Takeuchi's Free Hopf Algebra Construction Revisited

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Abstract

Takeuchi's famous free Hopf algebra construction is analyzed from a categorical point of view, and so is the construction of the Hopf envelope of a bialgebra. Both constructions this way appear as compositions of well known and natural constructions. This way certain partially wrong perceptions of these constructions are clarified and their mutual relation is made precise. The construction of Hopf envelopes finally is shown to provide a construction of a Hopf coreflection of bialgebras by simple dualization. The results provided hold for any commutative von Neumann regular ring, not only for fields.

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Introduction

In his seminal monograph on Hopf algebras [15] Sweedler already made the claims that (a) for any algebra A there exists a cofree Hopf algebra over A and (b) for any coalgebra C there exists a free Hopf algebra over C. He did not give any proofs and it took some years until Tackeuchi [16] proved claim (b). Proofs of (a) only were provided recently ([13], [2], [4]). Later the construction of a Hopf reflection of given bialgebra B, also called *Hopf envelope* of B, grew out of Takeuchi's construction. This is sometimes (see e.g. [14]) attributed to Manin [8]; the only careful description this author is aware of is [10].

The reasons to revisit Takeuchi's construction are twofold: firstly, his construction and its relation to that of a Hopf envelope seem not to be too well

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understood, as shown by the the quite recent survey [14] (see a more detailed comment in Section 3.2 below) and also by the quite general perception that Takeuchi's construction is a generalization of the free group construction (see the final remark in 3.2 correcting this). Secondly, the direct proof (that is, without using the existence of free Hopf algebras) of the fact that the category $\mathbf{Hopf}_R$ of Hopf algebras is closed under products in the category $\mathbf{Bialg}_R$ of bialgebras for any von Neumann regular ring R (see [13]) allows for a description of these constructions as compositions of extremely natural standard categorical constructions: The free Hopf algebra construction is the composition of the free bialgebra construction (a standard free monoid construction) with that of the Hopf envelope (= Hopf reflection) and the latter is the composition of special instances of the free algebra construction for functor algebras and that of so-called E-reflective subcategories. This categorical approach moreover allows for a construction of the coreflection of $\mathbf{Bialg}_R$ into $\mathbf{Hopf}_R$ (which exists for every ring R by [13]) for von Neumann regular rings by simple dualization, thus making the calculations of [4] superfluous.

The assumption on R to be von Neumann regular, used throughout in this note, might be too restrictive (see also [13]). What only is needed (in order to be able to lift image factorizations from the category $\mathbf{Mod}_R$ of R-modules to $\mathbf{Bialg}_R$) is the slightly (?) weaker property that for every injective R-linear map f its tensor square $f \otimes f$ is injective again.

1 Categorical prerequisites

For the sake of the reader who is not completely familiar with the categorical constructions we are going to use we will sketch these here briefly.

1.1 Free monoids

It is well known that the construction of the free monoid X^* over a set X as the word monoid over X only depends on the facts that the category **Set** of sets and maps has (finite and countable) coproducts (disjoint unions) and, for every set X, the functor "multiply by X" on **Set**, $X \times -$, preserves these coproducts (see [7]). Thus, this construction can be generalized to provide free monoids over any monoidal category $(\mathbf{C}, - \otimes -, I)$, provided that each functor "tensor by C" on $\mathbf{C}$, $C \otimes -$, preserves (finite and countable) coproducts. Applied to the monoidal category **Mod** of modules over any commutative ring (with its standard tensor product) this shows that the tensor algebra TM over a module M is the free R-algebra over M. $\mathbf{Coalg}_R$, the category of R-coalgebras, is monoidal (by the tensor product inherited from $\mathbf{Mod}_R$). Since coproducts in $\mathbf{Coalg}_R$ are created by the underlying functor $\mathbf{Coalg}_R \xrightarrow{V_c} \mathbf{Mod}_R$, the same argument describes the construction of the free bialgebra (= monoid in $\mathbf{Coalg}_R$) $T^\star C$ over a coalgebra C. $T^\star C$ is, algebraically, the tensor algebra TV_cC over the underlying module V_cC of C endowed with the unique coalgebra structure (Δ, ϵ) making the embedding of V_cC into TV_cC (the unit of the adjunction for T) a coalgebra morphism, which, thus, becomes the unit for the adjunction $T^* \dashv V_c$ (see e.g. [13]).

1.2 Free functor algebras

Throughout this section $\mathbf{C}$ denotes a category and F an endofunctor of $\mathbf{C}$. For details concerning this section we refer to [1].

