Equivalence for Varieties in General and for $\mathcal{BOOL}$ in Particular

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Abstract

The varieties $\mathcal{W}$ equivalent to a given variety $\mathcal{V}$ are characterized in a purely categorical way. In fact they are described as the models of those Lawvere theories which are Morita equivalent to the Lawvere theory of $\mathcal{V}$ which therefore are characterized first. Along this way the conceptual meanings of the n–th matrix power construction of a variety and McKenzie's σ–modification of classes of algebras [22] become transparent. Besides other applications not only the well known equivalences between the varieties $\mathcal{P}_m$ of Post algebras of fixed orders m and the variety $\mathcal{BOOL}$ of Boolean algebras are obtained; moreover it can be shown that the varieties $\mathcal{P}_m$ are the only varieties equivalent to $\mathcal{BOOL}$. The results then are generalized to quasivarieties and more general classes of algebras.

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Introduction

In the recent paper [22] R. McKenzie characterizes the varieties $\mathcal{W}$ equivalent to a given variety $\mathcal{V}$ and, correspondingly, those Lawvere theories $\mathbb{S}$ with model categories $\mathsf{Mod}\mathbb{S}$ equivalent to $\mathsf{Mod}\mathbb{T}$ for a given Lawvere theory $\mathbb{T}$ in a purely algebraic fashion. Since the first systematic study on equivalent varieties is probably what is now usually called Morita equivalence of module categories (see Example 3 below) Lawvere theories with equivalent model categories will be called *Morita equivalent.*

In view of the one–to–one translation between general algebra and categorical algebra, available via Lawvere theories, one would expect that categorical methods are at least as appropriate as algebraic ones to attack this problem. That these

have not been used might be surprising or — rather — is proof of the fact that categorists have not yet successfully conveyed their ideas across the borders of their own community. The purpose of this note is then to contribute to the dissemination of categorical methods in not only obtaining the results of [22] by purely categorical methods but also by clarifying at the same time the constructions used there: the
! construction of the n–th matrix power $\mathcal{V}^{[n]}$ and McKenzie's σ–modification $\mathcal{V}(\sigma)$ from a given (subclass of a) variety $\mathcal{V}$, seemingly ad hoc constructions in the algebraic context, turn out to be canonical constructions from the categorical point of view: each variety $\mathcal{W}$ equivalent to a given variety $\mathcal{V}$ corresponds, for categorical reasons, to a retraction $Q \xrightarrow{s} Fn \xrightarrow{r} Q = 1_Q$ where Fn denotes the free $\mathcal{V}$–algebra on n generators. The case $r = 1_{Fn}$ then corresponds to $\mathcal{V}^{[n]}$ while the nontrivial retractions $Q \xrightarrow{s} F1 \xrightarrow{r} Q = 1_Q$ precisely correspond to the modifications $\mathcal{V}(\sigma)$ with $\sigma = sr(1) \in F1$. As a byproduct one sees that quasivarieties and other classes of similar algebras are stable under these constructions.

It should be pointed out that a variety $\mathcal{V}$ is not only an abstract category but also a concrete category since it is equipped, by its very definition, with a canonical underlying functor $|-|\colon \mathcal{V} \longrightarrow \mathcal{SET}$ into the category $\mathcal{SET}$ of sets and mappings. Therefore two different notions of equivalence have to be distinguished: *equivalence* (as categories) and *concrete equivalence* as defined below. Conceptually here only the first notion is of interest, though the second one will be of crucial technical importance.

These notions as well as some other basic categorical concepts which are crucial for our work but possibly not familiar to every algebraist (such as coequalizer or generator) will be explained (and interpreted in the algebraic context) in a preliminary section. Categorical notions which need to be mentioned but are of lesser importance for our arguments are put into footnotes in order to avoid unnecessary technicalities at the beginning.

We then proceed by giving a survey on the categorical machinery describing general algebra, developed mainly by Lawvere, Isbell, Linton, and others. Based on these facts we characterize all Lawvere theories $\mathbb{S}$ Morita equivalent to a given one $\mathbb{T}$. The result, equivalent to [22, Corollary 6.3], is already contained in the apparently not too well known paper [10]; our approach however is a more direct one. We then identify the models of theories $\mathbb{S}$ Morita equivalent to $\mathbb{T}$. In other words, we describe the varieties $\mathcal{W}$ equivalent to a given variety $\mathcal{V}$. Finally these results are generalized to quasivarieties and more general classes of algebras extending similar results in [22] and also solving some problems posed therein.

Throughout this presentation the variety $\mathcal{BOOL}$ of Boolean algebras will be used as a sort of leading example illustrating the concepts and constructions. These examples are labeled separately as "Leading example" in order to signal their importance. In following up the leading example we arrive at a complete description of all varieties equivalent to $\mathcal{BOOL}$ (Theorem 6): we not only obtain the well known equivalences between the varieties of Post algebras $\mathcal{P}_m$ of fixed orders m

and the variety $\mathcal{BOOL}$ of Boolean algebras (see e.g. [5] or, implicitely, [14]); it can be shown, moreover, that any variety $\mathcal{V}$ equivalent to $\mathcal{BOOL}$ is—for some $m \in \mathbb{N}$—(concretely isomorphic to) the variety $\mathcal{P}_m$ of Post algebras of order m; it then follows that the varieties $\mathcal{P}_m$ are (up to concrete isomorphism) the only varieties generated by an m–element primal algebra (Theorem 7). The fact that all what is needed from algebra in order to obtain these results is *restricted Stone–duality*, i.e., the dual equivalence between the categories $\mathcal{SET}_{fin}$ of finite sets and $\mathcal{BOOL}_{fin}$ of finite Boolean algebras might be seen as an additional indication of the suitability of the categorical methods used.

Instead of using theories one alternatively could have used monads. This approach is indicated in [8]. In order to make the exposition comprehensible also for the non-category-theorist we refrain from using advanced categorical tools. How these can be used in order to obtain similar results and also to describe effectively the functors which *establish* the equivalences between varieties, can be seen in [7].

Preliminaries

Limits and Colimits

Since products are well known in algebra we only have to fix our notation: given a set indexed family $(K_i)_{i\in I}$ of objects, its *product* is denoted by $\prod_I K_i$ or more precisely by $(\prod_I K_i,\ (\pi_i)_I)$ where the morphisms $\pi_j \colon \prod_I K_i \longrightarrow K_j$ are the *projections*. The morphism $f \colon L \longrightarrow \prod_I K_i$ induced by a family $(f_i \colon L \longrightarrow K_i)_I$, i.e., the morphism f with $\pi_i f = f_i$ for each $i \in I$, is usually denoted by $\langle f_i \rangle$. The dual concept of product is that of *coproduct*, denoted by $(\coprod_I K_i, (\mu_i)_I)$ with $(\mu_i \colon K_i \longrightarrow \coprod K_i)_I$ its family of *injections*. The morphism $f \colon \coprod_I K_i \longrightarrow L$ induced by a family $(f_i \colon K_i \longrightarrow L)$ is denoted by $[f_i]$.

Coproducts always exist in quasivarieties, but are well known constructions only in particular cases: they are the *direct sums* in module categories, the *free products* in the category of groups; under their categorical name they are used in distributive lattices [5] in order to define Post algebras (as coproducts of a Boolean algebra and a finite chain). In a poset, considered as a category, a coproduct of a family of objects (elements) is their join. As one writes A^I for a product of a constant family on I with value A (i.e., of $\sharp I$ copies of A), we write $I \cdot A$ (or simply IA) for a coproduct of such a family, and similarly for morphisms. In any variety one has, for any set X, $FX = X \cdot F1$, where FX is the free algebra over X and $F1$ is the free algebra on one generator.

The coincidence set of a pair of homomorphisms $f, g \colon A \longrightarrow B$ in a variety is known to be a subalgebra of A and characterized by the fact that any morphism $D \xrightarrow{d} A$ with $f \circ d = g \circ d$ factors over it uniquely. Thus this subalgebra (together with its embedding into A) is an example of an *equalizer*. The dual concept is that of coequalizer. A *coequalizer* of the pair $f, g \colon A \longrightarrow B$ thus is (Q, q) with $q \colon B \longrightarrow Q$

a morphism with $q \circ f = q \circ g$ such that any $c\colon B \longrightarrow C$ with $c \circ f = c \circ g$ factors uniquely over q. In varieties coequalizers are closely related to congruence relations: if ϱ is a congruence on A the quotient homomorphism $q\colon A \longrightarrow A/\varrho$ is a coequalizer of the pair $\varrho \hookrightarrow A^2 \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} A$ where ϱ is considered as a subalgebra of A^2. In fact, the coequalizer of each pair $f, g\colon A \longrightarrow B$ is constructed this way in a variety: it is the quotient $q\colon B \longrightarrow B/\varrho_{f,g}$ where $\varrho_{f,g}$ is the congruence on B generated by all pairs $(f(a),\ g(a))$ with $a \in A$. Coequalizers do exist in quasivarieties as well, though by a different construction. For example, in the quasivariety $\mathcal{TORF}$ of torsion free abelian groups, $\mathbb{Z} \longrightarrow 0$ is a coequalizer of the pair $\mathbb{Z} \overset{n\cdot -}{\underset{0}{\rightrightarrows}} \mathbb{Z}$ (whose coequalizer in the variety of abelian groups is clearly the canonical map $\mathbb{Z} \longrightarrow \mathbb{Z}_n$). In fact the method of constructing coequalizers as described above characterizes varieties among SP–classes.[1]

Note that in a variety any idempotent homomorphism $u\colon A \longrightarrow A$ (i.e., any endomorphism u with $u \circ u = u$) splits as $u = A \xrightarrow{r} Q \xrightarrow{s} A$ with $r \circ s = 1_Q$ where $r\colon A \longrightarrow Q$ is a coequalizer of the pair $(u, 1_A)$.

A category admitting all products and equalizers (dually: all coproducts and coequalizers) is called *complete* (dually: *cocomplete*). It then has *all* limits respectively colimits. Varieties thus are complete and cocomplete. In particular they have *directed colimits* which are — as is well known — constructed on the level of sets.

Generators

It will become clear in the sequel that the problem to determine all varieties equivalent to a given one, $\mathcal{V}$, is equivalent to identifying all generators of a particular kind in $\mathcal{V}$. We therefore briefly discuss some concepts of generators in a category.

An object G in a category $\mathcal{K}$ is called *generator* provided the associated hom–functor $\hom_{\mathcal{K}}(G, -)\colon \mathcal{K} \longrightarrow \mathcal{SET}$ is faithful (i.e., does not identify any two distinct morphisms $f, g\colon A \longrightarrow B$). Clearly $F1$, the free algebra on one generator in a variety $\mathcal{V}$, has this property: its associated hom–functor is (isomorphic to) the canonical underlying functor of $\mathcal{V}$. The above condition on G can be expressed alternatively by saying that, for each K in $\mathcal{K}$, the family $\hom_{\mathcal{K}}(G, K)$ of all morphisms from G to K is right cancellable, i.e., an *episink*[2]. If $\mathcal{K}$ has coproducts this is equivalent to saying that for each K the *canonical map* $\hom_{\mathcal{K}}(G, K) \cdot G \longrightarrow K$, i.e., the mor-

[1] A categorical analysis of the relation between congruences and coequalizers in varieties results in saying that in a variety *congruence relations are effective*, i.e., that any congruence relation is a kernel pair. In a variety a congruence relation ϱ – identified with the pair $(\pi_1|_\varrho,\ \pi_2|_\varrho)$ as above – is the kernel pair of its quotient. An SP–class is a variety iff its congruence relations are effective.

[2] A family of $\mathcal{K}$–morphisms $(K_t \xrightarrow{e_t} K)_{t\in T}$ is called an *episink* provided that for any pair $r, s\colon K \longrightarrow L$ of $\mathcal{K}$–morphisms the implication $(\forall t \in T\colon re_t = se_t) \Longrightarrow r = s$ holds.

phism induced by the family $\hom_{\mathcal{K}}(G, K)$ is an epimorphism, which in turn can be shown to be equivalent to the statement that, for each K, there is *some* set X and *some* epimorphism $X \cdot G \longrightarrow K$.

The algebra $F1$ in a variety $\mathcal{V}$ has an even stronger property: for each V in $\mathcal{V}$ there is a set X and a *surjective* homomorphism $X \cdot F1 \longrightarrow V$. Since surjective homomorphisms are always epimorphisms but not conversely (e.g., the embedding $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epimorphism in the category of rings) the above observation requires a categorical concept modelling the surjective homomorphism in varieties. The most suitable notion for this is the following: an epimorphism $K \xrightarrow{e} L$ is called *extremal* provided it cannot be factored over a proper subobject of its codomain, i.e., if $e = m \circ g$ with a monomorphism m then m is an isomorphism. In varieties (and more generally in any SP–class) the extremal epimorphisms are precisely the surjective homomorphisms. The attribute "extremal" is used for episinks as well: an episink $(K_t \xrightarrow{e_t} K)_{t \in T}$ is called *extremal* provided that whenever $K_t \xrightarrow{e_t} K = K_t \xrightarrow{g_t} L \xrightarrow{m} K$ for all $t \in T$ with some monomorphism m, then m must be an isomorphism.

