \otimes S \otimes \mathsf{id}) \circ (e \circ \epsilon) \otimes \Delta\big) = (\epsilon \otimes \mathsf{id}) \circ \big((\mathsf{id} \otimes (S \star \mathsf{id})\big) \quad (11)$$

Since, by from the equalizing property of η, also

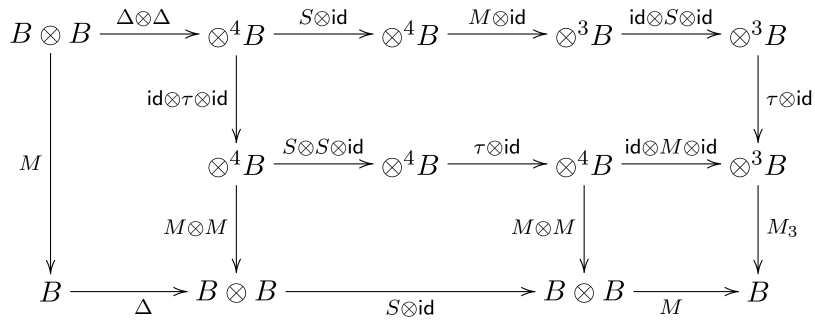
$$\big((S \star \mathsf{id}) \otimes \Delta\big) \circ (\eta \otimes \eta) = \big((e \circ \epsilon) \otimes \Delta\big) \circ (\eta \otimes \eta)$$

equations (10) and (11) imply (omitting the canonical isomorphism $I \otimes I \simeq I$)

$$\big((\epsilon \otimes \mathsf{id}) \circ (\mathsf{id} \otimes (S \star \mathsf{id})\big) \circ (\eta \otimes \eta) = \big((S \star \mathsf{id}) \circ M\big) \circ (\eta \otimes \eta)$$

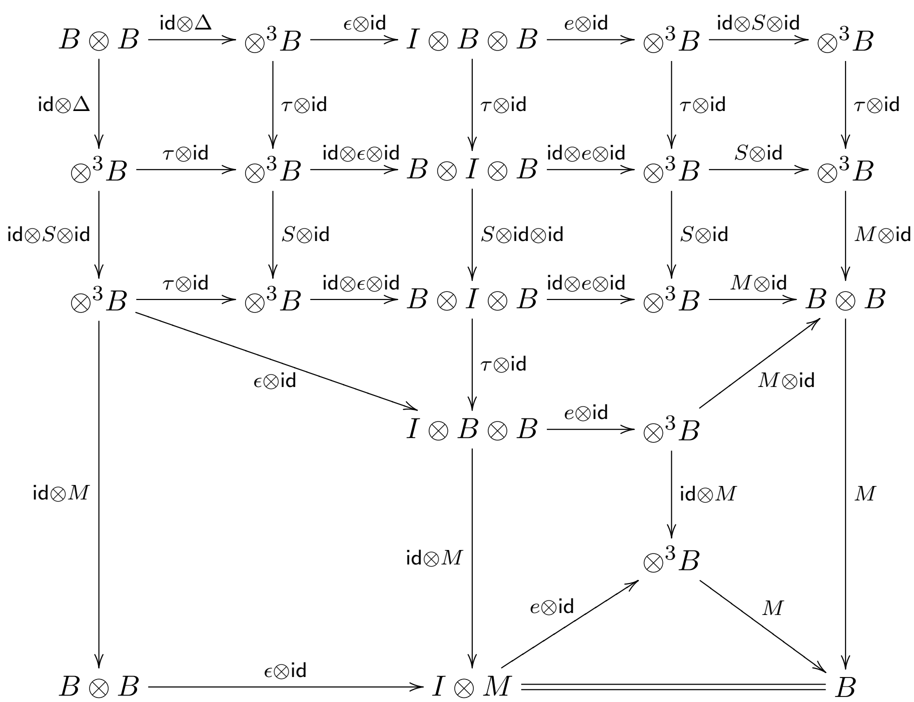
Since ϵ is a monoid homomorphism, one has $e \circ \epsilon \circ M = e \circ (\epsilon \otimes \epsilon) = \epsilon \otimes (e \circ \epsilon)$ which, together with the last equation, implies the first of the required equalities (9). It thus remains to prove the equalities (10) and (11) above.