1 Definition The category $\mathbf{Alg}F$ has as objects, called *F–algebras*, all pairs (C, α_C) where C is an object of $\mathbf{C}$ and $\alpha_C : FC \to C$ is a morphism. Morphisms $f : (C, \alpha_C) \to (D, \alpha_D)$ of $\mathbf{Alg}F$, called *F–algebra homomorphisms*, are morphisms $f : C \to D$ in $\mathbf{C}$ such that the square

$$\begin{tikzcd} FC \arrow[r, "\alpha_C"] \arrow[d, "Ff"'] & C \arrow[d, "f"] \\ FD \arrow[r, "\alpha_D"'] & D \end{tikzcd}$$

commutes. Composition and identities in $\mathbf{Alg}F$ are those of $\mathbf{C}$.

The paradigmatic example here is that of a set functor H_Ω induced by a signature Ω: Finitary universal algebras of a given (one–sorted) signature $\Omega = (\Omega_n)_{n\in\mathbb{N}}$, where Ω_n is the set of all n–ary operation symbols, can be viewed as F–algebras for the following endofunctor $F = F_\Omega$ of $\mathbf{Set}$: F_Ω assigns to a set X the set $\sum_{n\in\mathbb{N}} \Omega_n \times X^n$. Correspondingly F_Ω assigns to a map $f : X \to Y$ the map $\sum_{n\in\mathbb{N}} \Omega_n \times f^n$, i.e., the map $\sum_{n\in\mathbb{N}} \Omega_n \times X^n \to \sum_{n\in\mathbb{N}} \Omega_n \times Y^n$ mapping a pair $(\omega, (x_1, \ldots, x_n))$ to the pair $(\omega, (fx_1, \ldots, fx_n))$.

There is well known construction of free functor algebras as follows: Let $\mathbf{C}$ have countable colimits. Given an object X in $\mathbf{C}$ define an ω–chain $X_i^\sharp$ $(i < \omega)$ as follows:

$$0 \xrightarrow{!} F0 + X \xrightarrow{F! + X} F(F0 + X) + X \xrightarrow{F(F!+X)+X} F(F(F0 + X) + X) + X \cdots$$

If now F preserves colimits of ω–chains, then for every object X of $\mathbf{C}$,

$$X^\sharp = \operatorname*{colim}_{i<\omega} X_i^\sharp$$

is a free F–algebra on X.

Note that this construction not only is very general, being applicable to nearly all categories, but also extremely natural: specialized to the set functor $F_\Omega\colon \mathbf{Set} \to \mathbf{Set}$ induced by a signature Ω it describes the construction of term algebras, while specialization to a monotone map f on a complete lattice with $X = 0$ it is nothing but the Tarski-Knaster construction of the least fixpoint of f. In fact the above construction can be considered to be the "categorization" of that construction.

If F also preserves coproducts the above construction can be simplified as follows: the chain is simply

$$0 \xrightarrow{!} X \xrightarrow{F!+X} F(X) + X \xrightarrow{F(F!+X)+X} F(F(X)) + F(X) + X \cdots$$

with colimit $X^\sharp = \coprod_{n\in\mathbb{N}} F^n(X)$. In fact one has (even without F preserving colimits of chains — see [3, Thm. 2.1])

2 Fact *Let $\mathbf{C}$ have finite and countable coproducts and $F\colon \mathbf{C} \to \mathbf{C}$ preserve these. Then the free F-algebra $(X^\sharp, \alpha_X)$ over an object X in $\mathbf{C}$ is given by $X^\sharp = \coprod_{n\in\mathbb{N}} F^n(X)$ with action $\alpha_X\colon F(X^\sharp) \to X^\sharp$ determined by commutativity of the following diagram (for all $n > 0$), where we write $F^0 = \mathsf{id}_{\mathbf{C}}$, $F^{n+1} = F \circ F^n$ and where ι_n is the n^{th} coproduct injection.*

$$\begin{tikzcd}
F^n X \arrow[r, "\mathsf{id}_{F^n X}"] \arrow[d, "F\iota_{n-1}"'] & F^n X \arrow[d, "\iota_n"] \\
F X^\sharp \arrow[r, "\alpha_X"] & X^\sharp
\end{tikzcd}$$

The unit of the adjunction is ι_0, the 0^{th} coproduct injection.

Proof: Since F preserves coproducts the left column of the diagram is a coproduct, and this implies existence of α_X.