Now the following statements are equivalent for an object G in $\mathcal{K}$ (provided the required coproducts exist):

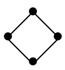
1. $\hom_{\mathcal{K}}(G, -)$ is faithful and reflects isomorphisms[3].

2. For each K in $\mathcal{K}$ the family $\hom_{\mathcal{K}}(G, K)$ of all morphisms from G to K is an extremal episink.

3. For each K in $\mathcal{K}$ the canonical map $\hom_{\mathcal{K}}(G, K) \cdot G \longrightarrow K$ is an extremal epimorphism.

4. For each K in $\mathcal{K}$ there exists a set X and an extremal epimorphism $X \cdot G \longrightarrow K$.

G is called an *extremal generator* provided it has these equivalent properties. Thus $F1$ is an extremal generator in any SP–class.

To avoid possible confusion we finally note that the categorical statement "G is a generator in the variety $\mathcal{V}$" is completely unrelated to the algebraic statement "$\mathcal{V}$ is generated by G". In fact the latter statement is, in many cases, just the dual of the first: e.g., the category $\mathcal{BOOL}$ is generated (in the algebraic sense) by the two-chain, and this is (categorically) a cogenerator in $\mathcal{BOOL}$. An (extremal) generator in $\mathcal{BOOL}$ is .

[3]A functor S is said to *reflect isomorphisms* if a morphism f is an isomorphism provided $S(f)$ is an isomorphism.

Equivalence, concrete equivalence, and isomorphism

Definition 1 (a) A *concrete category* is a pair $(\mathcal{K}, U)$ where $\mathcal{K}$ is a category and U is a faithful functor from $\mathcal{K}$ into the category $\mathcal{SET}$.

(b) A concrete category $(\mathcal{K}, U)$ is called *uniquely transportable* provided that for every $\mathcal{K}$-object K and every bijective map $UK \xrightarrow{k} X$ there exists a unique $\mathcal{K}$-object L with $UL = X$ such that k lifts to a $\mathcal{K}$-isomorphism $K \to L$.

Remark 1 Often we simply write $\mathcal{K}$ for the concrete category $(\mathcal{K}, U)$. In particular we always denote a variety by $\mathcal{V}$ since its underlying functor $|-|$ is canonically given. Moreover it will be clear from the context whether we consider the variety in question as an abstract or a concrete category. Note that varieties are uniquely transportable.

Definition 2 (a) A functor $F: \mathcal{K} \to \mathcal{L}$ is called

- an *equivalence* provided there exists a functor $G: \mathcal{L} \to \mathcal{K}$ and natural isomorphisms $\eta: 1_{\mathcal{K}} \to GF$ and $\epsilon: 1_{\mathcal{L}} \to FG$. G then is called an *equivalence inverse* of F.
- an *isomorphism* provided there exists a functor $G: \mathcal{L} \to \mathcal{K}$ such that $1_{\mathcal{K}} = GF$ and $1_{\mathcal{L}} = FG$. G then is called the *inverse* of F.

Categories $\mathcal{K}$ and $\mathcal{L}$ are called *equivalent* and *isomorphic* respectively provided there exists an equivalence (respectively an isomorphism) from $\mathcal{K}$ to $\mathcal{L}$.

(b) Given concrete categories $(\mathcal{K}, U)$ and $(\mathcal{L}, V)$, a functor $F: \mathcal{K} \to \mathcal{L}$ is called *concrete functor* provided $VF = U$.

(c) A concrete functor $F: (\mathcal{K}, U) \to (\mathcal{L}, V)$ is called *concrete equivalence* repectively *concrete isomorphism* provided it is an equivalence (respectively an isomorphism) as a functor. Concrete categories $(\mathcal{K}, U)$ and $(\mathcal{L}, V)$ are called *concretely equivalent* and *concretely isomorphic* respectively provided there exists a concrete equivalence (respectively a concrete isomorphism) from $(\mathcal{K}, U)$ to $(\mathcal{L}, V)$.

Remarks 2 1. The (standard categorical) terminology recalled above and used throughout in this paper differs from the terminology used in universal algebra (as e. g. in [22, 21]): concrete functors between varieties are known in algebra as *interpretations*; thus "equivalence" in the sense of [22, 21] is "concrete equivalence" while "equivalence" in the sense of Definiton 2 is called "categorical equivalence" in [22].

2. The inverse of an isomorphism is uniquely determined. An equivalence might have different equivalence inverses all of which are, however, naturally isomorphic.

3. The inverse of a concrete isomorphism is again a concrete functor. The equivalence inverses of a concrete equivalence will in general fail to be concrete functors again. By a suitable modification of the notion of concrete functor one can overcome this asymmetry. In particular, if $F\colon (\mathcal{K}, U) \to (\mathcal{L}, V)$ is a concrete equivalence one can always find an equivalence inverse G of F commuting with U and V up to isomorphism (i.e., such that UG and V are naturally isomorphic) (see [23]).

4. Uniquely transportable concrete categories which are concretely equivalent are even concretely isomorphic (see [1, Sect. 5]). In particular, if $\mathcal{V}$ and $\mathcal{W}$ are concretely equivalent varieties, then $\mathcal{V}$ and $\mathcal{W}$ are (concretely) isomorphic.

5. In the terminology just explained it then is the purpose of this paper to describe, for a given variety $\mathcal{V}$, all varieties equivalent to $\mathcal{V}$ up to concrete isomorphism.

Fundamentals of categorical algebra

The following well known results of categorical algebra, mainly due to Lawvere [18], Isbell [15], Linton [19], and others are fundamental for this work. We will only sketch the ideas behind these facts. Precise and easily accessible reference for each of them is added as well in the form *see e.g.*[x,y]. As a further general reference one might consult [3, 6, 20].

The fundamental concept is that of a Lawvere theory which emerges from extending the clone of variety (or an algebra) into a category such that what is called “clone composition” or “composition of operations” becomes composition in this category. The most intuitive (though not the most elegant) definition is the following:

Definition 3 A *Lawvere theory* $\mathbb{T}$ is a category of countably many objects enumerated as $C_0, C_1, \dots C_n, \dots$ together with distinguished families of morphisms $(\pi_i^n\colon C_n \longrightarrow C_1)_{1\le i\le n}$ which are products.

A *morphism* $R\colon \mathbb{T} \to \mathbb{T}'$ *of Lawvere theories* then is a functor R from $\mathbb{T}$ to $\mathbb{T}'$ which preserves the distinguished products.

Examples 1 1. The paradigmatic example of a Lawvere theory is the following: given a variety $\mathcal{V}$, choose, for each $n \in \mathbb{N}$, a free algebra Fn on n generators (pairwise distinct) and let $n_i\colon F1 \longrightarrow Fn$ be the homomorphic extension of the insertion of the i–th generator $1 \longmapsto x_i$. The dual of the full subcategory of $\mathcal{V}$ formed by these objects is a Lawvere theory with $C_n = Fn$ and $\pi_i^n = n_i$. This theory is called *theory of* $\mathcal{V}$ and will be denoted by $\mathsf{Th}\mathcal{V}$.

Algebraists might prefer to think of this theory in the following form: take as objects all finite powers $(F\omega)^n$ (ω a countable set of generators) and as morphisms $(F\omega)^n \longrightarrow (F\omega)^m$ the maps $\langle t_i\rangle$ induced by term operations

$t_i\colon (F\omega)^n \longrightarrow F\omega$ $(i = 1, \ldots, m)$. This defines a theory $\mathsf{Th}_1\mathcal{V}$ which clearly is isomorphic to $\mathsf{Th}\mathcal{V}$.

2. The Lawvere theories of the categories ${}_R\mathcal{MOD}$ of left (respectively $\mathcal{MOD}_R$ of right) R–modules for a unital ring R are particularly easy to describe. Since the finite copowers in these categories are at the same time finite powers the morphisms between them, i.e., the morphisms in the theories of these varieties, are completely determined by the endomorphisms of R in the respective variety. For further use we note the following ring isomorphisms (where $\hom_R$ and ${}_R\hom$ denote the hom–sets in $\mathcal{MOD}_R$ respectively ${}_R\mathcal{MOD}$, $\mathsf{M}_n(R)$ denotes the ring of $n \times n$–matrices over R and R^{op} the opposite ring of R):

   (a) $R \simeq \hom_R(R, R)$ and $R^{\mathsf{op}} \simeq {}_R\hom(R, R)$,
   (b) $\mathsf{M}_n(R) \simeq \hom_R(R^n, R^n)$ and $\mathsf{M}_n(R)^{\mathsf{op}} \simeq {}_R\hom(R^n, R^n)$

   obtained by right (respectively left) multiplication, and consequently

   c) $\mathsf{M}_n(R)^{\mathsf{op}} \simeq \mathsf{M}_n(R^{\mathsf{op}})$.

   It follows that, for the theory $\mathbb{T}_l^R$ of left R–modules (respectively the theory $\mathbb{T}_r^R$ of right R–modules), the following hold:

   (a) $\mathbb{T}_l^R(R, R) = R^{\mathsf{op}}$ and $\mathbb{T}_l^R(R^n, R^n) = \mathsf{M}_n(R)^{\mathsf{op}} = \mathsf{M}_n(R^{\mathsf{op}})$,
   (b) $\mathbb{T}_r^R(R, R) = R$ and $\mathbb{T}_r^R(R^n, R^n) = \mathsf{M}_n(R)$.

   With $\mathsf{Mat}(n \times m, R)$ denoting the set of all matrices over R with n rows and m columns a complete description of these theories then would be as follows:

   - the set of objects of $\mathbb{T}_l^R$ and $\mathbb{T}_r^R$ is $\mathbb{N}$,
   - for $n, m \in \mathbb{N}$ the set of morphisms $n \longrightarrow m$ is
     - $\mathsf{Mat}(n \times m, R)$ in $\mathbb{T}_l^R$,
     - $\mathsf{Mat}(m \times n, R)$ in $\mathbb{T}_r^R$,
   - composition of $A \in \mathsf{Mat}(n \times m, R)$, $B \in \mathsf{Mat}(m \times k, R)$ is given by
     - $B \circ_l A = A^t \cdot B^t$ in $\mathbb{T}_l^R$,
     - $A \circ_r B = A \cdot B$ in $\mathbb{T}_r^R$.

   Note in particular that $\mathbb{T}_r^R$ is (as a category) the dual of $\mathbb{T}_l^R$: $\mathbb{T}_r^R = (\mathbb{T}_l^R)^{\mathsf{op}}$.

Leading Example (a) The Lawvere theory of $\mathcal{BOOL}$ is, due to restricted Stone–duality, isomorphic to the full subcategory of $\mathcal{SET}$ spanned by the powersets of the finite sets $m = \{0, \ldots, m-1\}$. $\diamond$

The theory of a variety is its syntax (substituting the representation by operations and equations) given as a category (hence a mathematical structure). Observe that the $\mathsf{Th}\mathcal{V}$–morphisms $C_n \xrightarrow{\bar\sigma} C_1$ (i.e., the $\mathcal{V}$–morphisms $F1 \xrightarrow{\bar\sigma} Fn$) are in one-to-one correspondence to the elements of Fn, hence to equivalence classes $[\sigma]$ of terms σ in the language of $\mathcal{V}$ by

$$\bar\sigma \longleftrightarrow \bar\sigma(1) = [\sigma].$$

Now *algebras are functors* from the theory to $\mathcal{SET}$ preserving finite products: it is clear that every algebra $A = \big(|A|, (\sigma^A)\big)$ in a variety $\mathcal{V}$ can be considered as a functor $\tilde A\colon \mathsf{Th}\mathcal{V} \longrightarrow \mathcal{SET}$:

- on objects $\tilde A$ acts by mapping Fn to $|A|^n$;
- on morphisms $\tilde A$ acts by mapping
  - $C_n \xrightarrow{\bar\sigma} C_1$ to the A–interpretation $\sigma^A\colon |A|^n \longrightarrow |A|$ of σ;
  - and, finally, a morphism $C_n \xrightarrow{\langle\bar\sigma_i\ \rangle} C_m$ to the map $|A|^n \xrightarrow{\langle\sigma_i^A\ \rangle} |A|^m$.

$\tilde A$ is clearly a functor and, moreover, $\tilde A$ obviously preserves products; thus it is a model of $\mathsf{Th}\mathcal{V}$ in the sense of the following definition.

