Generalized Morita Theories

The power of categorical algebra

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Abstract

A solution to the problem posed by Isbell in the early 1970s of how to determine all varieties (in the sense of universal algebra) equivalent to a given variety is presented. The method is based on the use of Lawvere theories. It is shown how the known characterizations of rings, Morita equivalent to a given one, and of monoids, Morita equivalent to a given one, and also how (even an extension of) Hu's primal algebra theorem become simple consequences hereof. Finally, a couple of related approaches is discussed.

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Introduction

The question of whether or not two categories are "essentially the same" is at the heart of many important results in mathematics. It is, for example, useful to know that the category ${}_{\mathbb{R}}\mathsf{Vect}_{\mathsf{fin}}$ of finite-dimensional real vector space and $\mathbb{R}$-linear maps is "essentially the same" as its subcategory spanned by all finite sums of $\mathbb{R}$, or as its dual; similarly it can be helpful to know that the categories of Boolean algebras and Boolean rings are "essentially the same", or the category of finite Boolean algebras and the dual of the category of finite sets.

There are different degrees to which to categories $\mathcal{K}$ and $\mathcal{L}$ might be "essentially the same": they might be *isomorphic*, i.e., there might be functors $F\colon \mathcal{K} \to \mathcal{L}$ and $G\colon \mathcal{L} \to \mathcal{K}$

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with $FG = 1_{\mathcal{L}}$ and $GF = 1_{\mathcal{K}}$; they might be—a somewhat weaker but more useful notion—*equivalent*, i.e., related by a pair of functors as above where, however, GF and FG are only naturally isomorphic to the respective identity functors. Equivalences between $\mathcal{K}^{op}$ and $\mathcal{L}$ are called *dualities*. In case the given categories are even *concrete* in that they are equipped with underlying functors into some common base category as e.g. !! the category Set of sets and mappings (in that case $\mathcal{K}$ and $\mathcal{L}$ would be considered to be categories of "structured sets") the categories might be *concretely isomorphic* and *concretely equivalent* respectively, meaning that the functors F and G commute with the given underlying ones (that is, e.g. in the case of categories of "structured sets", that F and G don't change the underlying sets of the structures considered). Concrete equivalence between concrete categories as well as isomorphism of (abstract) categories will be denoted by the isomorphism symbol $\cong$. Note that concretely equivalent varieties (in the sense of universal algebra), i.e., equationally definable classes of algebras[1] are even concretely isomorphic[2].

It is the purpose of classical Morita theory to determine, for a given ring R, all rings S such that their respective categories ${}_R\mathsf{Mod}$ and ${}_S\mathsf{Mod}$ of (left) modules are equivalent. Since any variety equivalent to a category of modules is itself of the form ${}_S\mathsf{Mod}$ for some ring S, Morita theory can be viewed as describing all varieties $\mathcal{W}$ equivalent to a given variety ${}_R\mathsf{Mod}$. Stated this way Morita theory resembles a theory developed by Hu in the late 1960s, which characterizes those varieties which are equivalent to the variety Bool of Boolean algebras. It should be noted however that the methods employed in Morita theory and by Hu respectively are completely different: Morita theory depends on an analysis of the rôle played by certain generators in ${}_R\mathsf{Mod}$ while Hu exploits and extends Stone's famous duality between the categories of Boolean algebras and Boolean spaces.

Nevertheless, by invoking the basic results of categorical algebra as developed by Lawvere in the early 1960s we will describe in this paper a generalized Morita theory, i.e., a categorical method leading to a theorem which has both, Morita's and Hu's results as simple consequences. The method can be viewed as an abstract version of Morita's and is certainly not more demanding than this. It is in accordance with the slogan that the rôle played by a ring for its category of modules is played for an arbitrary variety by its Lawvere theory.

The outline of the paper is as follows: in the first section we simply state Morita's and Hu's theorems as well as a non-additive version of the Morita result, due independently to Banaschewski and Knauer. Sections 2 and 3 recall the fundamentals of universal algebra including the necessary definitions. While we explain in section 2 how varieties can be constructed, section 3 focusses on axiomatic characterizations of varieties, which then lead to the general Morita theorem. In section 4 it is shown how the results stated at the beginning can be deduced from the main theorem. In a final section we briefly discuss relations between the results presented here and similar ones obtained by using different methods, thereby also touching the problem of how to construct effectively the equivalence functors in question.

[1] The notion of variety as well as some technical notions in Section 1 concerning varieties will be explained in detail in section 2.

[2] The reader interested in a thorough discussion of the notions of concrete equivalence and concrete isomorphism might consult [14].