If now $f\colon X \to H$ is a $\mathbf{C}$-morphism where (H, α_H) is an F-algebra, define a family $(f_n)_{n\in\mathbb{N}}$ of $\mathbf{C}$-morphisms by

$$\begin{aligned}
f_0 \quad &:= F^0 X = X \xrightarrow{f} H \\
f_{n+1} &:= F^{n+1} X = F(F^n X) \xrightarrow{F f_n} FH \xrightarrow{\alpha_H} H
\end{aligned}$$

The coproduct property yields a unique $\mathbf{C}$-morphism $f^\sharp\colon \coprod_{n\in\mathbb{N}} F^n X \to H$ with $f^\sharp \circ \iota_n = f_n$ for all n.

By definition one has $f^\sharp \circ \iota_0 = f$. Moreover, $f^\sharp$ is a morphism $(X^\sharp, \alpha_X) \to (H, \alpha_H)$ in $\mathbf{Alg}F$, that is, the equation $f^\sharp \circ \alpha_X = \alpha_H \circ F f^\sharp$ holds, since, for each $n \in \mathbb{N}$

$$f^\sharp \circ \alpha_X \circ F\iota_n = f^\sharp \circ \iota_{n+1} = f_{n+1} = \alpha_H \circ Ff_n = \alpha_H \circ Ff^\sharp \circ F\iota_n$$

and $F\iota_n$ is a coproduct. $f^\sharp$ is unique with $f^\sharp \circ \iota_0 = f$, too. In fact, if $\phi\colon (X^\sharp, \alpha_X) \to (H, \alpha_H)$ is a homomorphism with $\phi \circ \iota_0 = f$, then

$$f^\sharp \circ \iota_n = \phi \circ \iota_n \implies f^\sharp \circ \iota_{n+1} = \phi \circ \iota_{n+1}$$

and thus $\phi = f^\sharp$, since $f^\sharp \circ \iota_{n+1} = f_{n+1} = \alpha_H \circ Ff_n = \alpha_H \circ Ff^\sharp \circ F\iota_n = \alpha_H \circ F\phi \circ F\iota_n = \phi \circ \alpha_X \circ F\iota_n = \phi \circ \iota_{n+1}$ for all n. □

1.3 E-reflective subcategories

Recall the following simple and easy to use criterion for reflectivity of a subcategory.

3 Fact ([6, 37.1]) *Let* **A** *be a category equipped with a factorization structure* (E, M) *for morphisms which, moreover, is* E*-co-wellpowered. Then the following are equivalent for any full subcategory* **B** *of* **A***.*

(1) **B** *is* E*-reflective in* **A***, that is,* **B** *is reflective in* **A** *and every reflection map belongs to* E*.*
(2) **B** *is closed in* **A** *under products and* M*-subobjects.*

In more detail, the reflection of an **A**-object A is given as follows: Let $\mathcal{S}$ be the class of all **A**-morphisms $f\colon A \to B_f$ into an **B**-object B_f.

Each such f as an (E, M)-factorization $A \xrightarrow{q_f} C_f \xrightarrow{m_f} B_f$. By hypothesis there is a set $\{A \xrightarrow{q_i} A_i \mid i \in I\} \subset E$ such that, for each $f \in \mathcal{S}$, there is some $i = i_f$ and an isomorphism $\phi_f\colon A_{i_f} \to C_f$ with $\phi_f \circ q_{i_f} = q_f$. The family $(q_i)_{i\in I}$ induces a morphism $q\colon A \to \prod_I A_i$ with (E, M)-factorization $A \xrightarrow{r} RA \xrightarrow{l} \prod_I A_i$.

Now $r\colon A \to RA$ is the **B**-reflection of A.

2 Some properties of $\mathbf{Hopf}_R$

2.1 Hopf algebras as functor algebras

We define a category $\mathbf{nHopf}_R$ of *near Hopf algebras* as follows: its objects are pairs (B, S) with a bialgebra B and a bialgebra homomorphism $S\colon B \to$

$B^{\mathsf{op,cop}}$ (equivalently $S\colon B^{\mathsf{op,cop}} \to B$). A morphism $f\colon (B,S) \to (B',S')$ then is a bialgebra homomorphism satisfying $S' \circ f = f \circ S$. In other words, $\mathbf{nHopf}_R$ is the category $\mathbf{Alg}H$ of functor algebras for the endofunctor H on $\mathbf{Bialg}_R$ sending B to $B^{\mathsf{op,cop}}$. The full embedding $\mathbf{Hopf}_R \hookrightarrow \mathbf{Bialg}_R$ now factors as

$$\mathbf{Hopf}_R \hookrightarrow \mathbf{nHopf}_R \to \mathbf{Bialg}_R$$

where the first arrow is a (full) embedding and the second one the forgetful functor.