Definition 4 For any Lawvere theory $\mathbb{T}$ the full subcategory $\mathsf{Mod}\mathbb{T}$ of the category $\mathcal{SET}^{\mathbb{T}}$ of all set–valued functors on $\mathbb{T}$, consisting of (finite) product–preserving functors, is called the *category of* $\mathbb{T}$*–models*.

Correspondingly, *homomorphisms are natural transformations.* Given any homomorphism $f\colon A \to B$ in $\mathcal{V}$, by the very definition of homomorphism, the family of maps

$$\tilde f = (\tilde f_{C_n})_{C_n \in \mathsf{Th}\mathcal{V}} = (|A|^n \xrightarrow{f^n} |B|^n)_{n\in\mathbb{N}}$$

is a natural transformation $\tilde f\colon \tilde A \to \tilde B$.

The following is now obvious:

Fact 1 *For every variety $\mathcal{V}$, by the assignments $A \mapsto \tilde A$ and $f \mapsto \tilde f$ there is defined a functor $\tilde{-}\colon \mathcal{V} \to \mathsf{Mod}\mathsf{Th}\mathcal{V}$.*

Remark 3 The category $\mathsf{Mod}\mathbb{T}$ is a concrete category by means of the underlying functor "*evaluation at* C_1", i.e., the functor

$$U^{\mathbb{T}} := ev_{C_1}\colon \mathsf{Mod}\mathbb{T} \longrightarrow\ \mathcal{SET}$$

mapping a model G to $G(C_1)$ and a natural transformation λ to its C_1-component λ_{C_1}. It is now seen easily that the functor $\tilde{-}$ even is a concrete functor.

The converse, i.e., interpreting product preserving functors $\mathbb{T} = \mathsf{Th}\mathcal{V} \longrightarrow \mathcal{SET}$ as algebras needs some precaution since products are unique only up to isomorphism, while algebras are defined with respect to a (chosen) notion of (cartesian) product in $\mathcal{SET}$. But clearly every functor $H \in \mathsf{Mod}\mathbb{T}$ is naturally isomorphic to a unique functor $\bar{H} \in \mathsf{Mod}\mathbb{T}$ which sends each product $(Fn \xrightarrow{\pi_i^n} F1)$ in $\mathbb{T}$ to the chosen powers $\bar{H}(Fn) = \bar{H}(F1)^n \xrightarrow{p_i^n} \bar{H}F1 = HF1$ in $\mathcal{SET}$.

We thus can assign to each functor $H \in \mathsf{ModTh}\mathcal{V}$ an algebra ΦH in the language of $\mathcal{V}$ by

$$\Phi H = \big(H(F1),\ \bar{H}(\bar{\sigma})\big)$$

where σ runs over all n–ary terms for all $n \in \mathbb{N}$. Since for each equation $\sigma = \tau$ defining $\mathcal{V}$ one has $[\sigma] = [\tau]$, hence $\bar{\sigma} = \bar{\tau}$, the ΦH–interpretations of these terms, i.e., $\bar{H}(\bar{\sigma})$ and $\bar{H}(\bar{\tau})$ coincide. Thus ΦH belongs to $\mathcal{V}$.

Moreover, it is easy to see that for each natural transformation $\lambda\colon H \to G$ in $\mathsf{ModTh}\mathcal{V}$ its C_1-component $\Phi\lambda\colon = \lambda_{F1}$ is a homomorphism $\Phi H \to \Phi G$. This way Φ becomes a functor.

The following results then are due to Lawvere [18]:

Fact 2 (see e.g. [6, 3.2.9]) *Every variety $\mathcal{V}$ is concretely equivalent to* $\mathsf{ModTh}\mathcal{V}$ *by means of the functors* $\tilde{-}$ *and* Φ*.*

This says in particular that for each algebra V in $\mathcal{V}$ and n–ary term σ in the language of $\mathcal{V}$ the following diagram commutes:

$$\begin{array}{ccccc}
|V|^n & \overset{\sim}{\longrightarrow} & \hom_{\mathcal{V}}(F1,V)^n & \overset{\sim}{\longrightarrow} & \hom_{\mathcal{V}}(Fn,V) \\
{\scriptstyle \sigma^V}\big\downarrow & & \big\downarrow{\scriptstyle \sigma^{\Phi Y V}} & & \big\downarrow{\scriptstyle \hom_{\mathcal{V}}(\bar{\sigma},V)} \\
|V| & \overset{\sim}{\longrightarrow} & \hom_{\mathcal{V}}(F1,V) & = & \hom_{\mathcal{V}}(F1,V)
\end{array}$$

where the horizontal isomorphisms are the obvious (natural) ones. In other words: up to the natural bijection $|V| \simeq \hom_{\mathcal{V}}(F1,V)$ the n-ary operations in the clone of an algebra V in the variety $\mathcal{V}$ are the maps $\hom_{\mathcal{V}}(\bar{\sigma},V)$.

Fact 3 (see e.g. [6, 3.3.4]) *For every Lawvere theory $\mathbb{T}$ there exists a variety $\mathcal{V}_{\mathbb{T}}$ — called the* variety determined by $\mathbb{T}$ *— such that* $\mathsf{Mod}\mathbb{T}$ *and $\mathcal{V}_{\mathbb{T}}$ are concretely equivalent*[4]*.*

Up to an isomorphism of algebras an algebra in $\mathcal{V}_{\mathbb{T}}$ thus has as its underlying set the set $H(C_1)$ for some finite product preserving functor $H\colon \mathbb{T} \to \mathcal{SET}$, and as the set of n-ary operations in its clone the set $\{H(\tau) \mid \tau \in \hom_{\mathbb{T}}(C_n, C_1)\}$.

[4]Technically $\mathcal{V}_{\mathbb{T}}$ is the uniquely transportable modification of $\mathsf{Mod}\mathbb{T}$ in the sense of [1, 5.36].

Fact 4 (see e.g. [6, 3.2.9, 3.8.5]) *The correspondence between varieties and Lawvere theories established above is essentially bijective, i.e.,*

1. *every variety $\mathcal{V}$ is concretely isomorphic to the variety $\mathcal{V}_{\mathsf{Th}\mathcal{V}}$ determined by $\mathsf{Th}\mathcal{V}$;*
2. *every Lawvere theory $\mathbb{T}$ is isomorphic to the theory $\mathsf{Th}\mathcal{V}_{\mathbb{T}}$ of the variety $\mathcal{V}_{\mathbb{T}}$ determined by $\mathbb{T}$.*

This results partly from the observation that the construction Mod is, in a sense, functorial: if $R\colon \mathbb{T} \longrightarrow \mathbb{S}$ is a theory morphism then for every $\mathbb{S}$–model $H\colon \mathbb{S} \longrightarrow \mathcal{SET}$ one obtains a $\mathbb{T}$–model $R^*(H) = H \circ R\colon \mathbb{T} \longrightarrow \mathcal{SET}$. This way one gets

Fact 5 *Each theory morphism $R\colon \mathbb{T} \longrightarrow \mathbb{S}$ determines a concrete functor $R^*\colon \mathsf{Mod}\mathbb{S} \longrightarrow \mathsf{Mod}\mathbb{T}$.*

For various reasons it is suitable to look at the results established so far from a slightly more abstract point of view:

Let $\mathcal{K}$ be any category and Q a $\mathcal{K}$–object which has finite copowers. Choose, for each $n \in \mathbb{N}$, pairwise distinct copowers nQ together with distinguished coproduct injections $\mu_i^n\colon Q \longrightarrow nQ$ $(1 \leq i \leq n)$. Take the full subcategory $\langle nQ \mid n \in \mathbb{N}\rangle$ of $\mathcal{K}$ spanned by these objects. Then $\mathsf{Th}_{\mathcal{K}}Q = \langle nQ \mid n \in \mathbb{N}\rangle^{\mathrm{op}}$ is a Lawvere theory.

Leading Example (b) Consider, in the category $\mathcal{BOOL}$, the powerset algebra $\mathsf{P}(m)$ of an m-element set. The theory $\mathsf{Th}_{\mathcal{BOOL}}\ \mathsf{P}(m)$ is — due to restricted Stone–duality — isomorphic to the full subcategory of $\mathcal{SET}$ spanned by the finite cartesian powers of m. ◇

Given $\mathcal{K}$ and Q as above there is the so-called Yoneda functor Y_Q from $\mathcal{K}$ into the functor-category $\mathcal{SET}^{\mathsf{Th}_{\mathcal{K}}Q}$, which maps an object K to $\hom_{\mathcal{K}}(-,K)|_{\mathsf{Th}_{\mathcal{K}}Q}$ and acts on morphisms similarly. Since hom-functors preserve products, Y_Q factors over $\mathsf{ModTh}_{\mathcal{K}}Q$; therefore we will in the sequel consider the Yoneda functor always as a functor

$$Y_Q\colon \mathcal{K} \longrightarrow \mathsf{ModTh}_{\mathcal{K}}Q.$$

Now the following is easily seen:

Fact 6 *For any variety $\mathcal{V}$ one has*

1. *$\mathsf{Th}\mathcal{V} = \mathsf{Th}_{\mathcal{V}}(F1)$;*
2. *the functors $\tilde{-}$ and Y_{F1} are naturally isomorphic.*

Since the canonical underlying functor of any variety $\mathcal{V}$ is naturally equivalent to the hom–functor $\hom_{\mathcal{V}}(F1,-)$ one now can read Fact 2 also as stating the concrete equivalence $Y_{F1}\colon (\mathcal{V},\hom_{\mathcal{V}}(F1,-)) \xrightarrow{\sim} (\mathsf{ModTh}_{\mathcal{V}}(F1), U^{\mathsf{Th}_{\mathcal{V}}(F1)})$. This leads to the question, for which objects Q instead of $F1$ the corresponding equivalence might hold. Another way of looking at this question is, in view of Fact 3, the following: can one characterize those concrete categories $(\mathcal{K},U)$ with representable underlying functor U which are concretely equivalent to a variety?

The answer — given as Fact 8 below — is due to Isbell [15] and based on the following categorical result which will also be of use later on.

Fact 7 (see e.g. [6, 3.8.5], [20, chap. 3, 1.29], [3, 1.26]) *Let $\mathcal{K}$ be a category admitting all finite copowers of some $\mathcal{K}$–object Q, and let Y_Q be the Yoneda functor $Y_Q\colon \mathcal{K} \longrightarrow \mathsf{ModTh}_{\mathcal{K}}Q$. Then the following hold:*

1. *$Y_Q(nQ)$ is the free object on n generators in $\mathsf{ModTh}_{\mathcal{K}}Q$.*
2. *Y_Q has a left adjoint provided $\mathcal{K}$ has coequalizers.*
3. *Y_Q is a full embedding iff $(\mathsf{Th}_{\mathcal{K}}Q)^{op}$ is dense[5] in $\mathcal{K}$.*
4. *Y_Q preserves directed colimits iff Q is finitely presentable[6].*

Note that Fact 7 can be used to prove (part of) Fact 2: in any variety $\mathcal{V}$ the full subcategory consisting of the finitely generated free algebras, i.e. $(\mathsf{Th}_{\mathcal{V}}(F1))^{op} = (\mathsf{Th}\mathcal{V})^{op}$ is dense, thus, by statement 3. above, Y_{F1} is full and faithful. But then the functor $\tilde{-}$, being naturally isomorphic to Y_{F1}, is full and faithful, too. Further, the left adjoint of Y_{F1} which exists by statement 2. above is nothing but the functor Φ of Fact 2.

Fact 8 (see e.g. [6, 3.9.1], [13, 32.21]) *For any object Q in a complete and cocomplete category $\mathcal{K}$ the following are equivalent:*

(i) *By means of $Y_Q\colon \mathcal{K} \longrightarrow \mathsf{ModTh}_{\mathcal{K}}Q$ the concrete category $\big(\mathcal{K},\hom_{\mathcal{K}}(Q,-)\big)$ is concretely equivalent to a full subcategory of the variety determined by $\mathsf{Th}_{\mathcal{K}}Q$, closed under products, subobjects, directed colimits, and free algebras.*

(ii) *Q is an extremal generator which is projective[7] and finitely presentable.*

---

[5] A small subcategory $\mathcal{D}$ of $\mathcal{K}$ is called *dense*, provided for each $\mathcal{K}$-object K the family of all morphisms $(D \to K)_{D\in \mathrm{Ob}\mathcal{D}}$ is a colimit of the canonical diagram of K with respect to $\mathcal{D}$ (for further details see e.g. [3, 0.4])

[6] A $\mathcal{K}$–object K is called *finitely presentable* iff $\hom_{\mathcal{K}}(K,-)$ preserves directed colimits; in any quasivariety this is equivalent to the algebraic meaning.

