On Tree Coalgebras and Coalgebra Presentations

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Abstract

For deterministic systems, expressed as coalgebras over polynomial functors, every tree t (an element of the final coalgebra) turns out to represent a new coalgebra A_t. The universal property of these coalgebras, resembling freeness, is that for every state s of every system S there exists a unique coalgebra homomorphism from a unique A_t which takes the root of t to s. Moreover, the tree coalgebras are finitely presentable and form a strong generator. Thus, these categories of coalgebras are locally finitely presentable; in particular every system is a filtered colimit of finitely presentable systems.

In contrast, for transition systems expressed as coalgebras over the finite–power–set functor we show that there are systems which fail to be filtered colimits of finitely presentable (= finite) ones.

Surprisingly, if λ is an uncountable cardinal, then λ–presentation is always well–behaved: whenever an endofunctor F preserves λ–filtered colimits (i.e., is λ–accessible), then λ–presentable coalgebras are precisely those whose underlying objects are λ–presentable. Consequently, every F coalgebra is a λ–filtered colimit of λ–presentable coalgebras; thus $\mathsf{Coalg}F$ is a locally λ–presentable category. (This holds for all endofunctors of λ–accessible categories with colimits of ω-chains.) Corollary: λ–accessible set functors are bounded at λ in the sense of Kawahara and Mori and, conversely, boundedness at λ implies λ^+–accessibility.

Key words: Coalgebras, Σ–labelled trees, λ–presentable objects and categories, accessible and bounded functors

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I Introduction

IA Coalgebras as Deterministic Systems

Coalgebras over a given endofunctor F of Set can be viewed as systems of a type determined by F, see Rutten's expository paper [18]. Recall that a coalgebra is a set S (of states) together with a structure map $\alpha\colon S \to FS$. For example, consider a signature Σ of operation symbols $\sigma_1, \dots, \sigma_k$ of arities $n_1, \dots, n_k$ respectively; the corresponding "polynomial" endofunctor H_Σ is given on objects S by

$$H_\Sigma S = S^{n_1} + S^{n_2} + \dots + S^{n_k}.$$

A coalgebra over H_Σ is a set S of states decomposed into k blocks, $S = S_1 + \dots, S_k$, where each block S_i has n_i inputs; the reaction of a state $s \in S_i$ to the n_i inputs is given by the n_i–tuple $\alpha(s) \in S^{n_i}$ of next states. This yields a function

$$\alpha\colon S \to S^{n_1} + S^{n_2} + \dots + S^{n_k}$$

where S_i is the preimage of the i–th summand under α. We call such systems Σ–*coalgebras* and we denote by

$$\mathsf{Coalg}\Sigma$$

the resulting category of Σ–coalgebras and coalgebra homomorphisms.

The main result of our paper is that categories of Σ–coalgebras exhibit behaviour that is surprisingly "algebraic". For example, whereas in classical universal algebra one works with free algebras (or term algebras), and moves to equations and equational presentations, for general F–coalgebras the corresponding concepts are cofree coalgebras and coequations, see, e.g., [5,12,18]. However, for the special case of $\mathsf{Coalg}\Sigma$ it turns out that, besides cofree coalgebras, there also exist multifree coalgebras — and they are closely related to the cofree ones. Recall that a final Σ–coalgebra, T_Σ, is the coalgebra of all Σ–labelled trees (see II.2 below). We now show that every tree $t \in T_\Sigma$ defines a Σ–coalgebra A_t per se, whose elements are the nodes of t and whose coalgebra structure is given by unfolding. And the collection of all these tree coalgebras is multifree on one generator, which means the following:

> for every Σ–coalgebra S and every state $s \in S$ there exists a unique coalgebra homomorphism $f\colon A_t \to S$ mapping the root of t to s, for a unique tree $t \in T_\Sigma$.

As a consequence, the natural forgetful functor

$$U\colon \mathsf{Coalg}\Sigma \to \mathsf{Set}$$

is not only a left adjoint (whose right adjoint assigns to a set of k elements the cofree Σ–coalgebra on k colors) but also a right multiadjoint (whose left multiadjoint assigns to a set of k elements the collection of all "forest coalgebras" made from forests of k trees in T_Σ).

We also study presentability of coalgebras. In algebra, objects A with a "finite description" (by generators and equations) are called finitely presentable—and they are precisely those, which are finitely presentable in the categorical sense (see the Appendix). For polynomial endofunctors of Set we prove that the above tree coalgebras A_t are finitely presentable. And we describe all finitely presentable coalgebras: these are precisely the coalgebras obtained from a finite coproduct of tree coalgebras by merging finitely many pairs of bisimilar states. (Thus, finitely presentable Σ-coalgebras are, again, precisely those with a finite "description" via tree coalgebras.) It then follows that the category of coalgebras is locally finitely presentable.

These results hold for all, not necessarily finitary, signatures.

IB Coalgebras as Nondeterministic Systems

The above results on polynomial endofunctors do by no means extend to general endofunctors of Set. As a concrete example, consider the finite power–set functor

$$\mathcal{P}_f \colon \mathsf{Set} \to \mathsf{Set}$$

assigning to every set the collection of all finite subsets. $\mathcal{P}_f$–coalgebras are precisely the finite–branching graphs (or non–labelled transition systems): for every coalgebra $\alpha\colon S \to \mathcal{P}_f S$ we consider the graph on S of all edges $x \to y$ with $y \in \alpha(x)$. However, coalgebra homomorphisms are much more restrictive than graph homomorphisms: they are the homomorphisms $h\colon S \to S'$ of non–labelled transition systems (i.e., such graph homomorphisms which for every edge $h(x) \to y$ in S' have an edge $x \to z$ in S with $h(z) = y$).

We are going to prove that finitely presentable $\mathcal{P}_f$–coalgebras are precisely the finite graphs (which sounds trivial, but does not seem to have a trivial proof). As a consequence, it is obvious that the following graph

$$\bullet \to \bullet \to \bullet \to \bullet \cdots$$

is not finitely presentable, nor is it a filtered colimit of finitely presentable coalgebras in $\mathsf{Coalg}\mathcal{P}_f$. Consequently $\mathsf{Coalg}\mathcal{P}_f$ is not a locally finitely presentable category.

The behavior of the categories $\mathsf{Coalg}F$ of coalgebras gets much more regular if we switch over from finite presentation to λ–presentation where λ is an uncountable regular cardinal. We are going to prove the following results for all uncountable regular cardinals λ. (See the appendix for the definition of presentability and accessibility.)

Theorem *If F is a λ–accessible endofunctor of a λ–accessible category A with colimits of ω–chains, then an F-coalgebra is λ-presentable iff its underlying object is λ-presentable in A. And the category of F–coalgebras is λ–accessible.*

Corollary *For every λ–accessible endofunctor of Set the category of F-coalgebras is locally λ–presentable.*

The above corollary has been proved by M. Barr, see [10, Proposition 1.3]. His proof works, in fact, for all locally λ–presentable categories. The above theorem concerning λ–accessible categories strengthens the result of Makkai and Paré that for every λ–accessible endofunctor the category of F-coalgebras, being a weighted limit (inserter) of F and the identity functor, is accessible, see [16]; from the general result it does not follow that $\mathsf{Coalg}F$ is λ–accessible (as the above example of $\mathcal{P}_f$ demonstrates). The proof of the above theorem, as presented in Section IV below, uses a similar technique of iteration which M. Barr has used in [10], but our proof is technically more involved. Barr's paper has been inspired by that of P. Aczel and M. Mendler [1].

Coming back to λ–accessible functors $F\colon \mathsf{Set} \to \mathsf{Set}$, here our result extends to all covarieties V of F–coalgebras:

1. λ–presentable objects of V are precisely the coalgebras of less than λ–elements, and
2. V is locally λ–presentable.

Finally we relate the concepts of accessibility and boundedness: We have proved in [5] that a Set functor is accessible iff it is bounded in the sense of Kawahara and Mori [14]. We sharpen our previous result by showing that for every regular cardinal λ (with the successor cardinal λ^+) we have

$$\lambda\text{–accessible} \;\Rightarrow\; \text{bounded at } \lambda \Rightarrow \lambda^+\text{–accessible.}$$

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II Σ–Coalgebras

II.1 We study first coalgebras over *polynomial functors* $H_\Sigma\colon \mathsf{Set} \to \mathsf{Set}$ where Σ is a given (not necessarily finitary) signature. That is, for every "operation" symbol σ in Σ a cardinal number $|\sigma|$, called the arity of σ, is given, and H_Σ is defined on objects by

$$H_\Sigma X = \coprod_{\sigma\in\Sigma, |\sigma|=n} X^n \;\simeq\; \bigcup_{\sigma\in\Sigma, |\sigma|=n} X^n \times \{\sigma\}.$$

H_Σ–coalgebras are simply called Σ*–coalgebras* and the corresponding category is denoted by

$$\mathsf{Coalg}\Sigma.$$

Thus, a Σ–coalgebra is a set A endowed with a function

$$\alpha_A\colon A \to \bigcup_{\sigma\in\Sigma, |\sigma|=n} (A^n \times \{\sigma\})$$

assigning to every element a a pair

$$\alpha_A(a) = ((a_i)_{i<n}, \sigma)$$

consisting of an n–tuple in A and an n–ary operation symbol σ in Σ.