Equation (10) means commutativity of the outer frame of the diagram



Here the left hand rectangle commutes, since Δ is a homomorphism of monoids; the lower middle rectangle commutes, since S is an anti-homomorphism of monoids; the lower right hand rectangle commutes by associativity of M. Commutativity of the upper right hand rectangle is a consequence of naturality of τ and τ's coherence property.

Equation (11) is equivalent to the commutativity of the outer frame of the diagram



which easily follows using naturality of τ, associativity of M and the axioms for the unit e.

The second of the required equalities (9) follows analogously.

It now remains to get a unit $e' \colon I \to E$, preserved by η. For this we need to verify the equation

$$(M \circ (S \otimes \mathsf{id}) \circ \Delta) \circ e = (e \circ \epsilon) \circ e \tag{12}$$

Then, by the equalizer property of (E, η), $e \colon I \to B$ will factor as

$$(I \xrightarrow{e} B) = (I \xrightarrow{e'} E \xrightarrow{\eta} B)$$

such that it finally remains to prove that e' acts as a unit for M'.

First, $\epsilon \circ e = \mathsf{id}_I$ because ϵ is an algebra homomorphism and hence $e \circ \epsilon \circ e = e$.

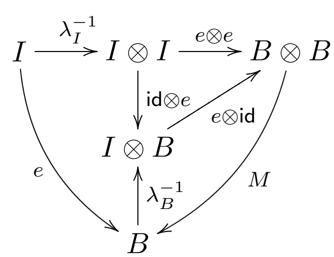
But also, since Δ is an algebra homomorphism, we have

$$\Delta \circ e = (I \xrightarrow{\lambda_I^{-1}} I \otimes I \xrightarrow{e \otimes e} B \otimes B)$$

and, since S is an algebra (anti) homomorphism, $S \circ e = e$. Therefore,

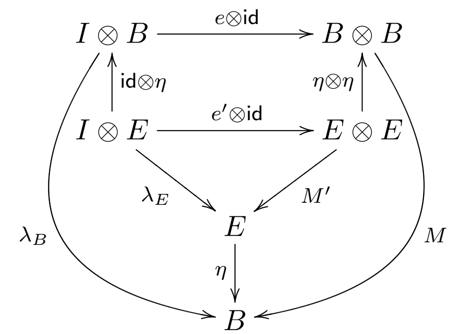
$$\begin{aligned} M \circ (S \otimes \mathsf{id}) \circ \Delta \circ e &= M \circ (S \otimes \mathsf{id}) \circ (e \otimes e) \circ \lambda_I^{-1} \\ &= M \circ ((S \circ e) \otimes e) \circ \lambda_I^{-1} \\ &= M \circ (e \otimes e) \circ \lambda_I^{-1} \\ &= e \end{aligned}$$

where the last equality follows from commutativity of the diagram



Here, the left triangle commutes since $\lambda^{-1} \colon \mathsf{id}_{\mathbf{C}} \to I \otimes -$ is natural, the upper right triangle since $- \otimes -$ is functorial, and the lower right triangle by the monoid axioms for (B, M, e).

Thus, equation (12) holds. Finally, (E, M', e') is a monoid in $\mathbb{C}$: In the diagram below the left hand triangle and the upper square commute trivially, the right hand triangle commutes by definition of M' and the outer triangle by the monoid axioms for (B, M, e). Now the desired equality $M' \circ (e' \otimes \mathsf{id}) = \lambda_E$ follows, since η is a monomorphism.