[7] Q is called *projective* iff every $\mathcal{K}$–morphism $Q \xrightarrow{f} L$ can be lifted against any extremal epimorphism $K \xrightarrow{e} L$; in SP–classes this is equivalent to being projective w.r.t. surjective homomorphisms. More correctly these objects should be called *extremally projective.*

If in $\mathcal{K}$, in addition, equivalence relations are effective[8], Y_Q *in (i) is even an equivalence if* Q *satisfies the conditions of (ii).*

Definition 5 An extremal generator Q in a category $\mathcal{K}$ which is (extremally) projective and finitely presentable is called a *varietal generator*[9].

Remark 4 Equivalences between categories map varietal generators to varietal generators.

The notion of a finitely presentable object models categorically the fact that the generator in question is finitely generated in the algebraic sense. In fact weaker notions (and equivalent only in the context of varieties — see e.g. [12]) would do here as well as, e.g., the notion of an abstractly finite object[10] (see [15, 18]). Replacing "finitely presentable" in (ii) of Fact 8 by "abstractly finite" requires only to delete the words "directed colimits" in (i). Note that both of these notions refer to colimits in the category under consideration. A colimit–free categorical description of finitely generated algebras which will be of use in the final sections of this note is given in the following definition. It is motivated by a remark in [22] and based on the fact that an algebra is finitely generated iff it is compact in its subalgebra lattice.

Definition 6 An object K in a category $\mathcal{K}$ is called *compact* provided that every extremal episink $(K_t \xrightarrow{e_t} K)_{t \in T}$ contains a finite one.

The proof of the following lemmata is an easy exercise.

Lemma 1 *Let* $\mathcal{V}$ *be a variety.*

1. *An algebra* A *in* $\mathcal{V}$ *is finitely generated (in the algebraic sense) iff* A *is compact in* $\mathcal{V}$.
2. *The regular projective finitely presentable algebras in* $\mathcal{V}$ *are precisely the extremally projective, compact* $\mathcal{V}$*–objects.* □

Lemma 2 *Let* $\mathcal{V}$ *be a variety.*

1. *The retracts of the finitely generated free algebras* Fn *are precisely the compact (= finitely presentable) regular projective objects in* $\mathcal{V}$.

[8] See Note 1

[9] These objects are misleadingly called "projective finitely generated co–generators" in [22]. In categorical language "co–generator" is the dual of "generator" (see also the last paragraph on generators).

[10] In a category $\mathcal{K}$ with copowers an object A is called *abstractly finite* provided any morphism $f\colon A \to X \cdot A$ factors through the natural injection $F \cdot A \to X \cdot A$ for some finite subset $F \subset X$.

2. *A retract Q of Fn is a varietal generator iff $F1$ (or equivalently some Fk) is a retract of some finite copower mQ of Q.* □

Examples 2 1. Every free algebra Fn on finitely many generators $n \neq 0$ in any (quasi–) variety is a varietal generator.

2. The varietal generators in the category of (abelian) groups are precisely the finitely generated free (abelian) groups. This is a special instance of the following more general but obvious fact: The varietal generators are precisely the finitely generated free objects in a variety $\mathcal{V}$ iff each finitely generated projective $\mathcal{V}$-algebra is free (hence free on finitely many generators).

3. The varietal generators in module–categories are precisely those modules which in commutative algebra are called *progenerators* or *faithful projective modules.*

Leading Example (c) The varietal generators in $\mathcal{BOOL}$ are precisely the powerset algebras of finite sets of cardinality greater 1 (use Lemma 2 and restricted Stone duality).

◇

Morita equivalent theories

Definition 7 Lawvere theories $\mathbb{T}$ and $\mathbb{S}$ are called *Morita equivalent* provided $\mathsf{Mod}\mathbb{T}$ and $\mathsf{Mod}\mathbb{S}$ are equivalent categories.

! Let $\mathbb{T}$ be any Lawvere theory and $\mathcal{V}$ the variety determined by $\mathbb{T}$. The problem of describing all theories $\mathbb{S}$ Morita equivalent to $\mathbb{T}$ is equivalent to describing all varietal generators Q in the variety $\mathcal{V}$. This follows from Facts 4 and 8 since, for any equivalence $\Phi\colon \mathcal{W} \to \mathcal{V}$, the algebra $Q = \Phi(F1)$ is a varietal generator and Φ in an obvious way induces an isomorphism of the Lawvere theories $\mathsf{Th}\mathcal{W}$ and $\mathsf{Th}_{\mathcal{V}}Q$. More explicitely, the following are equivalent for any variety $\mathcal{W}$ with theory $\mathbb{S}$:

- $\mathcal{W}$ is equivalent to $\mathcal{V}$,
- $\mathcal{W}$ is concretely equivalent to $(\mathcal{V}, \hom_{\mathcal{V}}(Q, -))$ — hence to $(\mathsf{ModTh}_{\mathcal{V}}Q, ev_Q)$ — for some varietal generator Q of $\mathcal{V}$.
- $\mathbb{S}$ is isomorphic to $\mathsf{Th}_{\mathcal{V}}Q$ for some varietal generator Q of $\mathcal{V}$.

Note that we do not claim that there is a one–to–one correspondence between varietal generators in $\mathcal{V}$ and varieties $\mathcal{W}$ equivalent to $\mathcal{V}$. In fact we do not know how to characterize the relation between two varietal generators P and Q having isomorphic theories (see [22, Problem 3]). The following, however, can be said already here (see [22, Theorem 6.3]):

Proposition 1 *The following are equivalent for varietal generators P and Q in any variety $\mathcal{V}$:*

(i) $\mathsf{ModTh}_{\mathcal{V}}P$ *and* $\mathsf{ModTh}_{\mathcal{V}}Q$ *are concretely equivalent.*

(ii) There exists an equivalence $\Psi\colon \mathcal{V} \xrightarrow{\sim} \mathcal{V}$ *with* $\Psi(P) = Q$.

Proof To prove that (i) implies (ii) consider a concrete equivalence Θ from $\mathsf{ModTh}_{\mathcal{V}}Q$ to $\mathsf{ModTh}_{\mathcal{V}}P$. Choose an equivalence inverse $\bar{Y}_P\colon (\mathsf{ModTh}_{\mathcal{V}}P, ev_P) \longrightarrow$! $\big(\mathcal{V}, \hom_{\mathcal{V}}(P,-)\big)$ of Y_P which can be assumed to commute with the underlying functors up to a natural isomorphism by Remark 2. Then $\Psi = \bar{Y}_P \circ \Theta \circ Y_Q\colon \mathcal{V} \longrightarrow \mathcal{V}$ is an equivalence, thus has an equivalence inverse $\bar{\Psi}$, and there is a natural isomorphism

$$\hom_{\mathcal{V}}(Q,-) \simeq \hom_{\mathcal{V}}(P,-) \circ \Psi \simeq \hom_{\mathcal{V}}(P, \Psi -);$$

hence, by adjunction

$$\hom_{\mathcal{V}}(\bar{\Psi}Q,-) \simeq \hom_{\mathcal{V}}(P,-).$$

Thus $\bar{\Psi}Q$ and P are isomorphic as are ΨP and Q. Clearly one now can modify Ψ to obtain the equality $\Psi P = Q$. □ !

In order to describe all varietal generators Q in $\mathcal{V}$ we have to find all retracts of the finitely generated free algebras in $\mathcal{V}$ satisfying condition 2. of Lemma 2. Now in every category a retraction $K \xrightarrow{s} L \xrightarrow{r} K = 1_K$ determines an idempotent morphism $u = sr$ and, conversely, every idempotent morphism $u\colon L \longrightarrow L$ determines, in the presence of coequalizers, a retraction $K \xrightarrow{s} L \xrightarrow{r} K$ by forming a coequalizer $L \xrightarrow{r} K$ of the pair $(u, 1_L)$. Hence, the possible retractions $Fn \longrightarrow Q$ can be determined within $\hom_{\mathcal{V}}(Fn, Fn) = \hom_{\mathsf{Th}\mathcal{V}}(Fn, Fn)$. So the only remaining problem is to express, within $\mathsf{Th}\mathcal{V}$, the fact that some Fk is, for some idempotent endomorphism $u\colon Fn \longrightarrow Fn$, a finite copower of the coequalizer of $(u, 1_{Fn})$. For this purpose the following notion is introduced.

Definition 8 An idempotent morphism $u\colon K \longrightarrow K$ in any category $\mathcal{K}$ admitting finite products of K is called *invertible* if there exist some $m \in \mathbb{N}$ and morphisms $p\colon K^m \longrightarrow K,\ d\colon K \longrightarrow K^m$ such that

$$K \xrightarrow{d} K^m \xrightarrow{u^m} K^m \xrightarrow{p} K = 1_K.$$

Remark 5 The notion of invertibility is, with minor differences, contained in [10] as well as in [22]. To show equivalence of their respective notions we note that the following conditions are equivalent for an idempotent morphism $u\colon K \longrightarrow K$.

(i) There exist, for some $m \in \mathbb{N}$, $p\colon K^m \longrightarrow K$, $d\colon K \longrightarrow K^m$ such that

$$K \xrightarrow{d} K^m \xrightarrow{u^m} K^m \xrightarrow{p} K = 1_K.$$

(ii) There exist, for some $k, m \in \mathbb{N}$, $p\colon K^m \longrightarrow K^k$, $d\colon K^k \longrightarrow K^m$ such that

$$K^k \xrightarrow{d} K^m \xrightarrow{u^m} K^m \xrightarrow{p} K^k = 1_{K^k}.$$

(iii) There exists, for some $m \in \mathbb{N}$, a retraction $L \xrightarrow{\bar{d}} K \xrightarrow{\bar{p}} L = 1$ such that $\bar{p} \circ u = \bar{p}$ or $u \circ \bar{d} = \bar{d}$.

Proposition 2 *Let $\mathcal{V}$ be a variety. For a $\mathcal{V}$–object Q the following are equivalent:*

(i) Q is a varietal generator in $\mathcal{V}$.

(ii) There exists, for some $n \in \mathbb{N}$, $1 \leq n$, a morphism $r\colon Fn \to Q$ such that (r, Q) is a coequalizer of a pair $(u, 1_{Fn})$ in $\mathcal{V}$, where u is an idempotent endomorphism of Fn which is invertible in $\mathsf{Th}\mathcal{V}$.

Proof Le Q be a varietal generator: By Lemma 2 there exists an idempotent morphism $u\colon Fn \longrightarrow Fn$, for some $n \in \mathbb{N}$, and a coequalizer diagram

! $Fn \underset{1}{\overset{u}{\rightrightarrows}} Fn \xrightarrow{r} Q$ where, in addition, u factors as $u = s \circ r$, and $r \circ s = 1$.

Moreover, there exists a retraction $Fk \xrightarrow{y} mQ \xrightarrow{x} Fk = 1$ in $\mathcal{V}$. With

$$p = Fk \xrightarrow{y} mQ \xrightarrow{m \cdot s} m \cdot Fn = F(n \cdot m) \quad \text{and} \quad d = F(n \cdot m) \xrightarrow{m \cdot r} mQ \xrightarrow{x} Fk$$

! one gets (in $\mathcal{V}$):

$$d \circ (m \cdot u) \circ p = x \circ (m \cdot r) \circ (m \cdot s) \circ (m \cdot r) \circ (m \cdot s) \circ y = 1$$

i.e., in $\mathsf{Th}\mathcal{V}$

$$Fk \xrightarrow{d} (Fn)^m \xrightarrow{u^m} (Fn)^m \xrightarrow{p} Fk = 1. \tag{*}$$

Conversely, if for an idempotent $u\colon Fn \longrightarrow Fn$ one has an equation (*) in $\mathsf{Th}\mathcal{V}$ where u splits as $Fn \xrightarrow{r} Q \xrightarrow{s} Fn$ in $\mathcal{V}$, let

$$x = mQ \xrightarrow{m \cdot s} m \cdot Fn \xrightarrow{d} Fk \quad \text{and} \quad y = Fk \xrightarrow{p} m \cdot Fn \xrightarrow{m \cdot r} mQ$$

in $\mathcal{V}$. It follows $x \circ y = d \circ (m \cdot s) \circ (m \cdot r) \circ p = d \circ (m \cdot u) \circ p = 1$. By Lemma 2 Q is a varietal generator. □

Leading Example (d) Let G be a varietal generator in $\mathcal{BOOL}$, i.e., $G \cong \mathsf{P}(m)$, the powerset algebra of the set $\{0, 1 \ldots, m-1\}$, $m \geq 2$ (see (c)). One can represent $\mathsf{P}(m)$ as in (ii) of the Proposition above as follows:
Let $F(m-1)$ be the free Boolean algebra on $m-1$ generators $x_1, \ldots, x_{m-1}$, $m \geq 2$. Denote by $u\colon F(m-1) \longrightarrow F(m-1)$ the Boolean homomorphism with $u(x_1) = x_1$, $u(x_i) = x_1 \wedge \ldots \wedge x_i$ $(i = 2, \ldots, m-1)$ and by $r\colon F(m-1) \longrightarrow \mathsf{P}(m)$ the Boolean homomorphism defined by $r(x_i) = \{1, \ldots, m-i\}$ for $1 \leq i \leq m-1$. Then

1. *u is idempotent and*

2. *$(r, \mathsf{P}(m))$ is a coequalizer of the pair $(u, 1_{F(m-1)})$.* ◇

An abstract description of the theories $\mathbb{S}$, Morita equivalent to a given one $\mathbb{T}$, can now be given. For this purpose it is convenient to introduce the following constructions.