II.2 Example (Terminal Σ–coalgebra) We briefly recall the description of a final Σ–coalgebra, T_Σ, as the set of all Σ–labelled trees where the structure map α_{T_Σ} is parsing, i.e., the inverse to the usual tree–tupling. An element t of T_Σ is thus a (finite or infinite) tree whose nodes are labelled by operation symbols so that a node with n children is labelled by an n–ary symbol. In case Σ is a finitary signature, a useful formalization of Σ–labelled trees uses the structure of the set ω^* of all finite sequences of natural numbers: a Σ–labelled tree is a partial function

$$t\colon \omega^* \to \Sigma$$

whose domain of definition, $\mathsf{Def}t$, has the following two properties:

(i) $\mathsf{Def}t$ contains the empty word $\in$ and is prefix–closed, i.e., if $uv \in \mathsf{Def}t$ then $u \in \mathsf{Def}t$

and

(ii) if $i_1 \dots i_k \in \mathsf{Def}t$ and $t(i_1 \dots i_k)$ has arity n, then for all $j < \omega$ we have

$$i_1 \dots i_k j \in \mathsf{Def}t \text{ iff } j < n.$$

Analogously for infinitary (bounded) signatures: choose any cardinal λ bigger than all arities. Consider λ as the set of all smaller ordinal numbers, and substitute ω^* above by λ^* (the set of all finite sequences of ordinals smaller than λ). Then a Σ–labelled tree is a partial function $t\colon \lambda^* \to \Sigma$ satisfying (i) and (ii) above.

T_Σ is equipped with an action, α_{T_Σ}, assigning to a tree $t \in T_\Sigma$ the element

$$\alpha_{T_\Sigma}(t) = ((t_0, \dots, t_{n-1}), \sigma) \in H_\Sigma(T_\Sigma)$$

where

- $\sigma = t(\epsilon)$, i.e., σ is the label of t's root;
- $x \in \mathsf{Def}t_i$ iff $ix \in \mathsf{Def}t$ and then $t_i(x) = t(ix)$, i.e., t_i is the i–th maximal subtree of t.

Given now a Σ–coalgebra (A, α_A) define, for all $a \in A$, trees $t_a \in T_\Sigma$ and elements $a_x \in A$ ($x \in \mathsf{Def}t_a$) inductively as follows

- $a_\epsilon\colon = a$ and $t_a(\epsilon) = \sigma \Leftrightarrow \alpha_A(a) = ((a_i)_{i<n}, \sigma)$
- if $x \in \mathsf{Def}t_a$ and $\alpha_A(a_x) = ((a_{xi})_{i<m}, \tau)$ then $xi \in \mathsf{Def}t_a$ for all $i < m$ and $t_a(x) = \tau = (t_{a_x}(\epsilon))$.

The assignment $a \mapsto t_a$ then is a homomorphism from (A, α_A) to $(T_\Sigma, \alpha_{T_\Sigma})$, and in fact the only one. Thus $(T_\Sigma, \alpha_{T_\Sigma})$ is a final Σ-coalgebra.

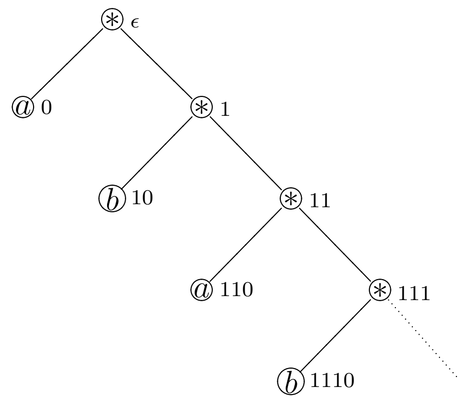
II.3 Example (Tree coalgebras) Given a Σ–labelled tree $t \in T_\Sigma$ the *tree coalgebra* A_t is the coalgebra whose elements are the nodes of t, and whose structure map α_t assigns to every node $i_1, \dots i_k$ with $t(i_1 \dots i_k) = \sigma$, an n–ary symbol, the n–tuple of its children, more precisely:

$$A_t = (\mathsf{Def}t, \alpha_t) \quad \text{with} \quad \alpha_t(i_1 \dots i_k) = ((i_1 \dots i_k j)_{j<n}, \sigma).$$

For example, consider

$\Sigma = \{*,\ a, b\}$ where $|*| = 2,\ |a| = 0 = |b|$.

The following tree t



defines a Σ–coalgebra whose elements are $\epsilon, 0, 1, 10, 11, 110, \ldots$ and whose structure map is defined as follows:

x	ϵ	0	1	10	11	$\ldots$
$\alpha_{A_t}(x)$	$((0,1),*)$	$((),a)$	$((10,11),*)$	$((),b)$	$((110,111),*)$	$\ldots$

Observe that this is *not* isomorphic to the sub–coalgebra of T_Σ generated by the tree t: that coalgebra B_t has only 4 elements, viz, all subtrees of t, which are:

$$t, a, s, b$$

where s is the subtree at the node 1 (i.e., $s(x) = t(1x)$). The structure map is

x	t	a	s	b
α_{B_t}	$((a,s)*)$	$((),a)$	$((b,t),*)$	$((),b)$

II.4 Proposition *For every element s of every Σ–coalgebra S there exists a unique Σ–labelled tree t such that some homomorphisms $h\colon A_t \to S$ maps the root to s. Moreover, h is also unique.*

Proof 1. *Uniqueness.* Let

$$h\colon A_t \to S \text{ and } \bar{h}\colon A_{\bar{t}} \to S$$

be homomorphisms with

$$h(\epsilon) = s = \bar{h}(\epsilon).$$

For every $x \in \omega^*$ we prove that

(i) $h(x) = \bar{h}(x)$

and

(ii) the fathers of x in $\bar{t}$ and t have the same label (if x is not the root)

by induction on the depth of x (= the length of the word x). That is, (i) states that if one side is defined then so is the other one, and then they are equal and (ii) holds.

Depth 0 For the root ϵ (the empty word),

$$h(\epsilon) = s = \bar{h}(\epsilon).$$

Depth $k+1$ Suppose that $y = xi$ where x has depth k. If $x \notin \mathsf{Def}t$, then $x \notin \mathsf{Def}\bar{t}$ (by induction hypothesis), so neither $h(y)$ nor $\bar{h}(y)$ is defined. Else, (i) holds by induction hypothesis. Put $h(x) = b = \bar{h}(x)$ and denote

$$\alpha_S(b) = ((b_j)_{j<n}, \sigma).$$

Since h is a homomorphism, it follows that $\alpha_{A_t}(x)$ has the form $((xj)_{j<n}, \sigma)$ and therefore

$$h(xi) = b_i \text{ and } t(x) = \sigma.$$

Analogously,

$$\bar{h}(xi) = b_i \text{ and } \bar{t}(x) = \sigma.$$

2. *Existence.* For the unique homomorphism $f\colon S \to T_\Sigma$ put

$$t = f(s).$$

We define a homomorphism

$$h\colon A_t \to S$$

in a node $x \in \mathsf{Def}t$ by induction on the depth of x so that $fh(x) = t(x-)$, the subtree of t at the node x, for every $x \in \omega^*$.

Depth 0 Put $h(\epsilon) = s$. Since $f(s) = t$, we know that $fh(\epsilon)$ is the subtree $t(\epsilon-) = t$.

Depth $k+1$ Let $y = xi$ where x is a node of depth k with $h(x) = b$. Since f is a homomorphism, from $\alpha_S(b) = ((b_j)_{j<n}, \sigma)$ where σ is an n–ary symbol we conclude

$$\alpha_{T_\Sigma}(f(b)) = (f(b_j), \sigma).$$

By induction hypothesis, $f(b)$ is the subtree $t(x-)$ of t, therefore, $f(b_j)$ is the subtree $t(xj-)$ (because $\alpha_{T_\Sigma}(y) = (y(j-), \sigma)$ for all trees y with $y(\epsilon) = \sigma$). Define

$$h(y) = b_i$$

and observe that

$$fh(y) = f(b_i) = t(xi-) = t(y-).$$

To verify that h is a homomorphism, one proves by easy induction on the depth of x that $h(x) = b$ and $\alpha_{A_t}(x) = ((xj), \sigma)$ imply $\alpha_S(b) = (h(xj), \sigma)$. $\diamond$

II.5 Recall the concept of a *right multiadjoint*: it is a functor $U\colon \mathsf{A} \to \mathsf{B}$ such that for every object $B \in \mathsf{B}$ there exists a "multifree collection on B", i.e., a collection of morphisms (on B) $h_i\colon B \to UA_i,\ i \in I$, with $A_i \in \mathsf{A}$, such that every morphism $h\colon B \to UA,\ A \in \mathsf{A}$, factorizes as $h = Uk \cdot h_j$ for a unique $j \in I$ and a unique morphism $k\colon A_j \to A$ of A.

Corollary *The collection*

$$A_t \ (t \in T_\Sigma)$$

of Σ–coalgebras is multifree on one generator. Thus, the forgetful functor

$$U\colon \mathsf{Coalg}\Sigma \to \mathsf{Set}$$

is a right multiadjoint (as well as a left adjoint).

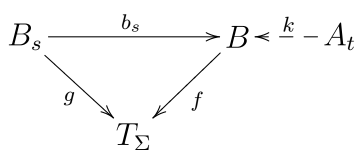
In fact, the first statement just rephrases II.4, and the latter follows from the fact that $\mathsf{Coalg}\Sigma$ has coproducts (in fact: U creates coproducts). Thus, a multifree collection on k generators is obtained from a multifree collection $(A_t)_{t\in T_\Sigma}$ on one generator by the formation of coproducts $\coprod_{i<k} A_{t(i)}$ for all k–tuples $(t(i))_{i<k}$ in T_Σ. The fact that U is a left adjoint, i.e., that cofree coalgebras exist, follows from H_Σ being a finitary functor, see [10].