We will need the following lemma.

4 Lemma *Let $f\colon (B,m,e,\mu,\epsilon,S) \to (B',m',e',\mu',\epsilon',S')$ be a homomorphism of near Hopf algebras. Then*

(1) $f \circ (S \star \mathsf{id}_B) = (S' \star \mathsf{id}_{B'}) \circ f$
(2) $f \circ (e \circ \epsilon) = (e' \circ \epsilon') \circ f$

Proof: Everything follows from commutativity of the diagrams

$$\begin{tikzcd} B \arrow[r, "\mu"] \arrow[d, "f"'] & B\otimes B \arrow[r, "B\otimes S"] \arrow[d, "f\otimes f"] & B\otimes B \arrow[r, "m"] \arrow[d, "f\otimes f"] & B \arrow[d, "f"] \\ B' \arrow[r, "\mu'"'] & B'\otimes B' \arrow[r, "B'\otimes S'"'] & B'\otimes B' \arrow[r, "m'"'] & B' \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "\epsilon"] \arrow[d, "f"'] & R \arrow[r, "e"] \arrow[d, "R"] & B \arrow[d, "f"] \\ B' \arrow[r, "\epsilon'"'] & R \arrow[r, "e'"'] & B' \end{tikzcd}$$

□

2.2 Image factorization of Hopf homomorphisms

The image factorization of homomorphisms in $\mathbf{Mod}_R$ lifts to a factorization system not only always in $\mathbf{Alg}_R$, but also in $\mathbf{Coalg}_R$, provided that R is von Neumann regular (recall that a commutative unital ring R is *von Neumann regular* iff, for each injective R-linear map f and each R-module M the map $f \otimes \mathsf{id}_M$ is injective). While the lifted factorization in $\mathbf{Alg}_R$ is the (regular epi, mono)-factorization, it is the (epi, regular mono)-factorization in $\mathbf{Coalg}_R$. Consequently, the surjections are precisely the epimorphisms in $\mathbf{Coalg}_R$, while the injections are the regular monomorphisms. If R is von Neumann regular image factorizations also provide a (surjective, injective)-factorization structure on $\mathbf{Bialg}_R$.

One certainly has, for a morphism f in $\mathbf{Bialg}_R$, the implications

(1) f is an extremal epic $\implies$ f is surjective $\implies$ f is an epimorphism.
(2) f is an extremal mono $\implies$ f is injective $\implies$ f is a monomorphism.

The following lemma, which is easy to prove (the required morphisms are obtained by diagonal fill ins and the required equations follow from the assumption that each $m \in M$ is a monomorphism), shows in particular that the categories $\mathbf{Hopf}_R$ and $\mathbf{nHopf}_R$ are closed in $\mathbf{Bialg}_R$ under image factorizations, if R is von Neumann regular.

5 Lemma *Let (E, M) be a factorization system for morphisms on $\mathbf{Bialg}_R$ with every $m \in M$ a monomorphism and every $e \in E$ an epimorphism in $\mathbf{C}$. Then (E, M) restricts to a factorization structure on $\mathbf{nHopf}_R$. Moreover, if $f : (B, S) \to (H, S_H)$ is a homomorphism of near Hopf algebras with (H, S_H) even a Hopf algebra, then the bialgebra A over which f can be factored is a Hopf algebra and, in fact, an M-subalgebra of (H, S_H).*

6 Lemma *Let R be a von Neumann regular ring and B be an R-bialgebra. Every subalgebra A of (the underlying algebra of) B contains a largest subcoalgebra C of (the underlying coalgebra of) B and this is a subbialgebra of B. If (B, S) is even a near Hopf algebra and A is S-invariant, then so is C (equivalently: C becomes a subobject of (B, S) in the category of near Hopf algebras).*

Proof: C exists since the subcoalgebras of an R-coalgebra from a complete lattices, provided that R is von Neumann regular (see e.g. [11]). Since the multiplication m of B is a coalgebra homomorphism, the image $m[C \otimes C]$ of $C \otimes C$ is a coalgebra (again by von Neuman regularity of R) and contained in A. The sup-assumption on C now shows that C is a subbialgebra.