The second equality follows analogously. □

8 Remark The (at first glance) unrelated statements 1 and 2 of Fact 6 are the duals of each other in the following sense: While 1 essentially expresses the statement of the previous lemma for the special case $\mathbb{C} = \mathbf{Mod}_R$, statement 2 expresses this statement in the special case $\mathbb{C} = \mathbf{Mod}_R^{\mathsf{op}}$ for any von Neumann regular ring R. In fact, in this case (see e.g. [6, 40.12]) 2 means $\Delta[I + J] \subset \ker \rho \otimes \rho$ such that there is an R-linear map $\Delta'\colon Q \to Q \otimes Q$ satisfying $(\rho \otimes \rho) \circ \Delta = \Delta' \circ \rho$, where $(Q, \rho\colon B \to Q)$ is the (multiple) coequalizer of $S \star \mathsf{id}$, $\mathsf{id} \star S$ and $e \circ \epsilon$ (note that the coequalizer of these maps is the quotient $\rho\colon B \to B/(I + J)$ with I and J as in statement 2 above). And this is nothing but the statement of the lemma above for $\mathbf{Mod}_R^{\mathsf{op}}$ (recall equation (3) and the fact that now the roles of e and ϵ and M and Δ respectively have to be changed).

The above-mentioned test on antipodes then can be generalized due to the following additional observation.

9 Lemma *Assume that $U\colon \mathbf{Mon}\mathbb{C} \to \mathbf{C}$ is regularly monadic.*

For any pair of $\mathbf{C}$–morphisms $f, g\colon UA \to UN$ one has $f = g$, provided that there exists a U-universal arrow $u\colon C \to UC^\sharp$ and a regular epimorphism $q\colon C^\sharp \to A$ (that is, there exists a representation of A as a regular quotient of a free monoid $C^\sharp$), such that

1. *the equalizer $(E, \eta\colon E \to UA)$ of f, g in $\mathbf{C}$ carries the structure of submonoid of A with embedding η, and*
2. *$f \circ q \circ u = g \circ q \circ u$.*

2. Limits and colimits

2.1. Limits and colimits in **Bimon**$\mathbb{C}$

The behaviour of the functors U_m and V_m in Diagram 4 with respect to limits is simple and well known — they create limits, because they are monadic. Dually U_c and V_c create colimits. This section is devoted to the behaviour of U_m and V_m towards colimits and that of U_c and V_c towards limits, respectively.

Recall the following colimit construction from ([1, 23.11, 23.20]), which is nothing but a categorical abstraction (due to Herrlich [9]) of the familiar colimit construction in Birkhoff varieties (see e.g. [11, Thm. 2.11]): Let $U\colon \mathbf{A} \to \mathbf{C}$ be a regularly monadic functor. Then a colimit of a diagram $D\colon \mathbf{I} \to \mathbf{A}$ (whith $D_i := D(i)$ for $i \in \mathsf{ob}\mathbf{I}$) can be constructed as follows:

1. Chose a colimit $\big(C, (UD_i \xrightarrow{\mu_i} C)_{i\in\mathsf{ob}\mathbf{I}}\big)$ of UD in $\mathbf{C}$.
2. Choose a U–universal morphism $u_C\colon C \to UC^\sharp$.
3. Form the collection of all $\mathbf{A}$–morphisms $f_j\colon C^\sharp \to A_j$ ($j \in J$) such that, for each i in $\mathsf{ob}\mathbf{I}$,
$$UD_i \xrightarrow{\mu_i} C \xrightarrow{u_C} UC^\sharp \xrightarrow{Uf_j} UA_j$$
is the U–image of some $\mathbf{A}$–morphism $h_{ij}\colon D_i \to A_j$ (note, that J might be a proper class).
4. Factorize the cone $\big(C^\sharp, (f_j)_{j\in J}\big)$ as
$$C^\sharp \xrightarrow{q} A \xrightarrow{m_j} A_j$$
with a regular epimorphism q and a mono–cone $\big(A, (m_j)_{j\in J}\big)$. This is possible by our assumptions.

Then, again by the assumptions on U, for each $i \in \mathsf{ob}I$, the morphism

$$UD_i \xrightarrow{\mu_i} C \xrightarrow{u_C} UC^\sharp \xrightarrow{Uq} UA \tag{13}$$

is the U–image of a (unique) $\mathbf{A}$–morphism $D_i \xrightarrow{\lambda_i} A$. The cocone $\big(A, (D_i \xrightarrow{\lambda_i} A)_i\big)$ then is a colimit of D.