II.6 Corollary *Each of the coalgebras A_t is finitely presentable.*

In fact, let $B = \mathrm{colim}_{s\in S} B_s$ be a filtered colimit in $\mathsf{Coalg}\Sigma$. For every homomorphism $k\colon A_t \to B$ there exists $s \in S$ such that the image of the root, $k(\epsilon)$, lies in the image of B_s under the corresponding colimit homomorphism $b_s\colon B_s \to B$.

We will show that k factors through b_s (say, as $k = h \cdot b_s$) and the factorization is essentially unique (i.e., if $k = h' \cdot b_s$, then some connecting morphism $b_{s\bar{s}}\colon B_s \to B_{\bar{s}}$ merges h and h'). This proves that A_t is finitely presentable.

We have $k(\epsilon) = b_s(x)$ for some $x \in B_s$. Now the unique homomorphisms $f\colon B \to T_\Sigma$ and $g\colon B_s \to T_\Sigma$ form a commutative triangle:



Since the uniqueness in II.4 implies $f(k(\epsilon)) = t$, we conclude $g(x) = t$. Let $h\colon A_t \to B_s$ be the unique homomorphism with $h(\epsilon) = x$. Then $k = b_s h$ by uniqueness in II.4, due to $k(\epsilon) = b_s h(\epsilon)$.

Also the fact that h is essentially unique follows from II.4: given a homomorphism $h'\colon A_t \to B_s$ with $k = b_s h'$, find a connecting morphism $b_{s\bar{s}}\colon B_s \to B_{\bar{s}}$ which merges $h(x)$ and $h'(x)$ (which exists since $b_s(h(x)) = b_s(h'(x))$. Then $b_{s\bar{s}}h = b_{s\bar{s}}h'$ by uniqueness in II.4.

II.7 Corollary *The category* $\mathsf{Coalg}\Sigma$ *is locally finitely presentable.*

In fact $\mathsf{Coalg}\Sigma$ is cocomplete. Thus, it is sufficient to show that it has a strong generator formed by finitely presentable objects, see [7, 1.11]. It follows from II.4 that $\{A_t;\ t \in T_\Sigma\}$ is a strong generator.

II.8 Remark Since every tree coalgebra A_t is finitely presentable, so is every forest coalgebra in the sense of the following

Definition A finite coproduct of tree coalgebras is called a *forest coalgebra.*

Which other Σ–coalgebras are finitely presentable? We are going to show that these are precisely the quotients of forest coalgebras obtained by identifying a finite number of bisimilar states. Recall here from [18] that, since polynomial functors preserve (weak) pullbacks, bisimulation equivalences are precisely the kernels of homomorphisms. Thus, for every Σ–coalgebra S the largest bisimulation equivalence is simply the kernel equivalence of the unique homomorphism $h\colon S \to T_\Sigma$. Given bisimilar states x and y, denote by

$$S/[x = y]$$

the quotient obtained of S by identifying x and y. This is just a coequalizer of the unique homomorphisms

$$\hat{x}, \hat{y}\colon A_t \to S$$

where $t = h(x) = h(y)$, $\hat{x}$ sends the root to x and $\hat{y}$ sends it to y.

We can describe the corresponding system $S/[x = y]$ more concretely: let $\sim$ be the smallest equivalence on S such that for every node of the tree t, if x' denotes the corresponding state in $\hat{x}[t]$ and y' the corresponding state in $\hat{y}[t]$, then $x' \sim y'$. Then the underlying set of $S/[x = y]$ is $S/{\sim}$, and the structure map is derived from α_S.

II.9 Proposition *Finitely presentable* Σ*–coalgebras are precisely the coalgebras* S *that can be presented via finitely many bisimilar pairs of states in a forest coalgebra.*

In other words, given pairs (x_i, y_i) of bisimilar states in a coalgebra $B = A_{t_1} + \ldots + A_{t_k}$ for $i = 1, \ldots n$, then the coalgebra

$$B/[x_1 = y_1, \ldots, x_n = y_n] = (\cdots (B/[x_1 = y_1])/[x_2 = y_2] \cdots) /[x_n = y_n]$$

is finitely presentable. And every finitely presentable coalgebra is isomorphic to one of that form.

Proof The first statement is obvious: denote by r_i the tree assigned to x_i (and y_i) by the unique homomorphism from B to T_Σ. This defines, by II.4, homomorphisms $\hat{x}_i, \hat{y}_i \colon A_{r_i} \to B$ for $i = 1, \ldots, n$. Let C denote the forest coalgebra $A_{r_i} + \ldots + A_{r_n}$, and let

$$\hat{x} \colon C \to B$$

be the homomorphism with components $\hat{x}_1, \ldots, \hat{x}_n$, analogously $\hat{y}$. Then we obviously have a coequalizer

$$C \underset{\hat{y}}{\overset{\hat{x}}{\rightrightarrows}} B \xrightarrow{\ c\ } B/\ [x_1 = y_1, \ldots, x_n = y_n]$$

in $\mathsf{Coalg}\Sigma$ (where c is the canonical map assigning to every element the equivalence class it lies in). Since C and B are finitely presentable, and the subcategory of all finitely presentable objects is (in every category) closed under finite colimits, it follows that $B/[x_1 = y_1, \ldots, x_n = y_n]$ is finitely presentable.

The latter statement that all finitely presentable coalgebras have the above form, follows from the fact, mentioned in II.7, that the set of all tree coalgebras is a strong generator of $\mathsf{Coalg}\Sigma$. Moreover, regular epimorphisms in $\mathsf{Coalg}\Sigma$ are precisely the surjective homomorphisms, see [6], thus, they are closed under composition. It follows from [11, 7.6] that the finitely presentable objects in $\mathsf{Coalg}\Sigma$ are precisely the coequalizers of parallel pairs between forest coalgebras. Now consider a parallel pair

$$f, g \colon C \to B$$

of homomorphisms between forest coalgebras $B = A_{t_1} + \ldots + A_{t_k}$ and $C = A_{r_1} + \ldots + A_{r_n}$. For the unique homomorphism $h \colon B \to T_\Sigma$ we have $hf = hg$, thus, $f(z)$ is bisimilar to $g(z)$ for every $z \in C$. Consequently, the n roots of trees in C give us n bisimilar pairs $x_i,\ y_i$ of states of B $(i = 1, \ldots, n)$ — and from Proposition II.4 we conclude that f has components $\hat{x}, \ldots, \hat{x}_n$ and g has components $\hat{y}_1, \ldots \hat{y}_n$. Thus, $f = \hat{x}$ and $g = \hat{y}$, and the coequalizer we consider has the form $B/[x_1 = y_1, \ldots, x_n = y_n]$. ◇

III Finite–Branching Transition Systems

III.1 Are categories $\mathsf{Coalg}F$ locally finitely presentable for all finitary endofunctors F of Set?

We show that the answer is negative even for the finite–power–set functor, $\mathcal{P}_f$. That is, we show that finite–branching transition systems are much less "algebraic" than the Σ–coalgebras. We start with an observation concerning the relationship between finiteness and finite presentability.

III.2 Lemma *Given a finitary endofunctor F, every finite coalgebra is finitely presentable in* $\mathsf{Coalg}F$.

Remark This lemma holds for finitary endofunctors of any category A: every coalgebra on a finitely presentable object of A is finitely presentable in $\mathsf{Coalg}F$.

Proof Let $\alpha\colon A \to FA$ be a coalgebra with A finitely presentable in A. Suppose that $B_i \xrightarrow{\beta_i} FB_i$ are coalgebras forming a filtered diagram with a colimit $(B_i,\ \beta_i) \xrightarrow{b_i} (B, \beta)$ $(i \in I)$. For every homomorphism $h\colon (A, \alpha) \to (B, \beta)$ there exists a factorization through some b_i (because A if finitely presentable and $B = \operatorname{colim} B_i$ in A), say $h = b_i \circ h'$. The parallel pair $\beta_i \circ h',\ Fh' \circ \alpha\colon A \to FB_i$ is merged by Fb_i:

$$Fb_i \circ (\beta_i \circ h') = \beta \circ b_i \circ h' = \beta \circ h = Fh \circ \alpha = Fb_i \circ (Fh' \circ \alpha).$$

Since A is finitely presentable and $FB = \operatorname{colim} FB_i$, it follows that for some morphism $u\colon (B_i, \beta_i) \to (B_j, \beta_j)$ of the given diagram the map Fu merges the pair, too: $Fu \circ \beta_i \circ h' = Fu \circ (Fh' \circ \alpha)$. Thus $h'' = u \circ h'\colon (A, \alpha) \to (B_j, \beta_j)$ is a homomorphism which forms the desired factorization:

$$\begin{aligned} h &= b_i \circ h' \\ &= b_j \circ u \circ h' \\ &= b_j \circ h''. \end{aligned}$$

The essential uniqueness of a homomorphism h'' with $h = b_i h' = b_j u h' = b_j h''$ is clear. ◇

III.3 Theorem *A coalgebra of $\mathcal{P}_f$ is finitely presentable iff it is finite.*

Proof I. The category $\mathsf{Coalg}\mathcal{P}_f$ is a subcategory of the category Gra of graphs (i.e., sets with a binary relation) and graph homomorphisms (i.e., functions preserving the binary relation, or simulations).

We first observe that the inclusion $E\colon \mathsf{Coalg}\mathcal{P}_f \hookrightarrow \mathsf{Gra}$ reflects colimits. That is: let $D\colon \mathsf{D} \to \mathsf{Coalg}\mathcal{P}_f$ be a diagram with a cocone $(Dd \xrightarrow{c_d} C)_{d\in \mathsf{D}^{obj}}$ in $\mathsf{Coalg}\mathcal{P}_f$. If the diagram ED has a colimit $(EDd \xrightarrow{Ec_d} EC)$ in Gra, then the given cocone is a colimit in $\mathsf{Coalg}\mathcal{P}_f$.