- For each Lawvere theory $\mathbb{T}$ and $n = 1, 2, 3, \ldots$, denote by $\mathbb{T}^{[n]}$ the following theory:
  - objects are the n-fold powers $\bar{C}_k\colon = C_k \times C_k \times \ldots \times C_k$ of all C_k in $\mathbb{T}$; observe that, for $n, k \in \mathbb{N}$, one has $(C_k)^n = (C_n)^k$.
  - morphisms are all $\mathbb{T}$–morphisms between them;
  - composition and identities are as in $\mathbb{T}$;
  - distinguished products are the families

$$(\bar{\pi}_j^k\colon \bar{C}_k \to \bar{C}_1)_{1 \leq j \leq k} = \big((\pi_j^k)^n\colon (C_k)^n \to (C_1)^n = C_n\big)_{1 \leq j \leq k}.$$

- For each Lawvere theory $\mathbb{T}$ and idempotent $\mathbb{T}$–morphism $u\colon C_1 \to C_1$ denote by $\mathbb{T}_u$ the following theory:
  - objects are all objects from $\mathbb{T}$;
  - morphisms $C_k \to C_j$ are those $\mathbb{T}$–morphisms from C_k to C_j which can be decomposed as

$$C_k \xrightarrow{u^k} C_k \xrightarrow{f} C_j \xrightarrow{u^j} C_j$$

    with a $\mathbb{T}$-morphism f; **!**
  - composition is as in $\mathbb{T}$ and u^k is chosen as the identity of C_k;
  - distinguished products are the families

$$\big(\bar{\pi}_j^k = u\pi_j^k u^k\colon C_k \longrightarrow C_1\big)_{1 \leq j \leq k}.$$

Remarks 6 1. Observe that "rising to the n–th power" is a theory morphism $\mathbb{T} \xrightarrow{(-)^n} \mathbb{T}^{[n]}$ while the embedding $\mathbb{T}^{[n]} \hookrightarrow \mathbb{T}$, though being product preserving is not. Similarly the idempotent u determines a theory morphism $\bar{u}$ by

$$\bar{u}(C_k \xrightarrow{f} C_j) = C_k \xrightarrow{u^j f u^k} C_j$$

though not from $\mathbb{T}$ into $\mathbb{T}_u$ (here this construction fails to be functorial) but if restricted to the subtheory $\mathbb{T}_{[u]}$ of $\mathbb{T}$, where a morphism $f \in \mathbb{T}(C_n, C_m)$ belongs to $\mathbb{T}_{[u]}$ iff $f \circ u^n = u^m \circ f$ (it is straightforward to check that $\mathbb{T}_{[u]}$ is a Lawvere theory, in fact a subtheory of $\mathbb{T}$).

2. It is clear from the definitions that, for any theory $\mathbb{T}$ whose dual $\mathbb{T}^{\mathsf{op}}$ is a theory again, one has for every $n \in \mathbb{N}$ and every idempotent $u \in \mathbb{T}$ (which then is idempotent in $\mathbb{T}^{\mathsf{op}}$, too):

$$(\mathbb{T}^{\mathsf{op}})^{[n]} = (\mathbb{T}^{[n]})^{\mathsf{op}} \quad \text{and} \quad (\mathbb{T}^{\mathsf{op}})_u = (\mathbb{T}_u)^{\mathsf{op}}$$

Proposition 3 *Let $\mathcal{K}$ be a category with finite colimits and G a $\mathcal{K}$–object. Let $\mathbb{T} = \mathsf{Th}_{\mathcal{K}} G$.*

1. *For each $n \in \mathbb{N},\ n > 1$ the Lawvere theory $\mathsf{Th}_{\mathcal{K}}(nG)$ is isomorphic to $\mathbb{T}^{[n]}$.*

2. *For each retraction $Q \xrightarrow{s} G \xrightarrow{r} Q = 1$ the Lawvere theory $\mathsf{Th}_{\mathcal{K}} Q$ is isomorphic to $\mathbb{T}_u$ for the idempotent morphism $u = sr$.*

Proof The proof of 1. is obvious by construction of $\mathbb{T}^{[n]}$. Concerning 2. we define a functor $\Lambda\colon \mathsf{Th}_{\mathcal{K}} Q \longrightarrow \mathbb{T}_u$ as follows:
Given a morphism $\tau\colon kQ \longrightarrow jQ$ in $\mathsf{Th}_{\mathcal{K}} Q$, i.e., a morphism $\tau\colon jQ \longrightarrow kQ$ in $\mathcal{K}$, let $t\colon jG \longrightarrow kG$ be the $\mathcal{K}$–morphism

$$jG \xrightarrow{j \cdot r} jQ \xrightarrow{\tau} kQ \xrightarrow{k \cdot s} kG$$

Then $(k \cdot u) \circ t \circ (j \cdot u) = t$, i.e., $t \in \mathbb{T}_u(kG, jG)$ such that we can define Λ by

$$\Lambda(jQ \xrightarrow{\tau} kQ) = jG \xrightarrow{(ks)\circ\tau\circ(jr)} kG.$$

Λ then is a functor, which obviously is bijective on objects. $(ks) \circ \tau \circ (jr) = (ks) \circ \tau' \circ (jr)$ implies $\tau = \tau'$ since jr is epic and ks monic; hence Λ is faithful. Given any $t\colon jG \longrightarrow kG$, one has $(k \cdot u) \circ t \circ (j \cdot u) = \Lambda\big((k \cdot r) \circ t \circ (j \cdot s)\big)$; thus Λ is full, hence an isomorphism. (All compositions and calculations were done in $\mathcal{K}$!) Also, Λ preserves the distinguished products by definition of those in $\mathbb{T}_u$. □

As a corollary we obtain the following theorem which is basically the main result of [10], here however presented in a form closer related to the results of [22].

Theorem 1 *The Lawvere theories $\mathbb{S}$, Morita equivalent to a given theory $\mathbb{T}$ are, up to isomorphism, precisely the theories $(\mathbb{T}^{[n]})_u$ for $n \in \mathbb{N}, n \geq 1$ and u an idempotent and invertible morphism in $\mathbb{T}^{[n]}(C_n, C_n)$.* □

Note that this theorem can be stated equivalently in the following way:

Theorem 2 *The Lawvere theories $\mathbb{S}$, Morita equivalent to a given theory $\mathbb{T}$ are, up to isomorphism, precisely the theories $\mathsf{Th}_{\mathcal{V}}Q$, where $\mathcal{V} = \mathcal{V}_{\mathbb{T}}$ is the variety determined by $\mathbb{T}$ and Q is a varietal generator in $\mathcal{V}$.*

In fact this result will be more useful in the latter form at least for the following reasons:

- It will be easier in general to determine all varietal generators in a given variety than all pairs (n, u) under discussion (see e.g. the following examples).
- For each pair (n, u) as in the first version of the theorem there are infinitely many different pairs (m, v) of the same kind with $\mathbb{T}^{[n]}_u \simeq \mathbb{T}^{[m]}_v$ (e.g. $(n \cdot k,\ u \times 1_{C_k})$).

Leading Example (e) Let $\mathbb{T}_{\mathcal{BOOL}}$ denote the theory of the variety $\mathcal{BOOL}$. As observed before the varietal generators of $\mathcal{BOOL}$ are the Boolean algebras $\mathsf{P}(m)$ for $m \geq 2$, where, for each m, the idempotent morphism $u\colon F(m-1) \longrightarrow F(m-1)$ specified in (d) splits over $\mathsf{P}(m)$. Thus
The varieties equivalent to $\mathcal{BOOL}$ are the varieties with Lawvere theories

$$\left(\mathbb{T}^{[m-1]}_{\mathcal{BOOL}}\right)_u \ \cong\ \mathsf{Th}_{\mathcal{BOOL}}\mathsf{P}(m).$$

We will identify the varieties determined by these theories in the next section. ◇

For further use we add the global characterization of Morita equivalent theories from [10] identifying, in addition, the Cauchy completion[11] in question. To this end we denote, for any variety $\mathcal{V}$, by $\mathsf{Proj}_{fp}\mathcal{V}$ the full subcategory of $\mathcal{V}$ spanned by its finitely presentable projective objects.

Theorem 3

1. *The Cauchy completion of any Lawvere theory $\mathbb{T}$ is given by the embedding $\mathbb{T} \hookrightarrow \left(\mathsf{Proj}_{fp}\mathcal{V}\right)^{\mathrm{op}}$, where $\mathcal{V} = \mathcal{V}_{\mathbb{T}}$ is the variety determined by $\mathbb{T}$.*
2. *Every variety $\mathcal{V}$ is equivalent to the category of all set–valued functors on $(\mathsf{Proj}_{fp}\mathcal{V})^{\mathrm{op}}$ which preserve finite products.*

[11] The *Cauchy completion* of a small category $\mathcal{K}$ is a full embedding $E\colon \mathcal{K} \longrightarrow \bar{\mathcal{K}}$ into a *Cauchy complete* category $\bar{\mathcal{K}}$ (i.e. a category $\bar{\mathcal{K}}$ where every idempotent morphism u splits as $u = sr$ with $rs = 1$), universal with this property. See [6] for an explanation of the term "Cauchy completion".

3. *Lawvere theories $\mathbb{S}$ and $\mathbb{T}$ are Morita equivalent if and only if $\mathbb{S}$ and $\mathbb{T}$ have equivalent Cauchy completions.*

Proof By Lemma 2 every idempotent morphism from $\mathbb{T}$ splits in $\big(\mathsf{Proj}_{fp}\mathcal{V}\big)^{\mathrm{op}}$, and so does any idempotent $u\colon Q \longrightarrow Q$ in $\mathcal{V}$, if there is a retraction $Fn \xrightarrow{r} Q$: u splits in $\mathcal{V}$ as $u = pq$ with a retraction $q\colon Q \longrightarrow P$, and $qr\colon Fn \longrightarrow P$ is a retraction. Hence $(\mathsf{Proj}_{fp}\mathcal{V})^{\mathrm{op}}$ is Cauchy complete.

Every functor $S\colon \mathbb{T} \longrightarrow \mathcal{D}$ into a Cauchy complete category $\mathcal{D}$ can be (essentially uniquely) extended to $\hat{S}\colon (\mathsf{Proj}_{fp}\mathcal{V})^{\mathrm{op}} \longrightarrow \mathcal{D}$. This is an easy exercise making use of the fact that for every morphism $P \xrightarrow{f} Q$ in $\mathcal{V}$ and retractions $Fn \xrightarrow{p} P$, $Fm \xrightarrow{q} Q$ there exists some $Fn \xrightarrow{\bar{f}} Fm$ with $q\bar{f} = fp$ (use here specifically $\bar{f} = sfp$ where $s\colon Q \to Fm$ satisfies $qs = 1$).

Statement 2. now follows easily: if $S\colon \mathsf{Th}\mathcal{V} \longrightarrow \mathcal{SET}$ is a $\mathsf{Th}\mathcal{V}$-model, i. e. preserves finite products, its (essentially unique) extension $\hat{S}$ (note that clearly $\mathcal{SET}$ is Cauchy complete) will preserve finite products as well (easy exercise). Thus the assignment $S \mapsto \hat{S}$ defines an equivalence between $\mathsf{ModTh}\mathcal{V}$ (hence $\mathcal{V}$) and the category of all set–valued functors on $(\mathsf{Proj}_{fp}\mathcal{V})^{\mathrm{op}}$ which preserve finite products.

Statement 3. now follows from this and the fact that the classes $\mathsf{Proj}_{fp}\mathcal{V}$ are stable under equivalence of categories. □

On the level of model categories of theories, i.e., varieties, Theorem 3 has the following interpretation — generalizing a familiar fact from module categories (e. g. [4, §22]) — which has been put into a broader perspective recently by Adámek, Lawvere and Rosický (see [2]).