1 Some theorems, some problems

Classical Morita theory deals with the following problem:

P1 Given a unital (not necessarily commutative) ring R, can one determine all rings S such that their respective categories of (left) modules, ${}_R\mathsf{Mod}$ and ${}_S\mathsf{Mod}$, are equivalent?

The answer, due to Morita, can be given in various equivalent ways, one of which is the following (see e.g. [10]):

Theorem 1 *The following are equivalent for rings R and S:*

(i) The variety ${}_S\mathsf{Mod}$ is equivalent to ${}_R\mathsf{Mod}$.

(ii) S is isomorphic to $u \cdot \mathsf{Mat}(n, R) \cdot u$ for some idempotent matrix u, which generates the ring $\mathsf{Mat}(n, R)$ of $n \times n$-matrices over R (as an ideal).

Remark 1 In this case an equivalence ${}_R\mathsf{Mod} \xrightarrow{F} {}_S\mathsf{Mod}$ is given by a hom-functor $F = \mathsf{hom}_{{}_R\mathsf{Mod}}(P, -)$ with an $R-S$-bimodule P which (as an R-module)

- is a finitely generated (regular) projective (regular) generator in ${}_R\mathsf{Mod}$,
- and satisfies $S \cong \mathsf{hom}_{{}_R\mathsf{Mod}}(P, P)$.

Dropping additivity in the problem above one ends up with a monoid M instead of a ring R and its associated category ${}_M\mathsf{Set}$ of M-sets, i.e., of sets X equipped with an M-action $M \times X \to X$. One then might—in complete analogy to (P1)—ask the following question:

P2 Given a monoid M, can one determine all monoids N such that the categories ${}_M\mathsf{Set}$ and ${}_N\mathsf{Set}$ are equivalent?

This question was answered independently by Banaschewski [2] and Knauer [11] in 1972 as follows:

Theorem 2 *The following are equivalent for monoids M and N:*

(i) The variety ${}_N\mathsf{Set}$ is equivalent to ${}_M\mathsf{Set}$.

(ii) N is isomorphic to $u \cdot M \cdot u$ for some idempotent element $u \in M$ for which there exist $d, p \in M$ with $p \cdot d = e_M$ and $u \cdot d = d$.

The following problem seems to be of a different nature:

P3 Determine all varieties $\mathcal{W}$ equivalent to the category Bool of Boolean algebras.

In fact, this problem originally was stated in a different—though equivalent—way. Starting from Stone's famous duality between Boolean algebras and Boolean spaces Hu asked whether there might be dualities between Boolean spaces and varieties different from Bool. Hu characterized these varieties (without explixcitely describing them) around 1970 as follows:

Theorem 3 *The following are equivalent for a variety $\mathcal{V}$:*

(i) $\mathcal{V}$ is equivalent to Bool*, the variety of Boolean algebras.*

(ii) $\mathcal{V}$ is generated by a primal algebra.

While at a first glance the problems and theorems above don't seem to have much in common—except for the analogy between (P1) and (P2) mentioned already and the appearance of the concept of equivalence—the picture changes in view of the following folklore proposition:

Proposition 1 *Every abelian variety $\mathcal{W}$ is of the form ${}_S\mathsf{Mod}$ for some ring S.*

We will refrain from explaining completely the notion of an *abelian* category here and also from giving a proof (which can be found, e.g., in [3] or [7]). For our purposes it is enough to know that abelianess is a categorical property—thus stable under equivalence—and is shared by any category of the from ${}_R\mathsf{Mod}$. It might be added moreover, for a better understanding of this result, that the ring S appearing therein is the endomorphism ring of the free algebra $F1$ on one generator in the variety $\mathcal{W}$. (The endomorphisms of $F1$ form a ring and not just a monoid since the abelianess condition forces in particular the hom-sets of the category to carry the structure of an Abelian group, where composition distributes over addition.)

As a consequence of this proposition problem (P1) is equivalent to the following one which now is of the same type as (P3):

P1' Determine, for a given ring R, all varieties $\mathcal{W}$ equivalent to the category ${}_R\mathsf{Mod}$ of left R-modules.

It is now not far fetched to ask the following question generalizing (P1') and (P3), which first was done (in a somewhat different but equivalent way) by Isbell around 1970 and left open until recently:

P Determine, for a given variety $\mathcal{V}$, all varieties $\mathcal{W}$ equivalent to $\mathcal{V}$.

A solution to this problem will be given at the end of Section 3.

2 Varieties

Before defining algebras and varieties in the way most suitable for our purposes we will (though somewhat sketchy only) recall Birkhoff's definitions from the 1930s, since these possibly provide a more intuitive approach to the subject.