10 Examples a. In order to construct a colimit of a diagram $D\colon \mathbf{I} \to \mathbf{Alg}_R$ one first forms a colimit $UD_i \xrightarrow{\mu_i} C$ of UD in $\mathbf{Mod}_R$ ($U\colon \mathbf{Alg}_R \to \mathbf{Mod}_R$ the forgetful functor), then builds the tensor algebra TC of C (this is the application of a left adjoint of U) and finally factors TC modulo an appropriate ideal I (since the regular epimorphisms in $\mathbf{Alg}_R$ are the surjective homomorphisms) — see [15] for an explicit description of I. This gives the colimit $(A, (\lambda_i))$ in $\mathbf{Alg}_R$.

Since the forgetful functor $V\colon \mathbf{Bialg}_R \to \mathbf{Alg}_R$ is comonadic, V creates colimits. Therefore, a colimit of a diagram $D\colon \mathbf{I} \to \mathbf{Bialg}_R$ can be constructed as follows: First form a colimit $VD_i \xrightarrow{\lambda_i} A$ of VD in $\mathbf{Alg}_R$ as above. Then the algebra A can be equipped with a unique pair of homomorphisms $(A \xrightarrow{\Delta} A \otimes A, A \xrightarrow{\epsilon} R)$ such that it becomes a bialgebra $\tilde{A}$ and each $\lambda_i\colon D_i \to \tilde{A}$ a bialgebra homomorphism. $\big(\tilde{A}, (\lambda_i)\big)$ then is a colimit of D. In particular, Δ and ϵ are determined by commutativity of the diagrams

$$\begin{array}{ccc} D_i & \xrightarrow{\lambda_i} & A \\ {\scriptstyle\Delta_i}\big\downarrow & & \big\downarrow{\scriptstyle\Delta} \\ D_i \otimes D_i & \xrightarrow[\lambda_i \otimes \lambda_i]{} & A \otimes A \end{array} \qquad\qquad \begin{array}{ccc} D_i & \xrightarrow{\lambda_i} & A \\ & {\scriptstyle\epsilon_i}\searrow & \big\downarrow{\scriptstyle\epsilon} \\ & & R \end{array} \tag{14}$$

Concerning coequalizers in $\mathbf{Bialg}_R$ this simply means that a coequalizer of a pair $f, g\colon B \to A$ — when performed in $\mathbf{Alg}_R$ as A/I with the ideal I generated by $\{f(b) - g(b) \mid b \in B\}$ — carries a unique bialgebra structure such that the quotient map also is a coalgebra homomorphism (in other words, I is a coideal), and that this then is a coequalizer in $\mathbf{Bialg}_R$.

b. For constructing limits in $\mathbf{Coalg}_R$ one can, by Lemma 5, make use of the dual of the above construction provided R is a von Neumann regular ring. Thus, a limit of the diagram $D\colon \mathbf{I} \to \mathbf{Coalg}_R$ is obtained from a limit $\big(A, (\pi_i\colon A \to VD_i)_i\big)$ of VD in $\mathbf{Mod}_R$ ($V\colon \mathbf{Coalg}_R \to \mathbf{Mod}_R$ the forgetful functor) by first forming the cofree coalgebra $VA^* \xrightarrow{\varrho} A$ on A. A limit L of D then is obtained by performing the (epi-sink, injective)–factorization of the family of all coalgebra homomorphisms $f_j\colon A_j \to A^*$ such that, for all $i \in \mathsf{ob}\mathbf{I}$, $\pi_i \circ \varrho \circ f_j$ is a coalgebra homomorphism.

Somewhat more explicitely, L is given by forming the sum of all subcoalgebras S_k of A^* such that the restriction of $\pi_i \circ \varrho$ to S_k is a coalgebra homomorphism.