Theorem 4 ([11, §11]) *Varieties $\mathcal{V}$ and $\mathcal{W}$ are equivalent iff their respective subcategories $\mathsf{Proj}_{fp}\mathcal{V}$ and $\mathsf{Proj}_{fp}\mathcal{W}$ are equivalent.* □

The varieties determined by $\mathbb{T}^{[n]}$ and $\mathbb{T}_u$

This section aims at two results. By identifying the varieties determined by $\mathbb{T}^{[n]}$ and $\mathbb{T}_u$ respectively we can, by means of Theorem 1, describe all varieties equivalent to $\mathcal{V}_{\mathbb{T}}$, the variety determined by $\mathbb{T}$. Along this way the status of the algebraic constructions $\mathcal{V}^{[n]}$ and $\mathcal{V}(\sigma)$ as described in [22] is clarified and certain properties of these constructions are obtained categorically. We start by recalling these constructions.

For any algebra A in a given variety $\mathcal{V}$ and any $n \in \mathbb{N}$ the *n–th matrix power* $A^{[n]}$ of A is the following algebra:

- the underlying set of $A^{[n]}$ is $|A|^n$;

- the r–ary operations $|A^{[n]}|^r = |A|^{nr} \longrightarrow |A|^n = |A^{[n]}|$ are those maps m, whose composition with the projections $\pi_i\colon |A|^n \longrightarrow |A|$ yield (derived) nr–ary $\mathcal{V}$–operations t_i^A $(i = 1, \dots, n)$ on A, i.e., $m = \langle t_i^A \rangle$.

For a subcategory $\mathcal{K}$ of $\mathcal{V}$ the category of all algebras isomorphic to some $A^{[n]}$ with A in $\mathcal{K}$ is denoted by $\mathcal{K}^{[n]}$.

Let $\mathcal{K}$ be a class of similar algebras and $\mathcal{V}$ the variety generated by $\mathcal{K}$. A unary $\mathcal{V}$–term ε is called *idempotent* and *invertible* respectively in $\mathcal{K}$ if the corresponding morphism $\bar{\varepsilon}\colon F1 \longrightarrow F1$ is idempotent (respectively invertible) in $\mathsf{Th}\mathcal{V}$. These notions are clearly equivalent to the corresponding ones in [22].

For any algebra A contained in a class of similar algebras $\mathcal{K}$ and any idempotent unary term ε in $\mathcal{K}$ the *ε–modification* $A(\varepsilon)$ of A is the following algebra:

- the underlying set of $A(\varepsilon)$ is $\varepsilon^A[|A|] \subset |A|$, i.e., the image of the A–interpretation of ε;
- the r–ary operations $(\varepsilon^A[|A|])^r \longrightarrow \varepsilon^A[|A|]$ are those maps t_ε^A which arise as restrictions to $(\varepsilon^A[|A|])^r$ of maps $|A|^r \xrightarrow{t^A} |A| \xrightarrow{\varepsilon^A} \varepsilon[|A|]$ where t^A is the A–interpretation of an r–ary operation in the similarity type of $\mathcal{K}$.

$\mathcal{K}(\varepsilon)$ denotes the category of all algebras isomorphic to some $A(\varepsilon)$ with A in $\mathcal{K}$.

Proposition 4 *Let $\mathcal{V}$ be any variety and $\mathbb{T}$ its Lawvere theory. Then $\mathcal{V}^{[n]}$ is the variety determined by $\mathbb{T}^{[n]}$. $\mathcal{V}$ and $\mathcal{V}^{[n]}$ are equivalent.*

Proof By Proposition 3 we may assume $\mathbb{T} = \mathsf{Th}_\mathcal{V}(F1)$ and $\mathbb{T}^{[n]} = \mathsf{Th}_\mathcal{V}(Fn)$.

Since Fn is a varietal generator of the exact category $\mathcal{V}$ we know by Fact 8 that the Yoneda map $Y_{Fn}\colon \big(\mathcal{V}, \hom_\mathcal{V}(Fn, -)\big) \longrightarrow \big(\mathsf{ModTh}_\mathcal{V}(Fn), ev_{Fn}\big)$ is a concrete equivalence. With notation as in Fact 2 and the remarks preceding it, this means in particular that the algebras of the variety $\mathcal{W}$ determined by $\overline{\mathbb{T}^{[n]}}$ are, up to isomorphism, the algebras $\Phi Y_{Fn}V = \overline{Y_{Fn}V} = \big(\hom_\mathcal{V}(Fn, V),\ (\sigma^{\overline{Y_{Fn}V}})_{\bar{\sigma}\in\mathbb{T}^{[n]}}\big)$ where for $\bar{\sigma} \in \mathbb{T}^{[n]}(\bar{C}_r, \bar{C}_1) = \mathbb{T}(F(nr), Fn)$ the operation $\sigma^{\overline{Y_{Fn}V}}$ is uniquely determined by the condition to make the diagram

$$\begin{array}{ccc} \hom_\mathcal{V}(Fn, V)^r & \xrightarrow{\ \sim\ } & \hom_\mathcal{V}(F(nr), V) \\ {\scriptstyle \sigma^{\overline{Y_{Fn}V}}}\big\downarrow & & \big\downarrow{\scriptstyle \hom_\mathcal{V}(\bar{\sigma}, V)} \\ \hom_\mathcal{V}(Fn, V) & = & \hom_\mathcal{V}(Fn, V) \end{array}$$

commute, where the top row is the canonical bijection. Now, by definition of $\mathbb{T}^{[n]}$, there exist $t_1, \dots, t_n \in \mathbb{T}\big(F(nr), F1\big)$ such that $\bar{\sigma} = \langle t_i \rangle$ in $\mathbb{T}$. Since $\hom_\mathcal{V}(-, V)$ preserves products in $\mathbb{T}$ it follows

$$\hom_\mathcal{V}(\bar{\sigma}, V) = \langle \hom_\mathcal{V}(t_i, V) \rangle.$$

This translates into

$$\bar{\sigma}^{\overline{Y_{Fn}V}} = \langle t_i^{\overline{Y_{F1}V}} \rangle$$

which identifies $\overline{Y_{Fn}V}$ as the n–th matrix power $\overline{Y_{F1}V}^{[n]}$ of $\overline{Y_{F1}V}$. Since, up to isomorphism, all $\mathcal{V}$–algebras are of the form $\overline{Y_{F1}V}$, $\mathcal{W}$ and $\mathcal{V}^{[n]}$ are concretely isomorphic. □

The proof shows in particular that the equivalence of $\mathcal{V}$ and $\mathcal{V}^{[n]}$ is given by sending an algebra to its n–th matrix power.

Using Example 2.2 in connection with Theorem 2 and Proposition 3 one obtains the following sharpening of [22, Theorem 6.4] (see also [17]):

Corollary 1 *For a variety $\mathcal{V}$ the following are equivalent:*

(i) The only varieties equivalent to $\mathcal{V}$ are (up to concrete isomorphism) the varieties $\mathcal{V}^{[n]}$, $n \in \mathbb{N}$.

(ii) Each finitely generated projective $\mathcal{V}$-algebra is free.

Remark 7 As observed before, "rising to the n–th power" is a theory morphism $\mathbb{T} \longrightarrow \mathbb{T}^{[n]}$. By Fact 5 there results a concrete (hence faithful) functor $\mathsf{Mod}\mathbb{T}^{[n]} \longrightarrow \mathsf{Mod}\mathbb{T}$ or, in view of Proposition 4, $\mathcal{V}^{[n]} \longrightarrow \mathcal{V}$. This functor assigns to a matrix power $A^{[n]}$ in $\mathcal{V}^{[n]}$ the n–th (cartesian) power of A in $\mathcal{V}$. Note that this is also clear from the very definition of $A^{[n]}$: among its r–ary operations are clearly the r–ary $\mathcal{V}$–operations of A^n. Thus the $\mathcal{V}$–operations induce $\mathcal{V}^{[n]}$–operations satisfying the same equations as in $\mathcal{V}$.

Moreover, the $\mathcal{V}^{[n]}$–algebras automatically have n unary operations $D_1, \ldots, D_n$ satisfying the equations $D_j D_i = D_i$. This is a consequence of the fact that in the theory $\mathbb{T}^{[n]}$ the object $\bar{C}_1 = C_1^n$ has, for each $i \in \{1, \ldots, n\}$, an endomorphism D_i induced by the constant family $(\pi_i, \ldots, \pi_i)$, i.e., $\pi_j \circ D_i = \pi_i$ for all $j = 1, \ldots, n$. Note that all other unary operations induced by families of projections can be obtained from the D_i's by clone–composition.

Other choices of (basic) unary operations for $\mathcal{V}^{[n]}$–algebras might be suitable as well. In order to describe $\mathcal{SET}^{[2]}$ as a variety of semigroups (see e.g. [22, Example 2]) one uses e.g. $s = \langle \pi_2, \pi_1 \rangle$.

Proposition 5 *Let $\mathcal{V}$ be a variety and $\mathbb{T}$ its theory. Let ε be an idempotent unary term in the language of $\mathcal{V}$ and $u = \bar{\varepsilon}$ the corresponding idempotent morphism in $\mathbb{T}(C_1, C_1) = \hom_{\mathcal{V}}(F1, F1)$. Then the following hold:*

1. *$\mathcal{V}(\varepsilon)$ is the variety determined by $\mathbb{T}_u$.*

2. *$\mathcal{V}(\varepsilon)$ and $\mathcal{V}$ are equivalent iff ε is invertible in $\mathcal{V}$.*

Proof We use the same notation as in Proposition 3 and its proof. Similarly as in the proof of Proposition 4 we now consider the Yoneda functor Y_Q. By Fact 8 and Proposition 2 Y_Q is an equivalence iff ε is invertible. Thus 2. is a consequence of 1.

For proving statement 1. of the proposition we first show that, for any $\mathcal{V}$–algebra V, $V(\varepsilon)$ and $\overline{Y_Q(V)}$ are isomorphic. Up to a bijection we may assume $|V| = \hom_{\mathcal{V}}(F1, V)$. Then $\varepsilon^V = \hom_{\mathcal{V}}(u, V)$ factors as

$$\varepsilon^V = \hom_{\mathcal{V}}(F1, V) \xrightarrow{\hom_{\mathcal{V}}(s,V)} \hom_{\mathcal{V}}(Q, V) \xrightarrow{\hom_{\mathcal{V}}(r,V)} \hom_{\mathcal{V}}(F1, V).$$

Since $rs = 1$, $\hom_{\mathcal{V}}(r, V)$ is, up to a unique bijection

$$i_V \colon \hom_{\mathcal{V}}(Q, V) \xrightarrow{\sim} \operatorname{Im} \varepsilon^V,$$

the embedding of the subset $\operatorname{Im} \varepsilon^V = |V(\varepsilon)|$, and we have, for each $k \in \mathbb{N}$, the commutative diagram

$$\begin{array}{ccccc}
\hom_{\mathcal{V}}(kQ, V) & \overset{\sim}{\longrightarrow} & \hom_{\mathcal{V}}(Q, V)^k & \overset{\sim}{\longrightarrow} & |V(\varepsilon)|^k \\
\Big\downarrow{\scriptstyle \hom_{\mathcal{V}}(kr,V)} & & \Big\downarrow{\scriptstyle \hom_{\mathcal{V}}(r,V)^k} & & \Big\downarrow \\
\hom_{\mathcal{V}}(Fk, V) & \overset{\sim}{\longrightarrow} & \hom_{\mathcal{V}}(F1, V)^k & = & |V|^k
\end{array}$$

Up to these natural bijections the operations of $V(\varepsilon)$ are, by definition, the maps

$$\begin{array}{ccccc}
\hom_{\mathcal{V}}(kQ, V) & \xrightarrow{\hom_{\mathcal{V}}(kr,V)} & \hom_{\mathcal{V}}(Fk, V) & & \\
 & & \Big\downarrow{\scriptstyle \hom_{\mathcal{V}}(t,V)} & & \\
 & & \hom_{\mathcal{V}}(F1, V) & \xrightarrow{\hom_{\mathcal{V}}(s,V)} & \hom_{\mathcal{V}}(Q, V).
\end{array}$$

Since by the proof of Proposition 3 the maps $kQ \longrightarrow Q$ in $\mathsf{Th}_{\mathcal{V}}Q$ are precisely the maps $kQ \xrightarrow{kr} Fk \xrightarrow{t} F1 \xrightarrow{s} Q$ with $Fk \xrightarrow{t} F1$ in $\mathsf{Th}\mathcal{V}$, we see that, up to isomorphism, $V(\varepsilon)$ is just $Y_Q(V)$.