In fact, given another cocone $(Dd \xrightarrow{c'_d} C')$ in $\mathsf{Coalg}\mathcal{P}_f$, we know that there exists a unique graph homomorphism $h\colon C \to C'$ with $hc_d = c'_d$ for each d, and it is sufficient to verify that h is a coalgebra homomorphism. Consider an edge $h(x) \to y$ in C'. Since the cocone (c_d) is a colimit in Gra, for the element $x \in C$ there exists $d \in \mathsf{D}$ and an element $\bar{x} \in Dd$ with $x = c_d(\bar{x})$. Now $c'_d = hc_d$ is a coalgebra homormophism, thus, for the edge

$$c'_d(\bar{x}) = h(x) \to y$$

of C' there exists an edge $\bar{x} \to \bar{z}$ in Dd with $y = c'_d(\bar{z})$. Put $z = c_d(\bar{z})$, then $x \to z$ is an edge in C (since c_d is a graph homomorphism with $c_d(\bar{x}) = x$ and $c_d(\bar{z}) = z$). Since $y = c'_d(\bar{z}) = hc_d(\bar{z}) = h(z)$, this proves that h is a coalgebra homomorphism, as requested.

II. We know from III.2 that every finite graph is finitely presentable in $\mathsf{Coalg}\mathcal{P}_f$. Let us prove the converse: assume that G is a finitely presentable coalgebra. To prove that G is a finite graph, it is sufficient to show that it has no infinite one–to–one paths (i.e., paths without repetitions of nodes); the finiteness of G then can be proved as follows. For every finite set $M \subseteq G$ we form the induced subgraph of G on the following set $\bar{M}$ of nodes reachable from M:

$$\bar{M} = \{x \in G; \text{ a path exists from a vertex in } M \text{ to } x\}$$

and find out that $\bar{M}$ is finite. (In fact, $\bar{M} = \bigcup_{m\in M} \overline{\{m\}}$ is a finite union of graphs $\overline{\{m\}}$ which a) are finite–branching, b) have no infinite one–to–one path, and c) every vertex is reachable from m by a path — then $\overline{\{m\}}$ is finite since it easily follows from König's Lemma that a), b) and c) imply finiteness.) Thus, we have a filtered diagram whose objects are the graphs $\bar{M}$ for all $M \subseteq G$ finite, and morphisms are the inclusions $\bar{M}_1 \hookrightarrow \bar{M}_2$ for all finite sets $M_1 \subseteq M_2 \subseteq G$. Observe that every edge $x \to y$ of G has the property that if $x \in \bar{M}$ then $y \in \bar{M}$; therefore, the inclusion map $i_M\colon \bar{M} \hookrightarrow G$ is a coalgebra homomorphism. Also the inclusion maps $\bar{M}_1 \hookrightarrow \bar{M}_2$ of the above filtered diagram are coalgebra homomorphisms, and we see immediately that the cocone $i_M\colon \bar{M} \to G$ is a colimit of that diagram. Since G is finitely presentable, $\mathsf{hom}(G, -)$ preserves that colimit. Consequently, the element id_G of $\mathsf{hom}(G, G)$ is an image of some element h of $\mathsf{hom}(G, \bar{M})$ under $\mathsf{hom}(G, i_M)$ for some finite $M \subseteq G$ — in other words, there exists $h\colon G \to \bar{M}$ with $i_M h = \mathrm{id}$. This implies that i_M is surjective, i.e., $G = \bar{M}$, and so G is finite.

It remains to prove that G has no infinite one–to–one paths. Suppose that, to the contrary, such a path

$$x_0 \to x_1 \to x_2 \to \cdots$$

is given. We derive a contradiction.

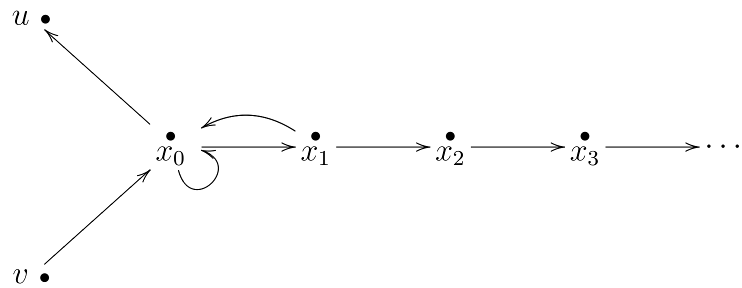
We define a new graph $\bar{G}$ by adding to G new pairwise distinct vertices x_0^*, x_1^*, $x_2^* \ldots$ and new edges as follows: for every edge $x_n \to y$ in G add to $\bar{G}$

(a) the edge $x_n^* \to y$,

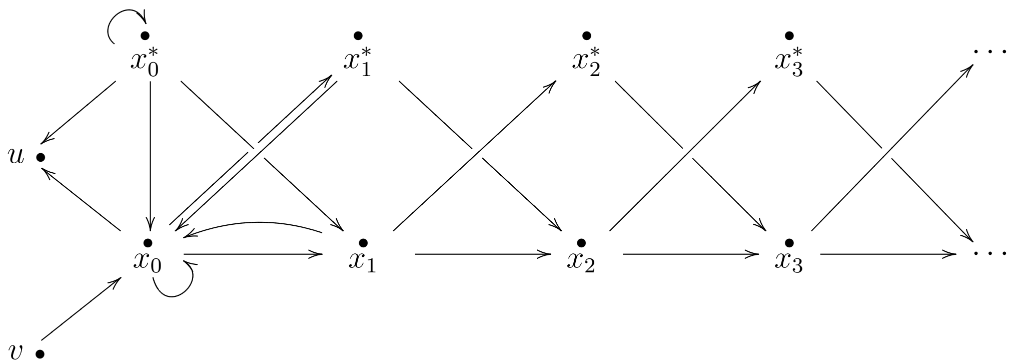
and

(b) the edge $x_n \to x_{n+1}^*$.

As an example, consider the following graph G:



then $\bar{G}$ is the following graph:



For every $k \in \omega$ denote by $\sim_k$ the equivalence relation on $\bar{G}$ with

$$x_0 \sim_k x_0^*,\ x_1 \sim_k x_1^*,\ \ldots,\ x_k \sim_k x_k^*$$

whereas all other equivalence classes are singleton sets. Let

$$\bar{G}_k = \bar{G} / \sim_k$$

be the quotient graph in the usual sense: nodes are the equivalence classes $[a]$ of nodes $a \in \bar{G}$, and there is an arrow $[a] \to [b]$ iff there is an arrow $a' \to b'$ in $\bar{G}$ with $a' \sim_k a$ and $b' \sim_k b$. Then we have canonical maps

$$c_k \colon \bar{G}_k \to G \text{ and } f_k \colon \bar{G}_{k+1} \to \bar{G}_k \; (k \in \omega)$$

merging x_n^* and x_n $(n \geq k+1)$ and leaving all other nodes unchanged, which are graph homomorphisms; this is clear for f_k and easy to check for c_k. But due to the construction of $\bar{G}$, they are also coalgebra homomorphisms. (In fact, c_k is a coalgebra homomorphism because for every edge $x_n = c_k(u) \to y$ in G there is a corresponding edge $x_n^* \to y$ in $\bar{G}$ which the canonical map $\bar{G} \to \bar{G}_k$ maps onto an edge $u \to v$ of $\bar{G}_k$ with $c_k(v) = y$; analogously for f_k.) Now the cocone $(c_k)_{k \in \omega}$ is a colimit of the ω–chain $(f_k)_{k \in \omega}$ in Gra, consequently, a colimit in $\mathsf{Coalg}\mathcal{P}_f$, too, see I. Since G is finitely presentable, $\mathsf{hom}(G, -)$ preserves that colimit. This implies that there exists n and a coalgebra homomorphism

$$h \colon G \to \bar{G}_n \text{ with } c_n h = \mathrm{id} \; .$$

In particular, from $c_n(h(x_{n+1})) = x_{n+1}$ we conclude that $h(x_{n+1}) = x_{n+1}$. However, the degree of x_{n+1} in G is smaller than in $\bar{G}_n$: besides all the edges with initial vertex x_{n+1} in G there is the extra edge $x_{n+1} \to x_{n+2}^*$. This is the desired contradiction: a coalgebra homomorphism is, obviously, nonincreasing on out–degrees. $\diamond$

III.4 Example The following transition system

$$G\colon \quad \underset{0}{\bullet} \to \underset{1}{\bullet} \to \underset{2}{\bullet} \to \cdots$$

is not finitely presentable in $\mathsf{Coalg}\mathcal{P}_f$.

Moreover, for every nonempty finite graph H there exists no coalgebra homomorphism from H to G. Thus, G is not a filtered colimit of finite graphs.

III.5 Corollary *The category $\mathsf{Coalg}\mathcal{P}_f$ is not locally finitely presentable.*

IV Infinitely Presentable Coalgebras

IV.1 Throughout this section λ denotes an uncountable regular cardinal.

We prove that for every λ–accessible endofunctor of Set the λ–presentable coalgebras are precisely the coalgebras on less than λ elements. Moreover, the category of coalgebras is locally λ–presentable. In particular, for every finitary endofunctor the category of coalgebras is locally ω_1–presentable (where $\omega_1 = \omega^+$ is the cardinal successor of ω).

We prove the result not only for endofuctors of Set, but for a class of accessible categories containing the locally presentable ones. See the Appendix for some basic definitions.

IV.2 Theorem *Let A be a λ–accessible category with colimits of ω–chains, where $\lambda > \aleph_0$. Then for every λ–accessible endofunctor F of A*

(1) a coalgebra is λ–presentable in $\mathsf{Coalg}F$ iff its underlying object is λ–presentable in A

and

(2) the category $\mathsf{Coalg}F$ is λ–accessible.