Definitions 1 1. A *signature* is a pair $(\Omega, \mathcal{E})$ where $\Omega = (\Omega_n)_{n\in\mathbb{N}}$ is a family of sets (of operational symbols) and $\mathcal{E}$ is a set of Ω-equations, i.e., a set of pairs of Ω-terms[3].

[3] It is beyond the scope of this paper to precisely explain the notion of *term*; somewhat imprecisely a term is an expression meaningfully built out of variables $x, y, \ldots$ and operational symbols. E.g., for the signature defining groups with m as multiplication symbol, $m(m(x, y), z)$ and $m(m(x, y), m(x, z))$ are terms but xy, m, and $m(y, m(x))$ are not.

2. An Ω-*algebra* is a pair $(A, (\omega^A)_\omega)$ where A is a set and, for $\omega \in \Omega_n$, ω^A is a map $A^n \to A$, the A-*interpretation of* ω. An Ω-algebra satisfying all equations[4] from $\mathcal{E}$ is called an $(\Omega, \mathcal{E})$-algebra.

3. An Ω-*homomorphism* $f\colon (A, (\omega^A)_\omega) \to (B, (\omega^B)_\omega)$ is a map $f\colon A \to B$ such that, for each $n \in \mathbb{N}$ and each $\omega \in \Omega_n$, the following diagram commutes:

$$\begin{array}{ccc} A^n & \xrightarrow{f^n} & B^n \\ {\scriptstyle \omega^A}\downarrow & & \downarrow{\scriptstyle \omega^B} \\ A & \xrightarrow{f} & B \end{array}$$

4. For a given signature $(\Omega, \mathcal{E})$ the category formed by all $(\Omega, \mathcal{E})$-algebras and Ω-homomorphisms is denoted by $\mathsf{Alg}(\Omega, \mathcal{E})$. A category of the form $\mathsf{Alg}(\Omega, \mathcal{E})$ is called a *variety.*

Clearly, ${}_R\mathsf{Mod}$ for any ring R, ${}_M\mathsf{Set}$ for any monoid M, and Bool are varieties. Note also that every variety has its canonical *underlying functor* $|-|\colon \mathsf{Alg}(\Omega, \mathcal{E}) \to \mathsf{Set}$, sending the algebra $(A, (\omega^A)_\omega)$ to its *underlying set* A, and thus is a concrete category over Set.

This concept of variety—as natural and intuitive it might appear—has its shortcomings. Firstly, to make everything explicit we put to the footnotes, is a somewhat cumbersome (though not difficult) task; secondly, different signatures $(\Omega, \mathcal{E})$ and $(\Omega', \mathcal{E}')$ might define the "same" variety: think e.g. of Boolean algebras and Boolean rings or of the left handed and right handed definition of groups. (Strictly speaking, the varieties of "left groups" and "right groups" are the same in the sense of being identical, while Boolean algebras and Boolean rings are only concretely isomorphic.)

A—maybe—less intuitive but conceptually more satisfactory way of defining varieties is due to Lawvere [12]. In this approach signatures and algebras are no longer just pairs of some unstructured mathmatical entities but well established mathematical structures: categories and functors respectively. The relation between this and Birkhoff's approach will be clarified by means of a simple example (for formal proofs see e.g. [3] or [12]) after the definitions have been provided.

Definitions 2 1. A *(Lawvere) theory* is a category $\mathbb{T}$ with countably many objects $T_0, T_1, \ldots T_n, \ldots$ such that, for each $n \in \mathbb{N}$, $T_n = (T_1)^n$ by means of prescribed projections $\pi_i^n\colon T_n \to \mathbb{T}_1,\ i = 1, \ldots n$.

2. A $\mathbb{T}$-*model* (or $\mathbb{T}$-*algebra*) is a functor $H\colon \mathbb{T} \to \mathsf{Set}$ which preserves the prescribed (thus all) finite products.

3. For a given Lawvere theory $\mathbb{T}$, the $\mathbb{T}$-models and natural transformations between them form the category $\mathsf{Mod}\mathbb{T}$.

Lawvere theories are abundant; here are two (types of) examples:

[4]We also refrain from stating formally what this means. In any case the formal definition captures the intuition; e.g, an algebra with a binary operation m satisfies the equation $(m(x, m(y, z)), m(m(x, y), z))$ iff m^A is associative.

I. Let G be an object in a category $\mathcal{K}$ admitting all its finite copowers. Take, for each $n \in \mathbb{N}$, an n-fold copower nG of G; these then span a full subcategory of $\mathcal{K}$ the dual of which is a Lawvere theory, denoted by $\mathsf{Th}_{\mathcal{K}}(G)$ and called the *Lawvere theory generated by* G.