It remains to show that every K in $\mathsf{Mod}\mathbb{T}_u$ is isomorphic to some $Y_Q V$, V in $\mathcal{V}$. Recall from Fact 7 that Y_Q has a left adjoint T and satisfies $Y_Q(X \cdot Q) = FX$ where FX is free in $\mathsf{Mod}\mathbb{T}_u$ over the finite set X. Since clearly $TFX \simeq X \cdot Q$ (by composition of adjoints) we get $Y_Q TFX \simeq Y_Q(X \cdot Q) \simeq FX$. Now each K in the variety $\mathsf{Mod}\mathbb{T}_u$ is a directed colimit of FX_λ's with each X_λ finite. Since T (as a left adjoint) and Y_Q (by Fact 7) preserve directed colimits $Y_Q TFX \simeq FX$ implies $Y_Q TK \simeq K$ for each K. □

Remark 8 As observed (see Remark 6.1) before the idempotent u determines a theory morphism $\bar{u} \colon \mathbb{T}_{[u]} \longrightarrow \mathbb{T}_u$, thus a concrete functor $\mathsf{Mod}\mathbb{T}_u \longrightarrow \mathsf{Mod}\mathbb{T}_{[u]}$ or, in view of Proposition 5, $\mathcal{V}(\varepsilon) \longrightarrow \mathsf{Mod}\mathbb{T}_{[u]}$ where the operations of the variety determined by $\mathbb{T}_{[u]}$ have precisely those $\mathcal{V}$–operations which commute with u. In view of the previous discussion this is equivalent to saying: it has precisely those $\mathcal{V}$–operations with respect to which all embeddings $A(\varepsilon) \hookrightarrow A$ are homomorphic.

As a corollary to Propositions 4 and 5 we get in view of Theorem 1:

Theorem 5 ([22]) *The varieties equivalent to a given variety $\mathcal{V}$ are, up to concrete isomorphism, precisely the varieties $\mathcal{V}^{[n]}(\varepsilon)$ for some $n \in \mathbb{N}$, $n \geq 1$, and some idempotent and invertible term ε for $\mathcal{V}^{[n]}$.* □

Remark 9 In view of the presentation in [22] it might be worth noticing that the equivalence $\mathcal{V} \xrightarrow{\sim} \mathcal{V}^{[n]}(\varepsilon)$ in the theorem above (modulo the concrete equivalence $\mathsf{ModTh}_{\mathcal{V}}Q \simeq \mathcal{V}^{[n]}(\varepsilon)$) is given by the Yoneda functor $Y_Q\colon \mathcal{V} \longrightarrow \mathsf{ModTh}_{\mathcal{V}}Q$ where Q is a coequalizer of the pair $Fn \overset{\bar{\varepsilon}}{\underset{1}{\rightrightarrows}} Fn$ in $\mathcal{V}$.

Leading Example (f) We now determine the varieties equivalent to $\mathcal{BOOL}$. What remains to be done is—in view of (e) and (d)—the identification of

$$\mathcal{P}'_m \colon = \mathcal{BOOL}^{[m-1]}(\varepsilon) \simeq \mathsf{ModTh}_{\mathcal{BOOL}}\mathsf{P}(m)$$

with $\varepsilon = (x_1, x_1 \wedge x_2, \ldots, x_1 \wedge \ldots \wedge x_{m-1})$.

Thus the $\mathcal{P}'_m$–algebras are up to isomorphism given as the subsets $\bar{\varepsilon}^{B^{[m-1]}}[B^{m-1}] \subset B^{m-1}$ with a Boolean algebra B and $\bar{\varepsilon}^{B^{[m-1]}}(b_1, \ldots, b_{m-1}) = (b_1,\ b_1 \wedge b_2, \ldots, b_1 \wedge \ldots \wedge b_{m-1})$, i.e. as

$$C(B^{m-1})\colon = \{(b_1, \ldots, b_{m-1}) \in B^{m-1} \mid b_{m-1} \leq \ldots \leq b_1\}.$$

Using Remarks 7 and 8 one checks first that there is a concrete functor $\mathcal{P}'_m \longrightarrow \mathcal{D}_{01}$ from $\mathcal{P}'_m$ to the category $\mathcal{D}_{01}$ of bounded distributive lattices. Since the lattice operations (including 0 and 1) commute with the idempotent, the $\mathcal{P}'_m$–algebras are first of all distributive lattices (see Remark 8). Moreover, also the unary operations D_i on $B^{[m-1]}$ (see Remark 7) clearly commute with u and thus are unary operations acting as

$$\mathrm{D}_i(b_1, \ldots, b_{m-1}) = (b_i, \ldots, b_i).$$

In addition the $\mathcal{P}'_m$–algebras inherit from B^{m-1} at least

- the nullary operations

$$c_i = \bar{\varepsilon}\ (0, \ldots, \underset{\substack{\uparrow \\ \text{i-th component}}}{0, 1}, \ldots, 1)$$

  for $i = 1, \ldots, m$ (in particular $c_0 = 0,\ c_m = 1$),

- the unary operation

$$\mathrm{C} = \bar{\varepsilon} \circ \neg \; (\neg \text{ the complementation in } B^{m-1})$$

which then acts as !

$$\begin{aligned} C(b_1, \ldots, b_{m-1}) &= (\neg b_1,\ (\neg b_1) \wedge (\neg b_2), \ldots, (\neg b_1) \wedge \ldots \wedge (\neg b_{m-1})) \\ &= (\neg b_1, \ldots, \neg b_1) \end{aligned}$$

It is now easy to verify that this way each $\mathcal{P}'_m$–algebra is a Post algebra of order m with equational presentation as in [5]. It is implicit in [24] that any Post algebra of order m is isomorphic to a $\mathcal{P}'_m$–algebra. If one doesn't want to refer to [24] one alternatively might check that the operations described above generate the theory $\mathsf{Th}_{\mathcal{BOOL}}\mathsf{P}(m)$, or that every $\mathcal{P}'_m$–algebra is a coproduct of a Boolean algebra and the m-chain in the category $\mathcal{D}_{01}$. Thus we obtain

Theorem 6 *The varieties equivalent to the variety $\mathcal{BOOL}$ of Boolean algebras are (up to concrete isomorphism) precisely the varieties $\mathcal{P}_m$ of Post algebras of order m for some $m \in \mathbb{N}$.*

As a corollary one might translate Hu's result on varieties generated by primal algebras (see [14]) as follows (a proof of this can also be given without using Hu's result).

Theorem 7 *For each $m \in \mathbb{N}$ the variety $\mathcal{P}_m$ of Post algebras of order m is (up to concrete isomorphism) the only variety generated by an m–element primal algebra.*

As a final example let us describe classical Morita theory by the methods developed above:

Example 3 Morita theory for module categories is concerned with the problem of characterizing — for a given unital ring R — all unital rings S (then called *Morita equivalent to R*) such that the varieties ${}_R\mathcal{MOD}$ and ${}_S\mathcal{MOD}$ are equivalent. This is a special case of the general problem dealt with here since characterizing all varieties equivalent to ${}_R\mathcal{MOD}$ only seems to be less specific: in fact, any variety equivalent to ${}_R\mathcal{MOD}$ is necessarily of the form ${}_S\mathcal{MOD}$ since module categories can be characterized as those varieties $\mathcal{V}$ which are Abelian categories[12] (see e.g. [13, Theorem 41.16]) and this property is preserved by equivalences.

This leads to a first result: ${}_R\mathcal{MOD} \simeq {}_S\mathcal{MOD}$ if and only if $\mathcal{MOD}_R \simeq \mathcal{MOD}_S$. Simply apply Theorem 1 and Remark 6.2 with a suitable pair (n, u): ${}_R\mathcal{MOD} \simeq {}_S\mathcal{MOD}$ iff $\mathbb{T}_l^S = (\mathbb{T}_l^R)_u^{[n]}$ iff $\mathbb{T}_r^S = (\mathbb{T}_l^S)^{\mathsf{op}} = ((\mathbb{T}_l^R)_u^{[n]})^{\mathsf{op}} = ((\mathbb{T}_l^R)^{\mathsf{op}})_u^{[n]} = (\mathbb{T}_r^R)_u^{[n]}$ iff $\mathcal{MOD}_R \simeq \mathcal{MOD}_S$.

[12] In our context this simply means that the hom–sets of $\mathcal{V}$ are Abelian groups such that composition is a group homomorphism in each variable (see [6]).

Now in view of Example 1.2 and statement 1. of Proposition 3 it is clear that for the theory $\mathbb{T}_l^R$ of ${}_R\mathcal{MOD}$ the theory $(\mathbb{T}_l^R)^{[n]}$ is just $\mathbb{T}_l^{\mathsf{M}_n(R)}$, the theory of the variety ${}_{\mathsf{M}_n(R)}\mathcal{MOD}$ of left modules over $\mathsf{M}_n(R)$. Thus $\mathsf{M}_n(R)$ is Morita equivalent to R and moreover, in case R is a PID, these matrix rings are the only rings with this property by Corollary 1.

In the general case S will be of the form $e\mathsf{M}_n(R)e$ with an idempotent matrix $e \in \mathsf{M}_n(R)$ by statement 2. of Proposition 3 and the definition of $\mathbb{T}_u$, again in view of the discussion in Example 1.2. Invertibility of e can be expressed by saying that the ideal generated by e is $\mathsf{M}_n(R)$.

A categorical description of the functors establishing the equivalence ${}_R\mathcal{MOD} \simeq {}_S\mathcal{MOD}$ for Morita equivalent rings R and S is given in [7]. For a classical presentation of these results see e.g. [4] or [16].

Equivalence for subcategories of varieties

In this section we will discuss equivalences between categories $\mathcal{K}$ of similar algebras which are not necessarily varieties. By $\mathcal{V}^{\mathcal{K}}$ we here denote the variety generated by $\mathcal{K}$.

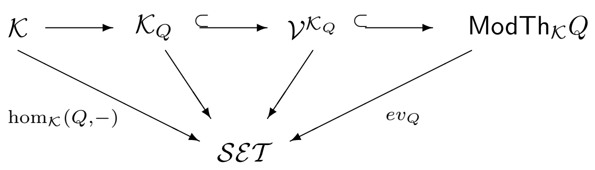
To relate our results to the corresponding ones of [22] it is useful to recall the following notation of [22] : If $\mathcal{K}$ is a class of similar algebras and Q an algebra from $\mathcal{K}$ having all its finite copowers in $\mathcal{K}$, the full subcategory of $\mathsf{ModTh}_{\mathcal{K}}Q$ spanned by the images Y_QK, $K \in \mathcal{K}$, of the Yoneda functor Y_Q is denoted by $\mathcal{K}_Q$ while

$$\hom(Q, -) : \mathcal{K} \to \mathcal{K}_Q$$

denotes the corresponding (co-)restriction of Y_Q to its image. Now in this context the following holds.

Proposition 6 *Let $\mathcal{K}$ be any category of similar algebras admitting all finite copowers of some $\mathcal{K}$–object Q. Then the variety $\mathcal{V}^{\mathcal{K}_Q}$ generated by $\mathcal{K}_Q$ is concretely equivalent to* $\mathsf{ModTh}_{\mathcal{K}}Q$.

Proof By definition $\mathcal{V}^{\mathcal{K}_Q}$ is a concrete subcategory of $\mathsf{ModTh}_{\mathcal{K}}Q$. Hence Y_Q factors as follows



where all triangles commute and the unlabeled underlying functors are restrictions of ev_Q.

By Fact 7 we know that $Y_Q(nQ)$ is free in $\mathsf{ModTh}_{\mathcal{K}}Q$ on n generators; in particular ev_Q (and its restrictions to $\mathcal{V}^{\mathcal{K}_Q}$ and $\mathcal{K}_Q$, respectively) are represented by $Y_Q(Q)$. In fact, by classical Yoneda Lemma, Y_Q fully embeds $(\mathsf{Th}_{\mathcal{K}}Q)^{op}$ into $\mathsf{ModTh}_{\mathcal{K}}Q$, hence into $\mathcal{V}^{\mathcal{K}_Q}$. Consequently $\mathsf{Th}_{\mathcal{K}}Q$ is the Lawvere theory of $\mathcal{V}^{\mathcal{K}_Q}$ which proves the claim. □

In order to generalize the results obtained in the previous section we now consider classes $\mathcal{K}$ of similar algebras suitably embedded in the variety $\mathcal{V}^{\mathcal{K}}$ generated by $\mathcal{K}$. The methods used until now suggest that "suitably embedded" should be understood as containing the subcategory $\mathsf{Proj}_{fp}\mathcal{V}^{\mathcal{K}}$ and satisfying the condition that an algebra K in $\mathcal{K}$ is compact in $\mathcal{K}$ iff it is compact in $\mathcal{V}^{\mathcal{K}}$. We therefore introduce the following notion:

Definition 9 A class $\mathcal{K}$ of similar algebras is called *varietally closed* provided that it satisfies the following conditions:

I. $\mathsf{Proj}_{fp}\mathcal{V}^{\mathcal{K}} \subset \mathcal{K}$

II. Every extremal episink $(K_t \xrightarrow{e_t} K)_{t\in R}$ in $\mathcal{K}$ is an extremal episink in $\mathcal{V}^{\mathcal{K}}$.