Concerning equalizers it would be simpler to proceed as follows. Since $\mathbf{Coalg}_R$ has (episink, regular mono)-factorizations (see Remark 4.2) an equalizer $E \xrightarrow{\eta} B$ of a pair $f, g\colon B \to A$ of homomorphisms is obtained by forming this factorization

$$C_h \xrightarrow{e_h} E \xrightarrow{\eta} B$$

of the cocone of all homomorphims $h\colon C_h \to B$ with $f \circ h = g \circ h$ (see [1, 15.7]). E thus is, as a module, the sum of all subcoalgebras of B contained in the kernel of $f - g$.

Since the forgetful functor $V\colon \mathbf{Bialg}_R \to \mathbf{Coalg}_R$ is monadic it creates limits. Therefore, a limit of a diagram $D\colon \mathbf{I} \to \mathbf{Bialg}_R$ can be constructed as follows: First form a limit $A \xrightarrow{\pi_i} VD_i$ in $\mathbf{Coalg}_R$ as above. Then the coalgebra A can be equipped with a unique pair of homomorphisms $(A \otimes A \xrightarrow{M} A,\ R \xrightarrow{e} A)$ such that it becomes a bialgebra $\tilde{A}$ and each $\pi_i\colon \tilde{A} \to D_i$ a bialgebra homomorphism. $(\tilde{A}, (\pi_i))$ then is a limit of D. In particular, M and e are determined by commutativity of the diagrams

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\pi_i \otimes \pi_i} & D_i \otimes D_i \\ {\scriptstyle M}\downarrow & & \downarrow{\scriptstyle M_i \otimes M_i} \\ A & \xrightarrow[\pi_i]{} & D_i \end{array} \qquad\qquad \begin{array}{ccc} A & \xrightarrow{\pi_i} & D_i \\ & {\scriptstyle e}\nwarrow & \uparrow{\scriptstyle e_i} \\ & & R \end{array} \tag{15}$$

Note that the condition on R to be von Neumann regular is only needed to construct limits *this way*. Their sheer existence is given for any ring (see Facts 1).

2.2. Limits and Colimits in $\mathbf{Hopf}_R$

We are now investigating the problem, whether limits and colimits respectively of Hopf algebras, taken in the category of bialgebras, again are Hopf algebras. The case of coproducts for R a field can already be found in [26]. The following is our main result.

11 Theorem *Let R be a commutative unital ring. Then the following hold:*

1. $\mathbf{Hopf}_R$ *is closed under colimits in* $\mathbf{Bialg}_R$.
2. $\mathbf{Hopf}_R$ *is closed under limits in* $\mathbf{Bialg}_R$, *provided that the ring R is von Neumann regular.*

Proof: In fact we prove a bit more: If $\mathbb{C}$ is a symmetric monoidal category such that

1. $U_m\colon \mathbf{Mon}\mathbb{C} \to \mathbf{C}$ is regularly monadic,
2. $U_c\colon \mathbf{Bimon}\mathbb{C} \to \mathbf{MonC}$ is comonadic,

then $\mathbf{Hopf}\mathbb{C}$ is closed under colimits in $\mathbf{Bimon}\mathbb{C}$.

Let $D\colon \mathbf{I} \to \mathbf{Hopf}\mathbb{C}$ be a diagram and $\big(A, (D_i \xrightarrow{\lambda_i} A)_i\big)$ its colimit in $\mathbf{Bialg}\mathbb{C}$. We need to construct an antipode $S\colon A \to A$. Since, clearly, each λ_i also is a bimonoid morphism $D_i^{\mathsf{op},\mathsf{cop}} \to A^{\mathsf{op},\mathsf{cop}}$ as is, for each $i \in \mathsf{ob}\mathbf{I}$, the antipode $S_i\colon D_i \to D_i^{\mathsf{op},\mathsf{cop}}$ of the Hopf monoid D_i, the colimit property guarantees the existence of a unique bialgebra morphism $S\colon A \to A^{\mathsf{op,cop}}$ such that the following diagrams commute.