II. Let $(A, (\omega^A)_\omega)$ be any algebra in a variety $\mathcal{V} = \mathsf{Alg}(\Omega, \mathcal{E})$. Each Ω-term t in variables $x_1, \ldots, x_n$ induces a *term operation* $t^A \colon A^n \to A$ in a (naively) obvious way ($t^A(a_1, \ldots, a_n)$ is what you get when substituting, in the term t, each occurence of an operational symbol ω by its A-interpretation ω^A and each occurence of the variable x_i by the element $a_i \in A$. Note that the term operation induced by x_i is just the i-th projection). Given terms $t_1, \ldots t_m$ in variables $x_1, \ldots, x_n$ denote by $\langle t_i^A \rangle$ the induced map $A^n \to A^m$; note that, for any term operation $s^A \colon A^m \to A$, the composition $A^n \xrightarrow{\langle t_i^A \rangle} A^m \xrightarrow{s^A} A$ is again a term operation. This shows that the finite powers of the set A, together with the term operations t^A and the maps $\langle t_i^A \rangle$ induced by those, form a Lawvere theory $\mathbb{A}$, called the *Lawvere theory determined by the algebra* $(A, (\omega^A)_\omega)$. (N.b. Universal algebra has for long—and rather unsuccessfully—tried to describe the structure of the set of term operations on an algebra; by the use of categorical language this is made possible.)

Note that the category $\mathsf{Mod}\mathbb{T}$ has a canonical underlying functor $U_\mathbb{T} \colon \mathsf{Mod}\mathbb{T} \to \mathsf{Set}$ which is simply evaluation at T_1:

$$U_\mathbb{T}(H \xrightarrow{\lambda} K) = H(T_1) \xrightarrow{\lambda_{T_1}} K(T_1)$$

In order to see how Birkhoff's and Lawvere's notion of algebra are related let us consider the variety Ab of Abealian groups. Any Abelian group $(A, +, -, 0)$ admits—as mentioned in the second example above—a lot of term operations, i.e., of maps $A^n \to A$ obtainable from $+, -$, and 0 by meaningfully combining them.

$$A^3 \to A \quad \text{with} \quad (x, y, z) \mapsto x - y + z \quad \text{and} \qquad A^4 \to A \quad \text{with} \quad (x, y, z, u) \mapsto 2x - 3y + 4u$$

might serve as illustrating examples. Each term operation $t^A \colon A^n \to A$ then is determined by an n-tuple $\tilde{t} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ via $t^A(x_1, \ldots, x_n) = k_1 x_1 + \ldots + k_n x_n$. Now $\tilde{t}$ corresponds bijectively to a homomorphism $\bar{t} \colon \mathbb{Z} \to \mathbb{Z}^n$. These correspondences can be formalised as follows:

Step 1 Form the full subcategory of Ab spanned by all finite copowers (sums) of $\mathbb{Z}$, i.e. of $0, \mathbb{Z}, \mathbb{Z}^2, \ldots, \mathbb{Z}^n, \ldots$ and call its dual, $\mathbb{T}_{\mathsf{Ab}}$, the *(Lawvere) theory of* Ab, i.e., $\mathbb{T}_{\mathsf{Ab}} = \mathsf{Th}_{\mathsf{Ab}}(\mathbb{Z})$.

Step 2 Now consider, for the Abelian group $(A, +, -, 0)$, the following assignment from the category $\mathbb{T}_{\mathsf{Ab}}$ to Set:

$$\begin{array}{rcl} \mathbb{Z}^n & \mapsto & A^n \\ \mathbb{Z}^n \xrightarrow{\bar{t}} \mathbb{Z} & \mapsto & A^n \xrightarrow{t^A} A \end{array}$$

It is checked easily that this assignment defines a $\mathbb{T}_{\mathsf{Ab}}$-model $\mathsf{A} \colon \mathbb{T}_{\mathsf{Ab}} \to \mathsf{Set}$ whose image is nothing but the Lawvere theory $\mathbb{A}$. Thus, for each Abelian group $(A, +, -, 0)$ there is an associated $\mathbb{T}_{\mathsf{Ab}}$-model A such that $\mathbb{A}$ is A's image. This assignment becomes functorial by sending a homorphism f to the family of maps $(f^n)_{n\in\mathbb{N}}$.

A certain subtlety arises when reversing this construction, due to the fact that in the definition of group (or of algebra in general) one is using particular (cartesian) powers of the underlying set as domains of operations while in Lawvere's approach these might be arbitrary ones since, categorically, products are determined up to isomorphism only. As a result of this $\mathsf{Mod}\mathbb{T}_{\mathsf{Ab}}$ and Ab are not isomorphic as categories but equivalent only; this equivalence however is concrete. The following theorem due to Lawvere [12] contains a complete description of the relation between theories and varieties.