The argument concerning S is analogous. $\square$

7 Remark The use of images in the proof above can be expressed more categorically by saying that $C \otimes C \to m[C \otimes C] \hookrightarrow B$ is the (E, M)-factorization of $C \otimes C \xrightarrow{i \otimes i} B \otimes B \xrightarrow{m} B$ with E all surjective and M all injective homomorphisms. Von Neumann regularity then guarantees that E and M are closed under tensor squaring which is what is needed to ensure that subcoalgebras form a complete lattice which is closed under formation of images. Therefore the above result can be dualized and one obtains

Let R be a von Neumann regular ring and B be an R-bialgebra. For every coalgebra quotient C of (the underlying coalgebra of) B there is a largest algebra quotient[1] *A of (the underlying algebra of) B smaller then C (as a quotient in $\mathbf{Mod}_R$) and this is a bialgebra quotient of B. If (B, S) is even a near Hopf algebra and S induces a morphism on C, then so it does on A and A becomes a quotient of (B, S) in the category of near Hopf algebras.*

[1] We here refer to the usual ordering of quotients: $(B \xrightarrow{q} Q) \geq (B \xrightarrow{q'} Q')$ iff there exists some $Q \xrightarrow{h} Q'$ with $q' = h \circ q$.

2.3 Limits and colimits of Hopf algebras

We recall from [12] and [13]

8 Fact *For every commutative unital ring R the category $\mathbf{Bialg}_R$ has products and coproducts.*

9 Fact *For every commutative unital ring R the following hold:*

(1) $\mathbf{Hopf}_R$ *is closed under colimits in* $\mathbf{Bialg}_R$.
(2) $\mathbf{Hopf}_R$ *is closed under limits in* $\mathbf{Bialg}_R$, *provided that the ring R is von Neumann regular.*

Since the underlying functor $\mathbf{nHopf}_R \to \mathbf{Bialg}_R$ creates limits (as every underlying functor of a category of functor algebras does) and colimits (since the functor H—see [1]—does), it thus follows that, for every von Neumann regular ring R, the category $\mathbf{Hopf}_R$ is closed under limits and colimits in $\mathbf{nHopf}_R$.

3 Some constructions of adjoints

3.1 Constructing the Hopf envelope

The best way to understand the construction of the Hopf envelope, i.e., the reflection from bialgebras into Hopf algebras, is to look at it as composition of left adjoints to the functors (see 2.1)

$$\mathbf{Hopf}_R \hookrightarrow \mathbf{Alg}(-)^{\mathsf{op,cop}} \quad \text{and} \quad \mathbf{Alg}(-)^{\mathsf{op,cop}} \to \mathbf{Bialg}_R$$

10 Lemma *The underlying functor $\mathbf{nHopf}_R = \mathbf{Alg}H \xrightarrow{|-|} \mathbf{Bialg}_R$ has a left adjoint given by the unit*

$$\iota_0 \colon B \to |(B^*, S^*)|$$

where $\Big(B^*, (\iota_n \colon B_n \to B^*)_{n\in\mathbb{N}}\Big)$ is the (countable) coproduct in $\mathbf{Bialg}_R$ of the family $(B_n)_{n\in\mathbb{N}}$, defined recursively by

$$B_0 := B,\ B_{n+1} := B_n^{\mathsf{op,cop}}.$$

and $S^* \colon (B^*)^{\mathsf{op,cop}} \to B^*$ is the unique bimonoid homomorphism such that the following diagrams commute, for all $n \in \mathbb{N}, n > 0$.

$$\begin{tikzcd}
B_n = B_{n-1}^{\mathsf{op,cop}} \arrow[r, "\mathsf{id}"] \arrow[d, "\iota_{n-1}^{\mathsf{op,cop}}"'] & B_n \arrow[d, "\iota_n"] \\
(B^*)^{\mathsf{op,cop}} \arrow[r, dotted, "S^*"] & B^*
\end{tikzcd}$$

Proof: The lemma follows from an application of Fact 2 to the functor $H\colon \mathbf{Bialg}_R \to \mathbf{Bialg}_R$ sending B to $B^{\mathsf{op,cop}}$ which, as an isomorphism, clearly preserves coproducts; these exist by 8. □

Note that the description of S^* given in [10] is obtained from the above by applying the isomorphism $(-)^{\mathsf{op,cop}}$.