Remark The following proof, as mentioned in the introduction, uses techniques similar to those of Barr [10] and Aczel and Mendler [1], in particular, the statement (*) below generalizes the small coalgebra lemma of [1].

Proof I. Every coalgebra (A, α_A) with $A \in \mathsf{A}_\lambda$ is λ–presentable in $\mathsf{Coalg}F$. The proof is completely analogous to that of Lemma III.2.

II. Next we prove that the collection $\mathsf{B} \subseteq \mathsf{Coalg}F$ of all coalgebras with underlying objects in A_λ has the following property

(*) every F–coalgebra is a λ–filtered colimit of coalgebras from B.

This concludes the proof of (1) and (2). In fact, for (1) we now only need to prove that every λ–presentable coalgebra (A, α_A) fullfils $A \in \mathsf{A}_\lambda$. Since (A, α_A) is λ–presentable, it follows from (*) that (A, α_A) is a retract of a coalgebra in B. But A_λ is easily seen to be closed under retracts. Thus, $A \in \mathsf{A}_\lambda$. But (*) also proves that $\mathsf{Coalg}F$ is λ–accessible (or locally λ–presentable, provided that A is). In fact, the existence of colimits lifts from A to $\mathsf{Coalg}F$, since the forgetful functor

$$U \colon \mathsf{Coalg}F \to \mathsf{A}$$

creates colimits. Next, from (*) we conclude that every coalgebra is a λ–filtered colimit of λ–presentable coalgebras (see (I)) and that the essentially small collection B presents all λ–presentable coalgebras up to isomorphism.

Proof of (*) Let

$$C \xrightarrow{\alpha_C} FC$$

be an arbitrary coalgebra. Since A is a λ–accessible category, the object C is a canonical colimit of all morphisms $A \xrightarrow{a} C$ with A λ–presentable. More precisely, consider the comma–category

$$\mathsf{A}_\lambda \downarrow C$$

of all such morphisms. The forgetful functor

$$D_C \colon \mathsf{A}_\lambda \downarrow C \to \mathsf{A} \text{ given by } (A \xrightarrow{a} C) \mapsto A$$

has as a colimit the object C and the colimit cocone of all $A \xrightarrow{a} C$, see [7, 2.1.5].

We create the analogous diagram $D_{(C,\alpha_C)} \colon \mathsf{B} \downarrow (C, \alpha_C) \to \mathsf{Coalg}F$ of all homomorphisms from coalgebras in B into (C, α_C). Now the forgetful functor $U \colon \mathsf{Coalg}F \to \mathsf{A}$ induces a forgetful functor

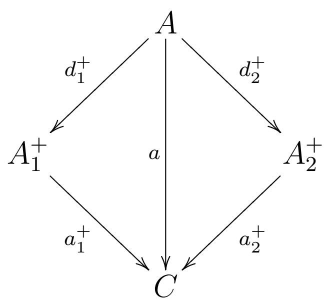
$$V \colon \mathsf{B} \downarrow (C, \alpha_C) \to \mathsf{A}_\lambda \downarrow C.$$

We are going to prove that V is *final* (i.e., for every object X of $\mathsf{A}_\lambda \downarrow C$ the comma–category $X \downarrow V$ is connected, in other words, $X \downarrow V$ is not a coproduct of two nonempty categories, [15]). This implies that from $C = \operatorname{colim} D_C$ it follows that $C = \operatorname{colim} D_C \cdot V$. Since U creates colimits, this proves $(C, \alpha) = \operatorname{colim} D_{(C,\alpha_C)}$ in $\mathsf{Coalg}F$.

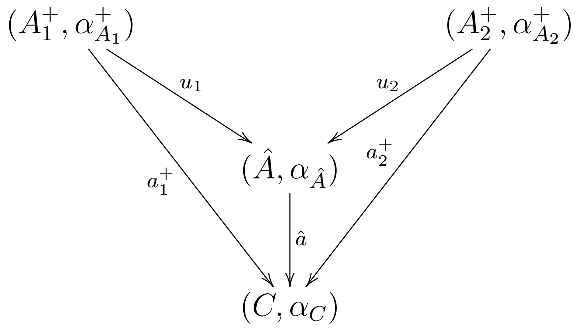
Thus our proof will be concluded by proving that

(a) for every morphism $a \colon A \to C$ with $A \in \mathsf{A}_\lambda$ there exists a coalgebra homomorphism $a^+ \colon (A^+, \alpha_{A^+}) \to (C, \alpha_C)$ with $(A^+, \alpha_{A^+}) \in \mathsf{B}$ through which a factors in A, and

(b) given two such factorizations through coalgebra homomorphisms a_1^+ and a_2^+ :



with A_1^+ and A_2^+ in B, there exists a third coalgebra homomorphism $\hat{a} \colon \hat{A} \to C$ with $\hat{A} \in \mathsf{B}$ together with factorizations of a_1^+ and a_2^+ in $\mathsf{B} \downarrow (C, \alpha_C)$:



In fact, (a) tells us that for every object $X = A \xrightarrow{a} C$ of $\mathsf{A}_\lambda \downarrow C$ the comma–category $X \downarrow V$ is nonempty. From (b) it follows that that comma-category is connected — thus, V is cofinal.

Proof of (a) The object A^+ will be defined as a colimit of an ω–chain

$$A = A_0 \xrightarrow{d_0} A_1 \xrightarrow{d_1} A_2 \cdots$$

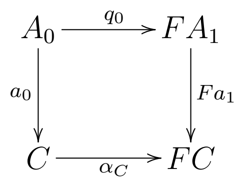
in A, where $d_n\colon (A_n, a_n) \to (A_{n+1}, a_{n+1})$ is an ω–chain in $\mathsf{A}_\lambda \downarrow C$ defined below. We will also define a natural transformation

$$q_n\colon A_n \to FA_{n+1}$$

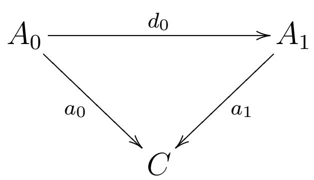
in A such that for all $n \in \omega$ we have

$$\alpha_C \circ a_n = Fa_{n+1} \circ q_n.$$

First Step Put $A_0 = A,\ a_0 = a$ and define (A_1, a_1) as follows. The morphism $\alpha_C a\colon A \to FC$ has a λ–presentable domain and the codomain is a λ–filtered colimit of $F \circ D_C$. Thus, $\alpha_C a$ factors, for some object (A_1, a_1) of $A_\lambda \downarrow C$, through the corresponding colimit morphism Fa_1:



Since $\mathsf{A}_\lambda \downarrow C$ is filtered, we can assume without loss of generality that (A_1, a_1) is chosen so that a morphism



exists in $\mathsf{A} \downarrow C$. This defines A_1, a_1, q_0, and d_0.

Induction Step Let $n > 0$. Given (A_n, a_n), we define (A_{n+1}, a_{n+1}) as follows. Factor the morphism

$$\alpha_C \circ a_n \colon A_n \to FC$$

whose domain is λ–presentable through some colimit map of the colimit $FC = \operatorname{colim} F \circ D_C$. That is, choose an object (A_{n+1}, a_{n+1}) of $\mathsf{A}_\lambda \downarrow C$ with the following factorization of $\alpha_C a_n$:

$$\begin{array}{ccc} A_n & \xrightarrow{q_n} & FA_{n+1} \\ {\scriptstyle a_n}\downarrow & & \downarrow{\scriptstyle Fa_{n+1}} \\ C & \xrightarrow[\alpha_C]{} & FC \end{array}.$$

For the desired naturality condition

$$q_n \circ d_{n-1} = Fd_n \circ q_{n-1}$$

we observe that the parallel pair in question

$$q_n \circ d_{n-1},\ Fd_n \circ q_{n-1} \colon A_{n-1} \to FA_{n+1}$$

is merged by the (colimit) map Fa_{n+1} of the λ–filtered colimit $FC = \operatorname{colim} F \circ D_C$:

$$\begin{aligned} Fa_{n+1} \circ (q_n \circ d_{n-1}) &= \alpha_C \circ a_n \circ d_{n-1} \\ &= \alpha_C \circ a_{n-1} \\ &= Fa_n \circ q_{n-1} \\ &= Fa_{n+1} \circ (Fd_n \circ q_{n-1}). \end{aligned}$$

Thus, we can assume without loss of generality that the desired equality holds: if not, we find a morphism $h \colon (A_{n+1}, a_{n+1}) \to (A'_{n+1}, a'_{n+1})$ of $\mathsf{A}_\lambda \downarrow C$ such that Fh merges the above parallel pair, and we substitute (A_{n+1}, a_{n+1}) with (A'_{n+1}, a'_{n+1}) (and q_n with $Fh \circ q_n$).

Finally, without loss of generality, our choice of (A_{n+1}, a_{n+1}) is such that a morphism

$$\begin{array}{ccc} A_n & \xrightarrow{d_n} & A_{n+1} \\ & {\scriptstyle a_n}\searrow \quad \swarrow{\scriptstyle a_{n+1}} & \\ & C & \end{array}$$

exists in $\mathsf{A}_\lambda \downarrow C$.