Theorem 4 *1. For each theory $\mathbb{T}$ there is a variety $\mathcal{V}_{\mathbb{T}}$ such that $\mathcal{V}_{\mathbb{T}} \cong \mathsf{Mod}\mathbb{T}$.*

2. For each variety $\mathcal{V}$ there is a theory $\mathbb{T}_{\mathcal{V}}$ such that $\mathcal{V} \cong \mathsf{Mod}\mathbb{T}_{\mathcal{V}}$.

3. The correspondence between varieties and theories is essentially bijective[5] and (contravariantly) functorial.

Remark 2 For any variety $\mathcal{V}$ the theory $\mathbb{T}_{\mathcal{V}}$ can always be chosen as in the example of Abelian groups, i.e., as $\mathsf{Th}_{\mathcal{V}}(F1)$ where $F1$ is the free algebra in $\mathcal{V}$ on one generator.

In particular, the theory of the variety ${}_R\mathsf{Mod}$ for some ring R not only is determined by R but conversely determines R, since one has $\mathsf{hom}_{\mathbb{T}_{{}_R\mathsf{Mod}}}(T_1, T_1) = \mathsf{hom}_{\mathsf{Th}_{{}_R\mathsf{Mod}}(F1)}(F1, F1) = \mathsf{hom}_{\mathsf{Th}_{{}_R\mathsf{Mod}}(R)}(R, R) = \mathsf{End}_{{}_R\mathsf{Mod}}(R)^{op}$ by the definitions above and thus an isomorphism $\mathsf{hom}_{\mathbb{T}_{{}_R\mathsf{Mod}}}(T_1, T_1) \cong R$ by means of the isomorphism $\mathsf{End}_{{}_R\mathsf{Mod}}(R)^{op} \cong R$ provided by assigning to a ring element the corresponding right translation. Similarly, $\mathsf{hom}_{\mathbb{T}_{{}_R\mathsf{Mod}}}(T_1, T_1) = \mathsf{End}_{{}_R\mathsf{Mod}}(R^n)^{op} \cong \mathsf{Mat}(n, R)$.

Analogously one has for varieties of M-sets $\mathsf{hom}_{\mathbb{T}_{{}_M\mathsf{Set}}}(T_1, T_1) = \mathsf{End}_{{}_M\mathsf{Set}}(M)^{op} \cong M$.

We finish this section by explaining the technical notions appearing in Hu's theorem as promised in footnote 1.

Definitions 3 Let A be an algebra in the variety $\mathcal{V} = \mathsf{Alg}(\Omega, \mathcal{E})$. A is said to

1. *generate* $\mathcal{V}$ iff the set $\mathcal{E}^A$ of Ω-equations holding in A equals $\mathcal{E}$.
2. be *primal* if A is finite with at least two elements and every map $A^n \to A$ is a term operation.

The paradigmatic example of a primal algebra generating a variety is the 2-chain in the variety Bool.

These notions—relatively involved in the language of classical universal algebra—become extremely simple and natural in the categorical setup (thus simpler in use) as the following is easily seen:

Proposition 2 *Let $\mathcal{V}$ be a variety and $\mathbb{T}_{\mathcal{V}}$ its theory. For an algebra A in $\mathcal{V}$ with associated model $\mathsf{A}\colon \mathbb{T}_{\mathcal{V}} \to \mathsf{Set}$ there are the following equivalences:*

1. A generates $\mathcal{V}$ iff A is faithful.

2. If A is finite with at least two elements then A is primal iff A is full.

[5] By this "bijective up to isomorphism of theories and concrete isomorphism of varieties repectively" is meant.

3 Characterizing varieties

Besides the fact that categorical language is extremely suitable in the context of varieties as indicated by Proposition 2 the categorical approach also allows for for an axiomatics of varieties, i.e, for characterizing varieties as categories with certain properties. We present two intimately related such characterizations: the first one axiomatizes varieties as abstract categories while the second one is an axiomatization of varieties considered as concrete categories over Set. The results presented in this section are essentially due to Lawvere, Isbell, Linton and Felscher and will serve as a basis for our main theorem.

As before we will refrain from giving complete proofs (which can be found e.g. in [3] or [12]). We will however try to convey the main ideas behind these results. The following notion turns out to be crucial (compare with the conditions imposed on the bimodule in the remark following Morita's theorem in Section 1):

Definition 4 An object G in a cocomplete category $\mathcal{K}$ is called *varietal generator* provided G is

a. a regular generator[6],

b. regularly projective[7],

c. finitely presentable[8].

The paradigmatic example of a varietal generator is a finitely generated free algebra in a variety. Varietal generators are preserved and reflected by equivalences (see [15]).

Theorem 5 *For any category $\mathcal{K}$ the following are equivalent:*

(i) $\mathcal{K}$ is equivalent to a variety.