11 Lemma *Let R be a von Neumann regular ring. Then $\mathbf{Hopf}_R$ is reflective in $\mathbf{nHopf}_R$ and the $\mathbf{Hopf}_R$-reflection of a near Hopf algebra (B, S) is given by a surjective homomorphism $r\colon (\mathsf{B}, S) \to (H_B, S_B)$.*

Denoting by $\mathcal{S}$ is the class of all near Hopf homomorphisms $(\mathsf{B}, S) \xrightarrow{f} (\mathsf{H}_f, S_f)$ into some Hopf algebra and by $\bar{\mathcal{S}}$ a representative set of all surjections in $\mathcal{S}$, the homomorphism r is characterized by any of the following equivalent properties:

(1) $B \xrightarrow{r} H_B$ *is the image of the morphism* $B \xrightarrow{q} \prod_{p\in\bar{\mathcal{S}}} H_p$ *induced by the family* $p \in \bar{\mathcal{S}}$.
(2) $\ker r = \cap_{f\in\mathcal{S}} \ker f$.

Proof: By Lemma 5 and Fact 9 we can apply Fact 3; this proves the first part of the Lemma, since, obviously (use Lemma 4), $\mathbf{Hopf}_R$ is closed in $\mathbf{nHopf}_R$ under subobjects carried by injective homomorphisms.

Statement 1 follows by applying the construction sketched in 1.3.

Since every $f \in \mathcal{S}$ factorizes through r one has $\ker r \subset \cap_{f\in\mathcal{S}} \ker f$; thus, $r \in \mathcal{S}$ implies $\ker r = \cap_{f\in\mathcal{S}} \ker f$. This proves 2. □

A more explicit (and even dualizable) description of r is possible. For this recall the following fact from [13].

12 Fact *Let (B, S) be a near Hopf algebra and $q\colon B \to \bar{B}$ the (multiple) coequalizer of the linear maps $S \star \mathsf{id}_B, \mathsf{id}_B \star S$ and $e \circ \epsilon$ in $\mathbf{Mod}_R$. Then $\bar{B}$ carries a (unique) coalgebra structure $(\bar{\mu}, \bar{\epsilon})$ such that q is a homomorphism of coalgebras.*

13 Remark Note that this is nothing but the dual of the well known fact (see e.g. [5, 4.3.3] or [13]) that the (multiple) equalizer of the linear maps

$S \star \mathsf{id}_B, \mathsf{id}_B \star S$ and $e \circ \epsilon$ in $\mathbf{Mod}_R$ is an algebra.

To be able to apply this fact recall that $\bar{B} = B/(I + J)$ with $I = \mathrm{im}(S \star \mathsf{id}_B - e \circ \epsilon)$ and $J = \mathrm{im}(\mathsf{id}_B \star S - e \circ \epsilon)$. We then can prove the following result, which in particular expresses the conceptual meaning of the final step of Takeuchi's construction.

14 Proposition *Let R be a von Neumann regular ring and (B, S) a near Hopf algebra over R. The Hopf reflection $r\colon (B, S) \to (H_B, S_B)$ is characterized by any of the following equivalent properties.*

(1) $r\colon B \to H_B$ *is the largest algebra quotient of B which factors (in $\mathbf{Mod}_R$) over the coequalizer $q\colon B \to \bar{B}$.*
(2) $\ker r$ *is the ideal $\langle M \rangle$ generated by*

$$M := \{(S \star \mathsf{id} - e\epsilon)(x), (\mathsf{id} \star S - e\epsilon)(x) \mid x \in B\}.$$

In particular, the ideal $\langle M \rangle$ is a bi-ideal and invariant under S.

Proof: For every near Hopf homomorphism $f\colon (B, S) \to (H, S_H)$ into a Hopf algebra one concludes from Lemma 4

$$f \circ (S \star \mathsf{id}_B) = f \circ (e \circ \epsilon) = f \circ (\mathsf{id}_B \star S)$$

Thus, by the definition of coequalizer, there exists a (unique) linear map $\bar{f}\colon \bar{B} \to H$ with $f = \bar{f} \circ q$. In particular, the reflection r is an algebra homomorphism, which factors over q.

Since r is the largest such quotient iff $\ker r = \langle M \rangle$, it only remains to prove that, if $r\colon B \to B'$ is the largest algebra quotient which factors over q as $r = s \circ q$, then B' not only is an algebra but even a Hopf algebra and r is a homomorphism of near Hopf algebras.

In fact, we get from Remark 7 that B' already is a near Hopf algebra and r homomorphism of near Hopf algebras. That B' then even is a Hopf algebra follows by means of Lemma 4 from the fact that it is a (linear) quotient of $\bar{B}$ and the latter's coequalizer description in Fact 12. □

15 Remark It is clear from the above that it would suffice, in the definition of the set M, to have x running through a subset of B which generates B as an algebra.

With notation as above we thus get, for every von Neumann regular ring R, the following result.