Let us form a colimit:

$$d_n^+ \colon A_n \to A^+ \quad (n < \omega)$$

of the ω–chain (A_n). Then A^+ carries an F–coalgebra structure $\alpha_{A^+} \colon A^+ \to FA^+$ defined by

$$\alpha_{A^+} \circ d_n^+ = Fd_{n+1}^+ \circ q_n \quad (n < \omega)$$

and, moreover, $a^+ = \operatorname{colim} a_n \colon A^+ \to C$ is a homomorphism. In fact the equality $\alpha_C \circ a^+ = Fa^+ \circ \alpha_{A^+}$ follows from

$$(\alpha_C \circ a^+) \circ d_n^+ = \alpha_C \circ a_n = Fa_{n+1} \circ q_n = Fa^+ \circ Fd_{n+1}^+ \circ q_n = (Fa^+ \circ \alpha_{A^+}) \circ d_n^+.$$

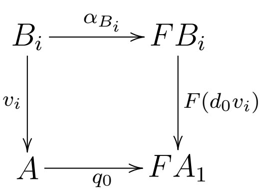
Since each A_n is λ–presentable and λ is assumed to be uncountable, it follows that A^+ is λ–presentable. And the desired factorization of a is

$$a = a^+ \circ d_0^+.$$

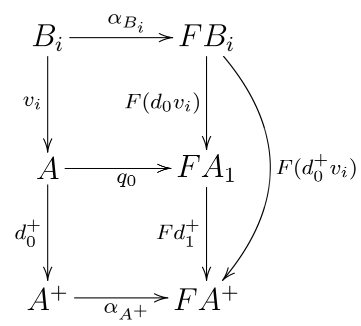
Before proving (b) let us observe that (a) generalizes to the following statement: suppose that, besides the morphism $a \colon A \to C$, also two coalgebras B_1, B_2 in B are given together with morphisms $v_i \colon B_i \to A$ in A such that $av_i \colon B_i \to C$ are homomorphisms for $i = 1, 2$. Then a^+ above can be constructed so that

$$d_0^+ \circ v_i \colon B_i \to A^+ \text{ are homomorphisms for } i = 1, 2.$$

In fact, all we have to do is to modify A_1 and q_0 in the first step of the above induction so that the following squares



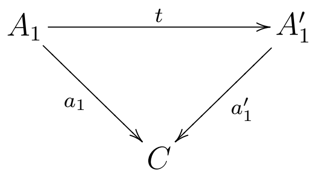
commute for $i = 1, 2$. It then follows that $d_0^+ v_i$ are homomorphisms because the lower square in the following diagram



commutes by definition of α_{A^+}. The above modification is trivial: recall that FC is a filtered colimit of the diagram $F \circ D_C$, and $Fa_1 \colon FA_1 \to FC$ is a colimit morphism. Now in the above squares the domains B_i are λ–presentable, and Fa_1 merges the two sides of the squares:

$$\begin{aligned} Fa_1(q_0 \circ v_i) &= \alpha_C \circ a \circ v_i && \text{by definition of } a_1, \\ &= Fa \circ Fv_i \circ \alpha_{B_i} && a \circ v_i \text{ is a homomorphism,} \\ &= Fa_1 \circ (Fd_0 \circ Fv_i \circ \alpha_{B_i}) && a = a_1 \circ d_0. \end{aligned}$$

Consequently, the two sides are also merged by some connecting morphism of the diagram $F \circ D_C$. That is, we have a morphism



of $\mathsf{A}_\lambda \downarrow C$ such that

$$Ft \circ (q_0 \circ v_i) = Ft \circ (Fd_0 \circ Fv_i \circ \alpha_{B_i}) \quad \text{for } i = 1, 2.$$

Now, if we substitute (A_1', a_1') for (A_1, a_1) in the above induction (and substitute d_0 by $t \circ d_0 \colon A_1 \to A_1'$, and q_0 by $Ft \circ q_0 \colon A \to FA_1'$, of course), then the last equation becomes

$$q_0 \circ v_i = Fd_0 \circ Fv_i \circ \alpha_{B_i},$$

which is the desired commutativity of the given squares.

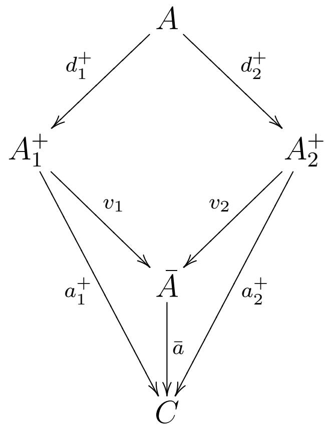
Proof of (b) Since A is λ–presentable, the category $A_\lambda \downarrow C$ is λ–filtered. Thus, for the objects $a_i^+ \colon A_i^+ \to C$ there exists an object

$$\bar{a} \colon \bar{A} \to C \quad \text{in } \mathsf{A}_\lambda \downarrow C$$

and morphisms

$$v_1 \colon a_1^+ \to \bar{a} \text{ and } v_2 \colon a_2^+ \to \bar{a} \text{ of } \mathsf{A}_\lambda \downarrow C.$$

Moreover, we can choose this object $\bar{a}$ so that the following diagram



commutes: in fact, since $C = \text{colim } D_C$ and the colimit morphism $\bar{a}\colon \bar{A} \to C$ merges $v_1 d_1^+$ and $v_2 d_2^+$, some morphism $t\colon (\bar{A}, \bar{a}) \to (\bar{A}', \bar{a}')$ of D_C also merges $v_1 d_1^+$ and $v_2 d_2^+$ (since $A \in \mathsf{A}_\lambda$) thus, a modification as above yields $v_1 d_1^+ = v_2 d_2^+$: substitute $(\bar{A}, \bar{a})$ by $(\bar{A}', \bar{a}')$ and v_i by $t v_i$ $(i = 1, 2)$.

Now apply the generalized statement (a) to $\bar{A} \xrightarrow{\bar{a}} C$ and the homomorphisms $\bar{a} v_i\colon A_i^+ \to C \quad (i = 1, 2)$. We obtain a coalgebra $(\hat{A}, \alpha_{\hat{A}})$ in B, a coalgebra homomorphism $\hat{a}\colon (\hat{A}, \alpha_{\hat{A}}) \to (C, \alpha_C)$, and a morphism $d_0^+\colon A \to \hat{A}$ such that

$$d_0^+ \circ v_i\colon (A_i^+, \alpha_{A_i^+}) \to (\hat{A}, \alpha_{\hat{A}})$$

are homomorphisms for $i = 1, 2$ and

$$\bar{a} = \hat{a} \circ d_0^+.$$

Put simply

$$u_i = d_0^+ \circ v_i.$$

Then

$$a_i^+ = \bar{a} \circ v_i = \hat{a} \circ d_0^+ \circ v_i = \hat{a} \circ u_i.$$

◇

Corollary *If λ is an uncountable regular cardinal, then for every λ–accessible endofunctor F of* Set *the category* $\mathsf{Coalg}F$ *is locally λ–presentable. And for every finitary functor F,* $\mathsf{Coalg}F$ *is locally ω_1–presentable.*

Example $\mathsf{Coalg}\mathcal{P}_f$ is locally ω_1–presentable, and so is the category $\mathsf{Coalg}\mathcal{P}_c$, where $\mathcal{P}_c$ is the countable–power–set functor. In fact, both of these functors are ω_1–accessible.

IV.3 Example of a λ–accessible functor whose category of coalgebras is not λ–accessible. This example has been used in [8]: see 3.4 and 3.5 there.

Consider an infinite regular cardinal λ and a set X of cardinality λ. Let A be the poset $\mathcal{P}(X)+\mathcal{P}_\lambda(X)$ (we denote, for $M \in \mathcal{P}(X)$ with cardinality less than λ, its copy in $\mathcal{P}_\lambda(X)$ by $\bar{M}$) whose order is the usual one on the components (inclusion) extended, for all $K \in \mathcal{P}(X)$ and for all $M \in \mathcal{P}(X)$ of cardinality $< \lambda$, as follows:

$$K < \bar{M} \iff K \subsetneq M$$
$$\bar{M} < K \iff M \subsetneq K$$

Then A is a λ–accessible poset and the functor (= order preserving function) $F\colon \mathsf{A} \to \mathsf{A}$ given by $FM = \bar{M}$, $F\bar{M} = M$, and $FK = K$ (for all $M \subset X$ of cardinality $< \lambda$, and all $K \subset X$ of cardinality λ) is λ–accessible. But $\mathsf{Coalg}F$ (the poset of all subsets of A of cardinality λ, ordered by inclusion) is not λ–accessible because no element K of $\mathsf{Coalg}F$ is λ–presentable.

Open problem Is there a λ–accessible endofunctor such that the category of coalgebras is not λ^+–accessible?

IV.4 Remark The above result extends, for λ–accessible endofunctors of Set, to covarieties of coalgebras. Recall from [18] that a full subcategory K of $\mathsf{Coalg}F$ is called a *covariety* provided that it is closed under coproducts, subcoalgebras and quotient coalgebras in $\mathsf{Coalg}F$. Here by a *subcoalgebra* of a coalgebra (A, α_A) is meant a coalgebra (B, α_B) where B is a subset of A and the inclusion function $B \hookrightarrow A$ is a homomorphism. Covarieties are, as shown in [5], precisely the coequationally specified classes of coalgebras, but we do not use this fact in the present paper.

In the proof of the next result we use the fact, established in [5], that if an endofunctor F of a category A preserves monomorphisms, then the forgetful functor

$$U\colon \mathsf{Coalg}F \to \mathsf{A}$$

creates (regular epi, mono)–factorizations. That is, given a homomorphism $h\colon (A, \alpha_C) \to (B, \alpha_B)$ of F–coalgebras and a factorization of h as a regular epimorphism $e\colon A \to C$ followed by a monomorphism $m\colon C \to B$, then C carries a unique structure of an F–coalgebra making both e and m homomorphisms.