(ii) a. $\mathcal{K}$ is cocomplete and exact[9].

b. $\mathcal{K}$ has a varietal generator.

Theorem 6 *For any concrete category $(\mathcal{K}, U)$ over* Set *the following are equivalent:*

(i) $(\mathcal{K}, U)$ is concretely equivalent to a variety.

(ii) a. $\mathcal{K}$ is cocomplete and exact,

b. U has a left adjoint,

c. U preserves and reflects regular epimorphisms.

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[6]In varieties this simply means that each object is a homomorphic image of some copower of G.

[7]In varieties this is: projective w.r.t. surjective homorphisms, often just called *projective* there.

[8]This categorical notion captures the idea of an algebra in a variety, which is presentable by finitely many generators and finitely many equations.

[9]Exactness is a categorical property—thus stable under equivalence—which models the way quotients are formed in varieties; every variety thus is exact.

In Theorem 6 the exactness condition can be replaced by the condition that U preserves and reflects kernel pairs.

To understand the relationship of these theorems one only needs to pass from an object G in a category $\mathcal{K}$ to its associated hom-functor $\mathsf{hom}(G,-)\colon \mathcal{K} \to \mathsf{Set}$. Then the defining properties of a varietal generator G (plus the given assumptions on $\mathcal{K}$ in the theorems) imply the properties of U stated in Theorem 6 and conversely[10].

It remains to indicate how the existence of a varietal generator G in a cocomplete and exact category $\mathcal{K}$ forces $\mathcal{K}$ to be concretely equivalent to a variety. In view of section 2 it is enough to provide a suitable Lawvere theory. One nearly might guess how to do this: one simply takes the theory generated by the varietal generator at hand! What then is at the heart of Theorems 5 and 6 is the following result (For the categorically experienced reader: the equivalence stated there is given by the Yoneda functor).

Proposition 3 *For $U \cong \mathsf{hom}_{\mathcal{K}}(G,-)\colon \mathcal{K} \to \mathsf{Set}$ with $\mathcal{K}$ cocomplete and exact and G a $\mathcal{K}$-object the following are equivalent:*

(i) $(\mathcal{K}, U)$ is concretely equivalent to the variety whose theory is $\mathsf{Th}_{\mathcal{K}}(G)$.

(ii) G is a varietal generator.

The following is now an immediate consequence and solves (in view of Theorem 4) Isbell's problem (P).

Theorem 7 *For varieties $\mathcal{V}$ and $\mathcal{W}$ the following are equivalent:*

(i) $\mathcal{W}$ is equivalent to $\mathcal{V}$ (as a category).

(ii) $\mathbb{T}_{\mathcal{W}} \cong \mathsf{Th}_{\mathcal{V}}(G)$ for some varietal generator G in $\mathcal{V}$.

Thus, in order to determine all varieties $\mathcal{W}$ equivalent to a given one, $\mathcal{V}$, one only has to determine the varietal generators of $\mathcal{V}$. As is easily seen these are precisely those regular generators which are retracts of finitely generated free algebras Fn; thus these are e.g.

in Set: the finite non-empty sets n;

in Ab: the finite non-zero (co)powers of $\mathbb{Z}$;

in ${}_R\mathsf{Mod}$: the progenerators;

in Bool: the powerset algebras $\mathcal{P}(n)$ of finite sets of cardinality > 1.

4 Applications

4.1 Varietal generators and idempotents

Before returning to the examples provided in the first section we will discuss a tool suitable to describe the theory generated by a varietal generator somewhat more explicitely (see [15]). This also provides the link to the results of [13] and explains the appearance of idempotents in Theorems 1 and 2.

[10] Note that every functor into Set with a left adjoint is naturally isomorphic to some hom-functor.

As mentioned towards the end of the previous section, any varietal generator G in a variety $\mathcal{V}$ is a retract of some finitely generated free algebra Fn. If now $G \xrightarrow{s} Fn \xrightarrow{r} G = 1_G$ is this retraction the endomorphism $u = s \circ r$ of Fn is idempotent (conversely, every idempotent endomorphism of Fn determines a retract of Fn). A retract G of Fn will be a varietal generator, provided $F1$ is a quotient (thus a retract by projectivity of $F1$) of some (finite) copower mG. With a little bit of thought one sees that these two retraction conditions in fact characterize the varietal generators and also can be linked together in suitable way. One then ends up with the following characterization of varietal generators which will prove to be useful.