Every varietally closed class $\mathcal{K}$ then contains the varietal generators of $\mathcal{V}^{\mathcal{K}}$. It is easy to see that every SF–class in the sense of [22] is varietally closed; therefore also all SP–classes and quasivarieties (= SP–classes closed under directed colimits) are varietally closed.

Let now $\mathcal{K}$ be any class of similar algebras. By Proposition 4 and Proposition 5 respectively there are equivalences

$$\begin{array}{rcl} Y_{Fn} : \mathcal{V}^{\mathcal{K}} & \longrightarrow & (\mathcal{V}^{\mathcal{K}})^{[n]} \\ Y_Q \ : \mathcal{V}^{\mathcal{K}} & \longrightarrow & \mathcal{V}^{\mathcal{K}}(\varepsilon) \end{array}$$

sending V to $V^{[n]}$ and $V(\varepsilon)$ respectively, for any $n \in \mathbb{N}$, $n \geq 1$, and any idempotent invertible unary term ε in the language of $\mathcal{V}^{\mathcal{K}}$. Clearly $\mathcal{K}^{[n]}$ and $\mathcal{K}(\varepsilon)$ respectively are the isomorphism closed full subcategories of $(\mathcal{V}^{\mathcal{K}})^{[n]}$ and $\mathcal{V}^{\mathcal{K}}(\varepsilon)$, respectively, spanned by the images under Y_{Fn} and Y_Q respectively of the objects of $\mathcal{K}$. These equivalences Y preserve and reflect subobjects, products, and directed colimits, and, in addition, free objects by Fact 8. We then immediately have as corollaries the following results of [22] characterizing, in particular, the quasivarieties $\mathcal{W}$ equivalent to a given quasivariety $\mathcal{V}$:

Corollary 2 *Let $\mathcal{K}$ be any category of similar algebras, ε an idempotent invertible unary term in the language of $\mathcal{K}$, and $n \in \mathbb{N}$. Then the following holds:*

1. *The following are equivalent:*

   (i) $\mathcal{K}$ is varietally closed (is an SF–class, SP–class, quasivariety, variety).

   (ii) $\mathcal{K}^{[n]}$ is varietally closed (is an SF–class, SP–class, quasivariety, variety).

   (iii) $\mathcal{K}(\varepsilon)$ is varietally closed (is an SF–class, SP–class, quasivariety, variety).

2. $\mathcal{K}$ is equivalent to both $\mathcal{K}^{[n]}$, and $\mathcal{K}(\varepsilon)$. □

Corollary 3 *The varietally closed classes equivalent to a given varietally closed class $\mathcal{K}$ are, up to concrete isomorphism, precisely the classes $\mathcal{K}^{[n]}(\varepsilon)$, for some $n \in \mathbb{N}$, $n \geq 1$, and some idempotent invertible unary term ε in $\mathcal{K}^{[n]}$.*

Proof Let $\mathcal{K}$ and $\mathcal{L}$ be varietally closed. Then every equivalence $\mathcal{K} \simeq \mathcal{L}$ restricts to an equivalence $\mathsf{Proj}_{fp}\mathcal{V}^{\mathcal{K}} \simeq \mathsf{Proj}_{fp}\mathcal{V}^{\mathcal{L}}$ by Lemma 1 since in a varietally closed class $\mathcal{K}$ an object K is extremally projective and compact iff K has these properties with respect to $\mathcal{V}^{\mathcal{K}}$. By Theorem 4 we get an equivalence $\mathcal{V}^{\mathcal{L}} \simeq \mathcal{V}^{\mathcal{K}}$. Theorem 5 yields a concrete isomorphism $Y_Q\colon \mathcal{V}^{\mathcal{L}} \xrightarrow{\sim} (\mathcal{V}^{\mathcal{K}})^{[n]}(\varepsilon)$ for suitable n and ε. Restriction to $\mathcal{L}$ proves the claim. □

Corollary 4 ([22, Thm 6.2]) *The following statements are equivalent for an object Q of a varietally closed class $\mathcal{K}$ admitting all finite copowers of Q:*

(i) Y_Q is an equivalence between $\mathcal{K}$ and a varietally closed class in $\mathsf{ModTh}_{\mathcal{K}}Q$.

(ii) Q is a varietal generator in $\mathcal{V}^{\mathcal{K}}$.

Proof $Y_Q(Q)$ is the free $\mathsf{ModTh}_{\mathcal{K}}Q$–object on one generator by Fact 7, hence a varietal generator; using Proposition 6 condition (i) implies that Q is a varietal generator, too. The converse is only a reformulation of parts of Corollary 2 (ii). □

Finally we are going to study the question to what extent interesting properties of the class $\mathcal{K}$ are stable under ε–modification, even if ε fails to be invertible. We already know from Proposition 5 that also in this case $\mathcal{K}(\varepsilon)$ is a variety provided $\mathcal{K}$ is — though not equivalent to $\mathcal{K}$. The following proposition strengthens conclusions of Corollary 2 and extends results of [22].

Proposition 7 *Let $\mathcal{K}$ be any category of similar algebras and ε an idempotent unary term in the language of $\mathcal{K}$. Then $\mathcal{K}(\varepsilon)$ is an SF–class (SP–class, quasivariety, variety) whenever $\mathcal{K}$ is.*

Proof The statement for varieties is clear by Proposition 5. Hence it remains to show that $\mathcal{K}(\varepsilon)$ has the closure properties defining the respective classes mentioned in the proposition in $\mathcal{V}^{\mathcal{K}}(\varepsilon)$, provided $\mathcal{K}$ has these properties in $\mathcal{V}^{\mathcal{K}}$. Since up to isomorphism of algebras the objects of $\mathcal{K}(\varepsilon)$ are the objects $Y_Q K, K \in \mathcal{K}$, this is a question of preservation properties of Y_Q with Q determined by ε as above.

Y_Q preserves products (since it has a left adjoint) and directed colimits (by Fact 7), hence $\mathcal{K}(\varepsilon)$ — being spanned (up to isomorphism) by the objects $Y_Q K,\ K \in \mathcal{K}$ — is closed under products and directed colimits. Also by Fact 7 for any free object F in $\mathcal{V}^{\mathcal{K}}$ the object $Y_Q(F)$ is free in $\mathcal{V}^{\mathcal{K}}(\varepsilon)$ on the same set of generators as F. Hence it only remains to prove that $\mathcal{K}(\varepsilon)$ is an S–class, provided $\mathcal{K}$ is.

To show this, we first observe that in any variety $\mathcal{V}$ with underlying functor $|-|$, a subalgebra S of an algebra V is generated by a set X iff S is the image of the homomorphic extension $i^\sharp\colon FX \longrightarrow V$ of the embedding $i\colon X \longrightarrow |V|$ with FX the free algebra over X.

Let now S be a subalgebra of Y_QK in $\mathsf{Mod}\mathbb{T}_u$ with $K \in \mathcal{K}$. Consider the free $\mathcal{V}^{\mathcal{K}}$–algebra F_S generated by the underlying set $|S|_\varepsilon$ of S (with respect to the underlying functor $|-|_\varepsilon\colon \mathcal{V}^{\mathcal{K}}(\varepsilon) \longrightarrow \mathcal{SET}$). By definition of Y_Q we have $|Y_QK|_\varepsilon = \hom_{\mathcal{V}}(Q,K)$ and by (the proof of) Proposition 5 we have $\hom_{\mathcal{V}}(Q,K) \subset |K|$. Hence $|S|_\varepsilon \subset |K|$ and this embedding admits a homomorphic extension $F_S \longrightarrow K$, which factors over its image as, say, $F_S \xrightarrow{s} L \overset{m}{\hookrightarrow} K$. Applying Y_Q gives $Y_QF_S = F_S^\varepsilon \xrightarrow{Y_Qs} Y_QL \xrightarrow{Y_Qm} Y_QK$ with F_S^ε the free $\mathcal{V}(\varepsilon)$–algebra over $|S|_\varepsilon$. Now Y_Qs is surjective (since Q is projective) and Y_Qm is injective (since right adjoints preserve monomorphisms)[13]. Hence $Y_QL \simeq S$, and $\mathcal{K}(\varepsilon)$ is closed under subalgebras provided $\mathcal{K}$ is. □

Some related questions and results

In this final section we will address the following problems posed by McKenzie [22, Problem 1] concerning the full subcategory $\mathcal{K}_Q$ of $\mathsf{ModTh}_{\mathcal{K}}Q$ spanned by the objects $Y_Q(K)$, K in $\mathcal{K}$ for any (varietally closed) class $\mathcal{K}$ and an object Q in $\mathcal{K}$ having all its finite copowers.

1. What can be said about Q if
   - 1.1 $\mathcal{K}_Q$ contains its (i.e. those of $\mathcal{V}^{\mathcal{K}_Q}$) finitely generated free algebras?
   - 1.2 $\mathcal{K}_Q$ is closed under subalgebras?
2. What can be said about Q if $Y_Q\colon \mathcal{K} \longrightarrow \mathcal{K}_Q$ is an equivalence (but $\mathcal{K}_Q$ isn't necessarily varietally closed as in Corollary 4)?

The answer to Question 1.1 follows immediately from Fact 7 and Proposition 6: !

Corollary 5 K_Q *always contains its finitely generated free algebras.* □

A partial answer to 1.2 is already contained in the proof of Proposition 7: it was projectivity of Q which made $\mathcal{K}_Q$ S–closed. It is easy to see that projectivity of Q is in fact necessary for this property in case where Y_Q is a full embedding. Now Y_Q is a full embedding iff $Y_Q\colon \mathcal{K} \longrightarrow \mathcal{K}_Q$ is an equivalence (see Question 2). By Fact 7 this is equivalent to the fact that the full category $\langle nQ \mid n \in \mathbb{N}\rangle$ of $\mathcal{K}$ spanned by the finite copowers of Q is dense in $\mathcal{K}$. This condition on Q is closely related to Q being a regular generator, a property however, which can be expressed easily only provided that *all* copowers of Q exist. In detail:

[13]In varieties the injective homomorphisms are precisely the monomorphisms.

Proposition 8 *Let $\mathcal{K}$ be a category and Q a $\mathcal{K}$–object having all copowers. Then the following hold:*

1. *If $\mathsf{Th}_{\mathcal{K}}Q$ is dense in $\mathcal{K}$ then Q is a regular generator*[14].

2. *If Q is a regular generator and finitely presentable then $\mathsf{Th}_{\mathcal{K}}Q$ is dense in $\mathcal{K}$.*

Proof For 1. see [11, §3] or [9, Theorem 3.5]; for 2. see (the proof of) [11, Satz 7.5]. □

By the remarks preceeding Proposition 8 we get the following (partial) answers to Questions 1.2 and 2 as corollaries:

Corollary 6 *Let $\mathcal{K}$ be any class of similar algebras and Q in $\mathcal{K}$ having all finite copowers. Then the following hold provided Q is finitely presentable:*

1. *$Y_Q \colon \mathcal{K} \longrightarrow \mathcal{K}_Q$ is an equivalence iff Q is a regular generator.*

2. *$\mathcal{K}_Q$ is S-closed provided Q is projective; S-closedness of $\mathcal{K}_Q$ implies projectivity of Q whenever Q is a regular generator.*

When applying these results one can relate the above conditions on Q in $\mathcal{K}$ to the corresponding conditions in $\mathcal{V}^{\mathcal{K}}$ provided the copowers of Q in $\mathcal{K}$ are copowers in $\mathcal{V}^{\mathcal{K}}$. This very restrictive condition is satisfied, if the finite copowers of Q are copowers in $\mathcal{V}^{\mathcal{K}}$ and if $\mathcal{K}$ is closed in $\mathcal{V}^{\mathcal{K}}$ against directed colimits as, e.g., in case of a quasivariety. One then should be aware of the following facts:

1. Q is a regular generator in $\mathcal{K}$ iff Q is a regular generator in $\mathcal{V}^{\mathcal{K}}$ provided $\mathcal{K}$ is varietally closed;

2. Finite presentability in $\mathcal{K}$ will be different from finite presentability in $\mathcal{V}^{\mathcal{K}}$ in general. Both notions coincide, however, provided $\mathcal{K}$ is a quasivariety.

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[14]An object G in a category $\mathcal{K}$ is called a *regular generator* provided for each $K \in \mathcal{K}$ the canonical map $\hom_{\mathcal{K}}(G,K) \cdot G \to K$ is a *regular epimorphism*, i.e., it is a coequalizer of some parallel pair of morphisms. In SP-classes the notions of extremal and regular epimorphism (hence of extremal and regular generator) coincide; this is not true in general.

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