$$\begin{array}{ccc} A & \overset{S}{\dashrightarrow} & A^{\mathsf{op,cop}} \\ {\scriptstyle \lambda_i}\uparrow & & \uparrow{\scriptstyle \lambda_i} \\ D_i & \underset{S_i}{\longrightarrow} & D_i^{\mathsf{op,cop}} \end{array} \tag{16}$$

In the following we omit the underlying functors $\mathbf{Bimon}\mathbb{C} \to \mathbf{Mon}\mathbb{C} \to \mathbb{C}$. By the discussion in section 2 each colimit map λ_i is the composition

$$\lambda_i = (D_i \overset{\mu_i}{\to} C \overset{u}{\to} TC \overset{q}{\to} A)$$

where $\big(C, (\mu_i)\big)$ is a colimit of D in $\mathbf{C}$, u is the universal morphism from C into the free algebra TC over C, and q is a regular epimorphism in $\mathbf{Mon}\mathbb{C}$.

Consider the first diagram below, where the upper square commutes by definition of Δ (see equation 14), the lower square commutes since λ_i is, in particular, a monoid homomorphism and the middle square commutes by definition of S (see equation 16) and functoriality of $- \otimes -$. Thus the outer frame of the diagram commutes. Similarly, the outer frame of the second diagram commutes, since the upper rectangle commutes by definition of ϵ (see equation 14), while the lower one again commutes since λ_i is an algebra homomorphism.

$$\begin{array}{ccccccc} D_i & \xrightarrow{\mu_i} & C & \xrightarrow{u} & TC & \xrightarrow{q} & A \\ {\scriptstyle \Delta_i}\downarrow & & & & & & \downarrow{\scriptstyle \Delta} \\ D_i \otimes D_i & & & \xrightarrow{\lambda_i \otimes \lambda_i} & & & A \otimes A \\ {\scriptstyle \mathsf{id}\otimes S_i}\downarrow & & & & & & \downarrow{\scriptstyle \mathsf{id}\otimes S} \\ D_i \otimes D_i & & & \xrightarrow{\lambda_i \otimes \lambda_i} & & & A \otimes A \\ {\scriptstyle M_i}\downarrow & & & & & & \downarrow{\scriptstyle M} \\ D_i & \underset{\mu_i}{\longrightarrow} & C & \underset{u}{\longrightarrow} & TC & \underset{q}{\longrightarrow} & A \end{array}$$

$$\begin{array}{ccccccc} D_i & \xrightarrow{\mu_i} & C & \xrightarrow{u} & TC & \xrightarrow{q} & A \\ {\scriptstyle \epsilon_i}\downarrow & & & & & & \downarrow{\scriptstyle \epsilon} \\ R & & & = & & & R \\ {\scriptstyle e_i}\downarrow & & & & & & \downarrow{\scriptstyle e} \\ D_i & \underset{\mu_i}{\longrightarrow} & C & \underset{u}{\longrightarrow} & TC & \underset{q}{\longrightarrow} & A \end{array}$$

From $e_i \circ \epsilon_i = M_i \circ (\mathsf{Id} \otimes S_i) \circ \Delta_i$ for all i it thus follows that

$$e \circ \epsilon \circ (q \circ u) = [M \circ (\mathsf{id} \otimes S) \circ \Delta \circ (q \circ u)].$$

Since q is a (regular) epimorphism in $\mathbf{C}$, we can conclude by Lemmas 7 and 9 the desired identities

$$e \circ \epsilon = M \circ (\mathsf{id} \otimes S) \circ \Delta = M \circ (S \otimes \mathsf{id}) \circ \Delta$$

Statement 2. now follows dually: $\mathbf{Hopf}_R$ is closed under limits in $\mathbf{Bialg}_R$ iff $\mathbf{Hopf}_R^{\mathsf{op}}$ is closed in $\mathbf{Bialg}_R^{\mathsf{op}}$ under colimits. This follows from the above by equations (3) and (8) and Lemma 5. □

Since, for any commutative unital ring R, the category $\mathbf{Hopf}_R$ is accessible (see [21]) and accessible and cocomplete categories are locally presentable we obtain as a corollary

12 Theorem *For every commutative unital ring R the category $\mathbf{Hopf}_R$ is locally presentable. In particular, $\mathbf{Hopf}_R$ has all limits and colimits, is wellpowered and co-wellpowered, has (epi, extremal mono)- and (extremal epi, mono)-factorizations of morphisms and a generator*[1].