Proposition 4 *The following are equivalent for an object G in $\mathcal{V}$:*

(i) G is a varietal generator;

(ii) There are retractions $G \xrightarrow{s} Fn \xrightarrow{r} G = id_G$ and $F1 \xrightarrow{p'} m \cdot G \xrightarrow{d'} F1 = id_{F1}$ for some natural numbers n and m;

(iii) There is an idempotent endomorphisms u of some Fn—splitting over G—together with a retraction $F1 \xrightarrow{p} F(nm) \xrightarrow{d} F1 = 1_{F1}$, for some m, such that

a. $d \circ (m \cdot u) = d$ *or*

b. $d \circ (m \cdot u) \circ p = 1$

The idempotent u encountered above can in fact be used to describe the theory $\mathsf{Th}_{\mathcal{V}}(G)$ in terms of $\mathbb{T}_{\mathcal{V}}$ (see [15]); in view of the applications to be discussed here we will consider only the following aspect here:

From the definition of $\mathbb{T}_{\mathcal{V}}$ as $\mathsf{Th}_{\mathcal{V}}(F1)$ one knows in particular that the endomorphism sets in $\mathbb{T}_{\mathcal{V}}$ are the duals of the endomorphism monoids of the algebras Fn in $\mathcal{V}$: $\mathsf{hom}_{\mathbb{T}_{\mathcal{V}}}(T_n, T_n) = \mathsf{End}_{\mathcal{V}}(Fn)^{op}$. Analogously, $\mathsf{hom}_{\mathsf{Th}_{\mathcal{V}}(G)}(T_n, T_n) = \mathsf{End}_{\mathcal{V}}(nG)^{op}$. The endomorphism monoids of Fn and G are now related by the idempotent endomorphism $u = s \circ r$ of Fn as follows:

The assignment $f \mapsto sfr$ provides an injective map $\mathsf{End}_{\mathcal{V}}(G) \to \mathsf{End}_{\mathcal{V}}(Fn)$ whose image is $u\mathsf{End}_{\mathcal{V}}(Fn)u$. By means of the multiplication of $\mathsf{End}_{\mathcal{V}}(Fn)$ the latter becomes a monoid with unit u. The bijection $\mathsf{End}_{\mathcal{V}}(G) \cong u\mathsf{End}_{\mathcal{V}}(Fn)u$ then is an isomorphism of monoids.

4.2 Classical Morita theory

Let now R be a ring and $\mathcal{V} = {}_R\mathsf{Mod}$. Then a ring S is Morita equivalent to R, i.e., the categories ${}_R\mathsf{Mod}$ and ${}_S\mathsf{Mod}$ are equivalent,

iff (by Proposition 1)

${}_S\mathsf{Mod}$ is a variety equivalent to ${}_R\mathsf{Mod}$,

iff (by Theorems 4 and 7)

$\mathbb{T}_{{}_S\mathsf{Mod}} \cong \mathsf{Th}_{{}_R\mathsf{Mod}}(G)$ for some varietal generator G in ${}_R\mathsf{Mod}$,

iff (by Remark 2)

$\mathsf{End}_{{}_S\mathsf{Mod}}(S) \cong \mathsf{End}_{{}_R\mathsf{Mod}}(G)$ for some varietal generator G in ${}_R\mathsf{Mod}$,

iff (by Proposition 4)

$\mathsf{End}_{{}_S\mathsf{Mod}}(S) \cong u\mathsf{End}_{{}_R\mathsf{Mod}}(R^n)u$ for some idempotent $u \in \mathsf{End}_{{}_R\mathsf{Mod}}(R^n)$ such that there are m, p, d with $d \circ (m \cdot u) \circ p = 1$.

Now the condition on u that there are m, p, d as required means—as a little calculation shows—that the ideal generated by u is the ring $\mathsf{End}_{_R\mathsf{Mod}}(R^n)$ itself. Employing finally the isomorphisms $S \cong \mathsf{End}_{_S\mathsf{Mod}}(S)^{op}$ and $\mathsf{Mat}(n, R) \cong \mathsf{End}_{_R\mathsf{Mod}}(R^n)^{op}$ we have got a proof of Theorem 1.

4.3 Morita theory for M-sets

Comparing Theorems 1 and 2 one observes an interesting difference. While the module case allows for an additional parameter $n \in \mathbb{N}$ besides u (and p, d), this parameter is missing in the case of M-sets—or rather: here n is fixed to be 1. This is easy to understand from the theory developed above. As opposed to the case of modules, a variety equivalent to a variety of M-sets need not be a variety of the form $_N\mathsf{Set}$ for some monoid N. As is not too difficult to show this will be the case if and only if n and m in Proposition 4 can be chosen as $n = m = 1$:

Varieties of the form $_N\mathsf{Set}$, for some monoid N, can be characterized by the fact that their underlying functors preserve coproducts. Thus, a varietal generator G in $_M\mathsf{Set}$ will, by Proposition 3, determine (via $\mathsf{Th}_{_M\mathsf{Set}}(G)$) a variety of the form $_N\mathsf{Set}$ (necessarily with $N \cong \mathsf{End}_{_M\mathsf{Set}}(G)$—see Remark 2) iff the functor $\mathsf{hom}_{_M\mathsf{Set}}(G, -)$ preserves coproducts.