Recall also from [9] that for every endofunctor F of Set there exists an endofunctor F' preserving monomorphisms and pullbacks of monomorphisms such that F and F' coincide on all nonempty sets and nonempty functions. Consequently, their categories of coalgebras are isomorphic

$$\mathsf{Coalg}F \cong \mathsf{Coalg}F'.$$

IV.5 Corollary *Let F be a λ–accessible endofunctor of* Set *where $\lambda > \aleph_0$. For every covariety K of F–coalgebras, the following holds:*

1. *A coalgebra is λ–presentable in* K *iff its underlying set has less than λ elements,*

and

2. K *is a locally λ–presentable category.*

Proof This can be derived from Theorem IV.2. Following the preceding remark, we can assume without loss of generality that F preserves monomorphims.

Every covariety K is closed in $\mathsf{Coalg}F$ under colimits. Consequently, every coalgebra of K whose underlying set has less than λ elements (and which is thus λ–presentable in $\mathsf{Coalg}F$) is λ–presentable in K. We will show that every coalgebra of K is a λ–filtered colimit of coalgebras of that type:

Given (A, α_A) in K, we know that there is a λ–filtered diagram D of F–coalgebras $(A_i, \alpha_{A_i}), i \in I$, with A_i λ–presentable with a colimit

$$c_i \colon (A_i, \alpha_{A_i}) \to (A, \alpha) \quad (i \in I)$$

in $\mathsf{Coalg}F$. Since F preserves monomorphisms, the forgetful functor $\mathsf{Coalg}F \to \mathsf{Set}$ creates epi–mono–factorizations of morphisms. Factoring c_i as an epimorphism $e_i \colon A_i \to B_i$ followed by a monomorphism $m_i \colon B_i \to A$ for each $i \in I$, we thus obtain, due to the diagonal fill–in, unique F–coalgebra structures α_{B_i} turning both e_i and m_i into homomorphisms. And the diagonal fill–in also provides us with a λ–filtered diagram D' of these coalgebras together with a natural transformation $D \to D'$ having components e_i and a cocone

$$m_i \colon (B_i, \alpha_{B_i}) \to (A, \alpha) \quad (i \in I).$$

Now D' is a diagram in K because K is closed under subcoalgebras in $\mathsf{Coalg}F$. And each B_i is λ–presentable in Set, being a quotient of A_i.

By the same argument as in the proof of Theorem IV.2 a λ–presentable coalgebra (C, α_C) in C is a split subobject of a coalgebra (D, α_D) in C which is λ–presentable in $\mathsf{Coalg}F$; thus (C, α_C) is λ–presentable in $\mathsf{Coalg}F$. The rest follows from Theorem IV.2. $\diamond$

V Bounded Functors

V.1 The present section is denoted to endofunctors of Set. We denote by λ an infinite regular cardinal (ω not excluded). Recall the concept of a polynomial endofunctor H_Σ in II.1; we call it λ*–ary* provided that all arities are smaller than λ. By a *quotient* of a functor F is meant a functor G for which a natural epi-transformation $F \to G$, i.e., a natural transformation with epimorphic components, exists.

V.2 Proposition *The following properties of set functors F are equivalent*

(i) F is λ–accessible;
(ii) every element of FS lies, for some subset $s\colon S' \hookrightarrow S$ of less than λ elements, in the image of Fs;
(iii) F is a quotient of a λ–ary polynomial functor.

Proof (i) $\Longrightarrow$ (iii): This is a consequence of the fact that every set is a λ–directed colimit of sets of of less than λ elements. Denote by Σ the signature whose n–ary equation symbols, for any cardinal $n < \lambda$, are precisely the elements of Fn. It follows easily from the Yoneda Lemma that the following defines an epitransformation $\varepsilon\colon H_\Sigma \to F$: given a set S and an element (f, σ) of $H_\Sigma S$, where $\sigma \in Fn$ and $f\colon n \to S$ is an n–tuple of elements, put $\varepsilon_S(f, \sigma) = Ff(\sigma)$.

(iii) $\Longrightarrow$ (ii): Every λ-ary polynomial functor has property (ii), and this is obviously inherited by all quotient functors.

Finally, we verify that (ii) implies (i). Let $D\colon \mathsf{D} \to \mathsf{Set}$ be a λ–filtered diagram with a colimit $(c_d\colon Dd \to C)_{d\in\mathsf{D}}$. In order to prove that the cone of $FC_d,\ d \in \mathsf{D}$ is a colimit of FD, we need to verify that

(a) every element of FC lies in the image of FC_d for some $d \in \mathsf{D}$

and

(b) for every $D \in \mathsf{D}$ and every subset $u\colon U \hookrightarrow FD_d$ of less than λ elements which Fc_d merges to one element there exists a morphism $f\colon d \to \bar{d}$ of D such that FDf also merges U to one element.

In fact (a) follows from (ii) because, for $C = S$, given a subset as in (ii), there exists $d \in \mathsf{D}$ such that s factorizes through c_d. To prove (b), use (ii) to obtain a subset $s\colon S \hookrightarrow Dd$ of less than λ elements such that $U \subset [FsFS]$. Since D is λ–filtered there exists $f\colon d \to \bar{d}$ such that $c_{\bar{d}}$ restricted to the image $\bar{s}\colon \bar{S} \hookrightarrow D\bar{d}$ of S under Df is one–to–one. Without loss of generality we can assume $S \neq \emptyset$, then $\bar{S} \neq \emptyset$, consequently, $c_{\bar{d}} \cdot \bar{s}\colon \bar{S} \to C$ is a split monomorphism — thus, $Fc_{\bar{d}} \cdot F\bar{s}$ is a monomorphism. Consequently, if Fc_d $(= Fc_{\bar{d}} \cdot FDf)$ merges two elements of the image of Fs, also FDf merges these elements. This proves that FDf merges U to one element. ◇

V.3 Remark We want to compare the concept of bounded functors of Kawahara and Mori, see Appendix, to that of accessibility. In order to do this, it is convenient to modify the former concept a bit.

Definition A set functor F called *strictly bounded at* λ, for a regular cardinal λ, provided that every element of an F–coalgebra lies in a subcoalgebra of less than λ elements.

Examples

1. H_Σ, for a finitary signature Σ, is bounded at ω. Unless all arities are 0, is is not strictly bounded at ω.
2. For λ uncountable, H_Σ is strictly bounded at λ iff Σ is a λ–ary signature. For example, the functor $H_\Sigma \colon S \mapsto S^\omega$ is strictly bounded at ω_1 (= bounded at ω).

V.4 Theorem *For an uncountable regular cardinal λ, a set functor is λ–accessible iff it is strictly bounded at λ.*

Proof I. Let F be λ–accessible. By Remark IV.4 we can assume, without loss of generality that F preserves monomorphisms. By Theorem IV.2 for every coalgebra C there exists a λ–filtered colimit $(C_i \xrightarrow{c_i} C)_{i\in I}$ where each C_i has less than λ elements. Since colimits of coalgebras are formed on the level of sets, every element $x \in C$ is contained in the image of c_i for some $i \in I$. This image is a subcoalgebra C_i' of C (because F preserves monomorphisms), thus $x \in C_i'$ where C_i' has less than λ elements since C_i does.

II. For a functor F strictly bounded at λ we form the diagram of all "elements of F on sets of cardinalities $< \lambda$" and show that F is the corresponding colimit of hom–functors. Each of the hom–functors $\mathsf{hom}(X,-)$, where $\mathsf{card} X < \lambda$, is obviously λ–accessible, thus, F is λ–accessible since colimits commute with colimits. By Remark IV.4 we can assume that F preserves monomorphisms and pullbacks of monomorphisms.

Let D be the (essentially small) category of all pairs (X,x) where X is a set of cardinality $< \lambda$ and $x \in FX$, with morphisms $f : (X,x) \to (X',x')$ all functions $f : X' \to X$ with $Ff(x') = x$. We prove that F is a colimit of the diagram $V \colon \mathsf{D} \to [\mathsf{Set},\mathsf{Set}]$ where $V(X,x) = \mathsf{hom}(X,-)$ with the colimit cocone $f_{(X,x)}$ having components

$$f^Y_{(X,x)} \colon \mathsf{hom}(X,Y) \to FY, \quad q \longmapsto Fq(x) \ \text{ for all } \ q \colon X \to Y.$$

That is, we prove that for every set Y

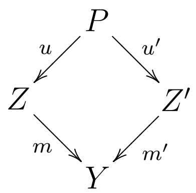
(a) the maps $f^Y_{(X,x)}$ are collectively epimorphic, and
(b) whenever $f^Y_{(X,x)}(q) = f^Y_{(X',x')}(q')$ then q is connected with q' by a zig–zag in the diagram of elements of V composed with the evaluation–at–Y, $eval_Y\colon [\mathsf{Set}, \mathsf{Set}] \to \mathsf{Set}$.

Proof of (a): Given $y \in FY$, for the coalgebra $(Y, const(y))$ there exists a homomorphism $h : (D, \alpha_D) \to (Y, const(y))$ with card $D < \lambda$ which fulfills $D \neq \emptyset$ if $Y \neq \emptyset$. For $Y \neq \emptyset$ choose $d_0 \in D$, then $(D, d) \in \mathsf{D}$ with $d = \alpha_D(d_0)$

$$f^Y_{(D,d)}(h) = Fh(\alpha_D(d)) = const(y) \cdot h(d) = y.$$

The case $Y = \emptyset$ is trivial since $(Y, y) \in \mathsf{D}$.