This generalizes a result of [21], where we had shown that the category of Hopf algebras over a field is locally presentable. Note in particular that the proof given above does not make use of the existence of free Hopf algebras as does the argument used in [21].

13 Remark Concerning the presentablity degree of $\mathbf{Hopf}_R$ we can say more, provided that R is von Neumann regular. Since, in this case, $\mathbf{Hopf}_R$ is closed in $\mathbf{Bialg}_R$ under limits and colimits and, moreover, $\mathbf{Bialg}_R$ is finitary monadic over $\mathbf{Coalg}_R$, $\mathbf{Hopf}_R$ is locally λ-presentable provided that $\mathbf{Coalg}_R$ is (use [4, 2.48]). By [3, IV.5] $\mathbf{Coalg}_R$ is locally $\aleph_1$ presentable, since it is a covariety (see [19]) and the relevant functor $\otimes^2 \times R$ preserves monomorphisms by our assumption on R.

The category of Hopf algebras over a field k even is locally finitely presentable: By the so-called *Fundamental Theorem of Coalgebras* (see e.g. [8, 1.4.7]) every coalgebra is a directed colimit of finitely dimensional vector spaces, which form a set of finitely presentable objects in the category of coalgebras (use [3]). This proves that $\mathbf{Coalg}_k$ and, thus, $\mathbf{Hopf}_k$ is locally finitely presentable.

[1] As opposed to [17] a generator here in general is not a singleton.

3. Free and Cofree Hopf algebras

As mentioned before Theorem 11 implies existence of cofree Hopf algebras on algebras (of free Hopf algebras on coalgebras) for any (regular) ring R by means of results of [21]. We recall the main arguments here as follows:

Since, for every ring R, the embedding $E\colon \mathbf{Hopf}_R \hookrightarrow \mathbf{Bialg}_R$ preserves colimits by Theorem 11 and the underlying functor $\mathbf{Bialg}_R \to \mathbf{Alg}_R$ is comonadic (and therefore preserves colimits), the underlying functor $U\colon \mathbf{Hopf}_R \to \mathbf{Alg}_R$ — being the composition of these — preserves colimits. Moreover, $\mathbf{Hopf}_R$ has a generator according to [17] or, independently, by [21]. Thus, existence of a right adjoint to U (as well as to E) follows by the Special Adjoint Functor Theorem.

Concerning the underlying functor $V\colon \mathbf{Hopf}_R \to \mathbf{Coalg}_R$, that is, the composition of E and the forgetful functor $\mathbf{Bialg}_R \to \mathbf{Coalg}_R$ we observe that by composition of adjoints and the fact that the latter functor has a left adjoint (see Facts 1) V has a left–adjoint provided that E has one. Now E preserves limits by Theorem 1, provided that R is a von Neumann regular ring. Since E also preserves colimits and both categories, $\mathbf{Bialg}_R$ and $\mathbf{Hopf}_R$ are locally presentable (see Facts 1 and Corollary 12), E has a left adjoint by [4, 1.66].

We thus arrive at our second main result

14 Theorem *Let R be a commutative unital ring. Then the following hold:*

1. $\mathbf{Hopf}_R$ *is coreflective in* $\mathbf{Bialg}_R$ *and the underlying functor* $\mathbf{Hopf}_R \to \mathbf{Alg}_R$ *has a right adjoint.*
2. $\mathbf{Hopf}_R$ *is reflective in* $\mathbf{Bialg}_R$ *and the underlying functor* $\mathbf{Hopf}_R \to \mathbf{Coalg}_R$ *has a left adjoint, provided that the ring R is von Neumann regular.*

By Beck's Theorem and its dual these results imply in view of Theorem 11

15 Corollary *For every von Neumann regular ring R the underlying functors* $\mathbf{Hopf}_R \to \mathbf{Coalg}_R$ *and* $\mathbf{Hopf}_R \to \mathbf{Alg}_R$ *are monadic and comonadic respectively.*

Occasionally it might be desirable to have a construction of the adjoints — we just proved to exist — at hand. We close this section in sketching them; details will appear elsewhere. Our construction of a reflection of $\mathbf{Bialg}_R$ into $\mathbf{Hopf}_R$ will essentially be a revision Manin's approach as presented in [23] and [16].