16 Proposition $B \xrightarrow{\iota_0} B^* \xrightarrow{r} H_B$ *is a Hopf reflection of any bialgebra* B.

This is, by the above, the description of the Hopf envelope as given in [10] (with $\cup_n \iota_n[B_n]$ as generating set for B^*).

3.2 Constructing the free Hopf algebra

By composition of adjoints and the description of free bialgebras as given in 1.1 we thus obtain Takeuchi's description of the free Hopf algebra over a coalgebra as follows.

Let C be an R-coalgebra, where R is a von Neumann regular ring. Then the free Hopf algebra $H(C)$ over C is the Hopf reflection of the free bialgebra T^*C. Having in mind that the left adjoint of the forgetful functor $\mathbf{Bialg}_R \to \mathbf{Coalg}_R$ commutes with coproducts, the above description of the Hopf reflection gives Takeuchi's original construction of $H(C)$. This also clarifies completely the relation between Takeuchi's construction and that of the Hopf envelope.

In view of [14, 13.2] it might be worthwhile to clarify Takeuchi's original notation: Where he writes B^{op} for a bialgebra B he certainly means, in todays notation, the bialgebra $\mathsf{B}^{\mathsf{op,cop}}$ (otherwise his antipode would be an algebra homomorphism, not an anti-homomorphism as required). Thus, also taking the opposite algebraic structure into account in his construction is *not* 'superfluous/invisible' as claimed in [14], but used explicitely (only somewhat hidden by notation).

Our analysis moreover shows that Takeuchi's free Hopf algebra construction is *not* a generalization of the free group construction as often perceived. One certainly *could* construct free groups also analogously to the construction just presented, that is, by composing the word monoid construction with the reflection from monoids into groups; the typical free group construction however represents a free group differently, namely as a quotient of the term algebra for the signature $(\times, (-)^{-1}, 1)$.

3.3 Constructing the Hopf coreflection

As shown in [13] the category of Hopf algebras also is coreflective in the category of bialgebras, and this even for *any* commutative ring R. In case R is von Neumann regular, we can provide a construction of the coreflection by simple categorical dualization of the above construction of the Hopf envelope.

For a precise formulation of this dualization process recall that $\mathbf{Bialg}_R$ is nothing but the category $\mathbf{Bimon}\mathbb{C}$ of bimonoids in the symmetric monoidal category $\mathbb{C} = \mathbf{Mod}_R$ and, similarly, $\mathbf{Hopf}_R$ is the category $\mathbf{Hopf}\mathbb{C}$ of Hopf monoids in $\mathbb{C} = \mathbf{Mod}_R$. Moreover, for each symmetric monoidal category $\mathbb{C}$, the (categorical) dual of $\mathbf{Bimon}\mathbb{C}$ is the category of bimonoids in the dual $\mathbb{C}^{\mathsf{op}}$ of $\mathbb{C}$ (the dual of the category $\mathbf{C}$ underlying $\mathbb{C}$, equipped with the same tensor product), that is, $(\mathbf{Bimon}\mathbb{C})^{\mathsf{op}} = \mathbf{Bimon}(\mathbb{C}^{\mathsf{op}})$, and also $(\mathbf{Hopf}\mathbb{C})^{\mathsf{op}} = \mathbf{Hopf}(\mathbb{C}^{\mathsf{op}})$. Note that the convolution monoid of a bimonoid B over $\mathbb{C}$ coincides with the convolution monoid of B, if considered a bimonoid over $\mathbb{C}^{\mathsf{op}}$.

Thus, a coreflection from $\mathbf{Bialg}_R$ into $\mathbf{Hopf}_R$ is nothing but a reflection from $\mathbf{Bimon}(\mathbf{Mod}_R^{\mathsf{op}})$ into $\mathbf{Hopf}(\mathbf{Mod}_R^{\mathsf{op}})$.

As an analysis of the construction of the Hopf envelope above shows, we only used, besides their very definition as categories of bi- and Hopf monoids respectivley, the following properties of the categories $\mathbf{Bialg}_R$ and $\mathbf{Hopf}_R$ (which certainly depend on the underlying category $\mathbb{C} = \mathbf{Mod}_R$): (1) $\mathbf{Bialg}_R$ has products, (2) $\mathbf{Hopf}_R$ is closed in $\mathbf{Bialg}_R$ under products, and (3) The image factorization from $\mathbf{Mod}_R$ lifts to a factorization structure for $\mathbf{Bialg}_R$, to which Lemma 5 can be applied.