Since $\mathsf{hom}_{_M\mathsf{Set}}(F1, -)$ preserves coproducts (this is up to equivalence the underlying functor of $_M\mathsf{Set}$) the same holds for $\mathsf{hom}_{_M\mathsf{Set}}(G, -)$ where G is a retract of $F1$. Thus suppose that G is a retract of $F2$ (for the sake of simplicity), but not of $F1$. The decomposition $F2 = F1 + F1$ then induces a decomposition $G = G_1 + G_2$ into non-empty M-sets G_1, G_2. If $\mathsf{hom}_{_M\mathsf{Set}}(G, -)$ preserves coproducts, this decomposition implies

$$\mathsf{hom}_{_M\mathsf{Set}}(G, G) \cong \mathsf{hom}_{_M\mathsf{Set}}(G, G_1 + G_2) \cong \mathsf{hom}_{_M\mathsf{Set}}(G, G_1) + \mathsf{hom}_{_M\mathsf{Set}}(G, G_2)$$

which clearly is a contradiction since the identity on G cannot have only G_1 or G_2 as its codomain if both are non-empty. This implies $n = 1$; the same argument applied to m shows $m = 1$.

With this in mind Theorem 2 follows the same way as Morita's theorem above.

4.4 Hu's theorem

By Theorem 7 and the last example following it a variety $\mathcal{W}$ is equivalent to Bool iff $\mathbb{T}_{\mathcal{W}} = \mathsf{Th}_{\mathsf{Bool}}(\mathcal{P}(n))$ for some $n > 1$. Due to the representation theorem for finite Boolean algebras—which in fact rather should be called the duality between finite sets and finite Boolean algebras—the contravariant powerset functor sets up an isomorphism between $\mathsf{Th}_{\mathsf{Bool}}(\mathcal{P}(n))$ and the full subcategory $\mathbb{T}_n$ of Set spanned by the finite powers $n^k,\ k \in \mathbb{N}$. Thus, $\mathbb{T}_{\mathcal{W}} \cong \mathbb{T}_n \hookrightarrow \mathsf{Set}$ is a finite full and faithful model of $\mathbb{T}_{\mathcal{W}}$. In other words (see Proposition 2): $\mathcal{W}$ is generated by a primal algebra. Conversely, if $\mathcal{W}$ is generated by a primal algebra, i.e., if its theory $\mathbb{T}_{\mathcal{W}}$ has a finite full and faithful model A, its associated model A provides—being faithful—an isomorphism from $\mathbb{T}_{\mathcal{W}}$ to $\mathbb{A}$ which—since A is full—is some $\mathbb{T}_n$; thus $\mathbb{T}_{\mathcal{W}} \cong \mathsf{Th}_{\mathsf{Bool}}(\mathcal{P}(n))$ for some $n > 1$ and $\mathcal{W}$ is equivalent to Bool, again by Theorem 7. Since it is known that the n-chain generates the variety Post_n of Post algebras of order n (see e.g. [1]; for an alternative argument using Proposition 4 see [15]) we have proven the following extension of Hu's theorem (see [16]):

Theorem 8 *The following are equivalent for a variety $\mathcal{W}$:*

(i) *$\mathcal{W}$ is equivalent to* Bool*;*

(ii) *$\mathcal{W}$ is generated by a primal algebra;*

(iii) *$\mathcal{W} \cong \mathsf{Post}_n$ for some natural number $n > 1$.*

5 Some related results

Instead of invoking Lawvere theories one can use the theory of monads instead in order to attack Isbell's problem (see [5]). For the non-categorist this is somewhat harder to understand because it necessarily has to be more technical from the very beginning; it has the advantage, however, to allow for a discussion of generalized varieties defined by operations without any bound to their arities.

For the same reason generalized Lawvere theories (Linton theories) are used in [6]. This paper has been the first to successfully deal with the problem discussed here. It is closer to our presentation in content than in style.

A different approach (see [4]) focusses on the functors providing the equivalences between varieties. Here one starts generalizing the observations of Remark 1. More advanced categorical methods are needed to find a suitable categorical notion of bimodel (generalizing the notion of bimodule).

Finally there is an algebraic solution to Isbell's problem as well (see [13]) which is closely related to the one presented here, but lacking the conceptual lucidity inherent to the categorical approach. A comprehensive account of this paper is contained in [15].

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