Our completely categorical approach will not only show that this construction is nothing but the composition of two standard categorical constructions, it is moreover dualizable to the extent that it provides also a construction for the coreflection (though only in the case of a von Neumann regular ring).

The construction of a free adjunction of an antipode to a bialgebra can best be understood as a composition of of two adjunctions. To make this precise we define a category $\mathbf{nHopf}\mathbb{C}$ of *near Hopf monoids* over $\mathbb{C}$ as follows: its objects are pairs (B, S) with a bimonoid B and a bimonoid homomorphism $S\colon B \to B^{\mathsf{op,cop}}$ (equivalently $S\colon B^{\mathsf{op,cop}} \to B$). A morphism $f\colon (B, S) \to (B', S')$ then is a bimonoid homomorphism satisfying $S \circ f = f \circ S'$. In other words, $\mathbf{nHopf}\mathbb{C}$ is the category $\mathbf{Alg}H$ of functor algebras for the endofunctor H on $\mathbf{Bialg}_R$ sending B to $B^{\mathsf{op,cop}}$.

The first step of the free adjunction of an antipode (see [15] or [26]), constructing a near Hopf algebra B^* out of given bialgebra B, then is nothing but the application of the standard construction of free functor algebras as described e.g. in [2] to this situation where one in particular uses the fact that the functor H also preserves finite coproducts.

The second step of our construction then is the construction of a reflection of $\mathbf{nHopf}\mathbb{C}$ into $\mathbf{Hopf}\mathbb{C}$. And this can be obtained by using [10, 37.1] with $e \in E$ iff e is surjective and $m \in M$ iff m is injective (this however requires the restriction to von Neumann regular rings R). This provides us with a surjective bialgebra homomorphism $q\colon B \to RB$ for every near Hopf algebra (B, S) as its Hopf reflection. Note that it is here were we use closure of $\mathbf{Hopf}_R$ in $\mathbf{Bialg}_R$ under products. Finally, one then can show that this quotient is given by the ideal described by Takeuchi.

A constructive description of the Hopf coreflection of a bialgebra B then can be obtained by duality, provided that R is von Neumann regular.

16 Remark Our approach is also applicable to the monoidal category of sets with cartesian product as tensor product. In that case the category of bimonoids is (isomorphic to) the category of (ordinary) monoids and the category of Hopf monoids is the category of groups. We thus get the familiar facts that the category of groups is reflective and coreflective in the category of monoids. There is, however, a notable difference between this situation and the case of Hopf algebras: While the coreflection from groups to monoids is a mono-coreflection (the coreflection of a monoid M is its subgroup of invertible elements) this is not the case for Hopf algebras. If the Hopf-coreflection of a bialgebra B always were a sub-bialgebra of B, this would imply that every bialgebra quotient of a Hopf algebra is a Hopf algebra (use the dual of [10, 37.1]); but this is not the case (not every bi-ideal in a Hopf algebra is a Hopf ideal). This answers a question left open in [5].

17 Problem Whenever the condition on R was used to be von Neumann regular, this was to ensure that, for an R-linear map $f\colon A \to B$, its tensor square $f \otimes f$ is injective again (see Remark 4), a condition for which injectivity of $f \otimes \mathsf{id}_A$ and $f \otimes \mathsf{id}_B$ would be sufficient. Von Neumann regularitiy of R, that is injectivity of $f \otimes \mathsf{id}_M$ for *any* R-module M for such f, thus is (at least formally) a too restrictive assumption. It would then in this context be interesting to be able to characterize those rings satisfying the condition really needed and to know to what extent these rings are really more general then the von Neumann regular ones.

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