By Facts 8 and 9 properties 1. and 2. are shared by $(\mathbf{Hopf}_R)^{\mathsf{op}}$ and $(\mathbf{Bialg}_R)^{\mathsf{op}}$ (and this even for every commutative ring R!). Further, the image factorization structure of $\mathbf{Mod}_R$ also is a factorization structure in $\mathbf{Mod}_R^{\mathsf{op}}$ which obviously lifts to one in $\mathbf{Bimon}(\mathbf{Mod}_R^{\mathsf{op}}) = (\mathbf{Bialg}_R)^{\mathsf{op}}$ (provided that R is von Neumann regular), to which then Lemma 5 can be applied, too. Observing Fact 12 and the remark following it, our construction, thus, yields a reflection from $\mathbf{Bimon}(\mathbf{Mod}_R^{\mathsf{op}})$ into $\mathbf{Hopf}(\mathbf{Mod}_R^{\mathsf{op}})$, that is, a coreflection from $\mathbf{Bialg}_R$ into $\mathbf{Hopf}_R$. Hence, somewhat more explicitly, the following holds. Note that, due to the dualization process just described, not only coproducts have to be replaced by products but also quotients by subobjects.

17 Proposition *Let R a commutative unital ring which, in addition, is von Neumann regular. Then a Hopf coreflection of a bialgebra B can be constructed as follows*

Step 1: *Form the product $\hat{B} := \prod B_n$ of the family $(B_n)_{n\in\mathbb{N}}$ from the proof of Lemma 10. Let $\hat{S}$ be the unique homomorphism, such that the following diagram commutes.*

$$\begin{tikzcd}
B_n = B_{n-1}^{\mathsf{op,cop}} & B^n \arrow[l, "\mathsf{id}"'] \\
(\hat{B})^{\mathsf{op,cop}} \arrow[u, "\pi_{n-1}^{\mathsf{op,cop}}"] & \hat{B} \arrow[u, "\pi_n"'] \arrow[l, dotted, "\hat{S}"]
\end{tikzcd}$$

Step 2: *Form the image factorization $f\colon H \xrightarrow{p_f} C \xrightarrow{j_f} \hat{B}$ of each bialgebra* *homomorphism from a Hopf algebra* (H, S_H) *into* $\hat{B}$ *satisfying*

$$f \circ S_H = \hat{S} \circ f$$

and choose a representative set $\{C_k \mid k \in K\}$ *of these* C*'s.*

Step 3: *Form the image factorization of* j*, the morphism induced by the family* j_k*,*

$$j = \coprod_K C_k \xrightarrow{s} H^B \xrightarrow{c} \hat{B}.$$

Then $H^B \xrightarrow{c} \hat{B} \xrightarrow{\pi_0} B$ *is a Hopf coreflection of* B*.*

H^B *is, alternatively, the largest subcoalgebra (which then automatically is a Hopf algebra) of* $\hat{B}$ *contained in*

$$E := \{x \in \hat{B} \mid \hat{S} \star \mathsf{id}_{\hat{B}}(x) = \hat{e} \circ \hat{\epsilon}(x) = \mathsf{id}_{\hat{B}} \star \hat{S}(x)\},$$

that is, the (multiple) equalizer of the linear maps $\hat{S} \star \mathsf{id}_{\hat{B}}, \hat{e} \circ \hat{\epsilon}$ *and* $\mathsf{id}_{\hat{B}} \star \hat{S}$ *in* $\mathbf{Mod}_R$*.*

18 Remark In [4] a Hopf coreflection H^B of a bialgebra B is constructed as the largest subcoalgebra of $\hat{B}$ contained in the (multiple) equalizer E. The author does not observe that this is nothing but the categorical dual of the familiar Hopf envelope construction as in Proposition 14 above and thus provides an independent proof.

19 Remark Our approach is also applicable to the monoidal category of sets with cartesian product as tensor product. In that case the category of bimonoids is (isomorphic to) the category of (ordinary) monoids and the category of Hopf monoids is the category of groups.

We thus get the familiar facts that the category of groups is reflective and coreflective in the category of monoids. There is, however, a notable difference between this situation and the case of Hopf algebras: While the coreflection from groups to monoids is a mono-coreflection (the coreflection of a monoid M is its subgroup of invertible elements) this is not the case for Hopf algebras. If the Hopf-coreflection of a bialgebra B always were a sub-bialgebra of B, this would imply that every bialgebra quotient of a Hopf algebra is a Hopf algebra (use the dual of [6, 37.1]); but this is not the case (not every bi-ideal in a Hopf algebra is a Hopf ideal — see [9]).

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