Proof of (b): We have $Fq(x) = Fq'(x')$ for some $q\colon X \to Y$ and $q'\colon X' \to Y$. Factor q as an epimorphism $e\colon X \to Z$ followed by an injection $m\colon Z \hookrightarrow Y$ and put $z = Fq(x)$; analogously e', m', and z'. By assumption, F preserves the pullback



The equality $Fm(z) = Fq(x) = Fq'(x') = Fm'(z')$ thus guarantees that there exists $p \in FP$ with $z = Fu(p)$ and $z' = Fu'(p')$. And since $\mathsf{card}P \leq \mathsf{card}(Z \times Z') \leq \mathsf{card}(X \times X') < \lambda^2 = \lambda$, we obtain an object (P, p) of D with morphisms

$$(X, x) \xleftarrow{e} (Z, z) \xrightarrow{u} (P, p) \xleftarrow{u'} (Z', z') \xrightarrow{e'} (X', x')$$

forming the desired zig-zag. $\diamond$

V.5 Remark For $\lambda = \omega$ all we can say is F finitary ($= \omega$–accessible) $\Rightarrow F$ bounded at $\omega \Leftrightarrow F$ ω_1–accessible. This follows from V.4 since "bounded at ω" means "strictly bounded at ω_1".

Corollary *For every regular cardinal λ we have*

$$F\ \lambda\text{–}accessible \Rightarrow F\ bounded\ at\ \lambda \Rightarrow F\ \lambda^+\text{–}accessible.$$

V.6 As a converse to Corollary IV.2 we obtain from the previous result

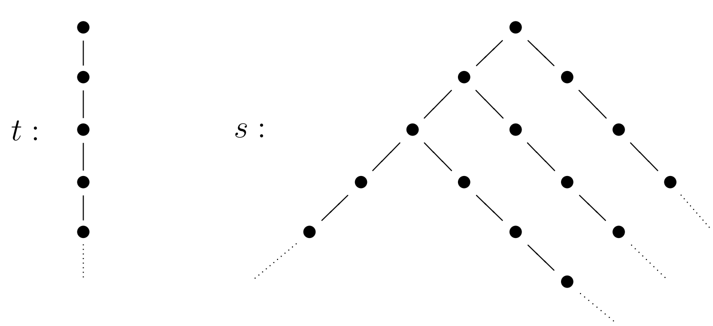
Corollary *Let F be a* Set*-functor such that* $\mathsf{Coalg}F$ *is a locally presentable category. Then F is accessible.*

Proof Let $\mathsf{Coalg}F$ be locally λ-presentable. Denote by Gen_λ a representative set of strong quotients of the λ-presentable coalgebras; choose an infinite regular cardinal $\mu \geq \lambda$ which is an upper bound of the cardinalities of the coalgebras from Gen_λ. By [7, 1.70] each F-coalgebra (C, α_C) is a λ-directed colimit of subcoalgebras which belong to Gen_λ. Since colimits in $\mathsf{Coalg}F$ are formed as in Set, each $c \in C$ is contained in a subcoalgebra of (C, α_C) of cardinality at most μ. Thus, F is bounded at μ, hence accessible. $\diamond$

VI Conclusions and Further Research

For coalgebras of a given polynomial type Σ we have seen that final coalgebras (of Σ–labelled trees) play a certain double role: every element t of the final coalgebra can be viewed as a coalgebra per se, where the states now are the nodes of the tree t. These coalgebras have a universal property resembling that of free algebras on one generator. As a consequence, we have shown that the category $\mathsf{Coalg}\Sigma$ exhibits a surprisingly "algebraic" behavior. We expect that the role of these tree coalgebras will be investigated further. For example, the multiequational semantics studied in [3] for algebraic specifications might have an analogy in case of Σ–coalgebras.

For the finite–power–set functor $\mathcal{P}_f$ the behavior of the category $\mathsf{Coalg}\mathcal{P}_f$ is much less algebraic. And there is no evident way how elements of a final coalgebra could be turned into $\mathcal{P}_f$–coalgebras. In fact, recall from [10] that a final $\mathcal{P}_f$–coalgebra is the set of all finite–branching extensional trees modulo the following congruence: two trees t and s are congruent iff the cuttings $t\,|_n$ and $s\,|_n$ at level n have isomorphic extensional quotients for all $n = 1, 2, 3, \ldots$. (Recall that a tree is extensional iff different children of one parent always represent non–isomorphic subtrees. Every tree t has a smallest congruence such that the quotient tree is extensional; that quotient is called the extensional quotient of t.) Example of congruent extensional trees:



Now although t and s represent the same element of the final $\mathcal{P}_f$–coalgebra, they are very different as trees. So we have no intuition of a $\mathcal{P}_f$–coalgebra associated with the element $[t] = [s]$. It then seems that the above procedure of formation of tree Σ–coalgebras has no generalization to arbitrary endofunctors of Set.

This leaves open the following questions: which endofunctors of Set have the property that multifree coalgebras exist?

The second main topic of our paper is presentability of coalgebras. In universal algebra this is important because to say that an algebra A is finitely presentable within a given equational class means precisely the possibility of using finite data to specify A (up to isomorphism) in the equational class. There are two equivalent ways of how to say that A is a finitely presentable algebra:

(a) A is a quotient of a free algebra on finitely many generators modulo a finitary generated congruence,

or

(b) $\mathsf{hom}(A, -)$ is a finitary functor.

We have chosen (b) as our basis for studying finitely presentable coalgebras. And we have shown a surprising parallel between algebra and coalgebra in case of polynomial functors: finitely presentable Σ–coalgebras are precisely the quotients of members of a multifree coalgebra on finitely many generators modulo a finitely generated congruence. On the other hand, the name "finitely presentable" suggests that a presentation by finite data is possible — this is true for Σ–coalgebras if (and only if) the elements of the final Σ–coalgebra are considered (in spite of being infinite trees) as data units. That is, considering an implementation of trees as granted, finitely presentable Σ–coalgebras are given by finitely many of those plus a finitely generated congruence.

Infinite presentability of coalgebras reduces to the corresponding size restriction on the underlying set of the coalgebra. This is hardly surprising, e.g., the same holds for Σ–algebras of any countable finitary signature Σ: a Σ–algebra is λ–presentable, where λ is an uncountable regular cardinal, iff it has less than λ elements. What is surprising is how involved the proof of that (simple?) fact for coalgebras has turned out to be. It would be interesting to search for a simpler proof — so far, we have not found any.

The final topic of our paper concerned the relationship between the concept of accessible functors, introduced by Makkai and Paré [16] and widely used in category theory, and bounded functor of Kawahara and Mori [14]. These concepts are equivalent, as we have proved in [5] for all set functors. In the present paper we have sharpened that to

$$\lambda\text{–accessible} \;\Rightarrow\; \text{bounded at } \lambda \Rightarrow \lambda^+\text{–accessible.}$$

An interesting question is whether the concept of boundedness for set functors can be generalized to endofunctors of arbitrary categories, such that the above implications remain valid.

Appendix

We recall some concepts from [11], [14] and [16]. Below λ always denotes an infinite regular cardinal, i.e., an infinite cardinal λ which is not a sum of less than λ cardinals smaller than λ. A category D is called *λ–filtered* if every subcategory of less than λ morphisms has a cocone in D; colimits of diagrams over small λ–filtered categories are called *λ–filtered colimits* (example: colimits of λ–chains).

Definition A functor F preserving λ-filtered colimits is called a *λ-accessible functor*. For $\lambda = \omega$ we call F *finitary.*
An endofunctor of Set is called *bounded at* λ provided that every element of every coalgebra is contained in some subcoalgebra of at most λ elements.

Examples

1. A polynomial functor of a (possibly infinitary) signature Σ is λ–accessible iff all arities of operations in Σ are smaller than λ. And it is bounded at λ iff all arities are smaller or equal to λ.
2. A hom-functor $\mathsf{hom}(A,-)\colon \mathsf{Set} \to \mathsf{Set}$ is λ-accessible iff A has less than λ elements. It is bounded at λ iff A has at most λ–elements.
3. The functor $\mathcal{P}_\lambda$ of all subsets of less than λ elements is λ–accessible. E.g., $\mathcal{P}_f$ is finitary and the countable power-set functor $\mathcal{P}_c$ is bounded at ω but not finitary.
4. The power-set functor $\mathcal{P}$ is not accessible.

Definition An object A of a category A is called *λ-presentable* provided that the hom–functor $\mathsf{hom}(A,-)\colon \mathsf{A} \to \mathsf{Set}$ is λ-accessible. For $\lambda = \omega$ we call A *finitely presentable.*

Examples

1. A set is λ–presentable in Set iff it has less than λ elements.

2. An algebra is λ–presentable in a variety iff it can be presented by less than λ generators and equations. If the variety is finitary this is, for all uncountable cardinals λ, equivalent to having less than λ elements.

Definition A category A is called λ–*accessible* provided that it has λ–filtered colimits and a set of λ–presentable objects whose closure under λ–filtered colimits is all of A. If, moreover, A has colimits, then it is called *locally* λ–*presentable* .
For $\lambda = \omega$ we call A *finitely accessible* and *locally finitely presentable*, respectively.

Examples Set is locally finitely presentable, the full subcategory of all nonempty sets is finitely accessible. The category ω–CPO of all posets with a least element and joins of ω–chains, where morphisms are functions preserving joins of ω–chains, is ω_1–accessible.
Locally finitely presentable categories are precisely the categories of models of finitary essentially algebraic theories (see [7]).

References

[1] P. Aczel and M. Mendler. A final coalgebra theorem. *Lect. Notes Comp. Sci.* **389**, Springer 1989, 357–365.

[2] J. Adámek. Free algebras and automata realizations in the language of categories. *Comment. Math. Univ. Carolinae* **15** (1974), 589–602.

[3] J. Adámek, M. Hébert and J. Rosický. On abstract data types presented by multiequations, *Theoretical Computer Science* **275** (2002), 427–462.

[4] J. Adámek and V. Koubek. On the greatest fixed point of a functor. *Theoretical Computer Science* **150** (1995), 57–75.

[5] J. Adámek and H.-E. Porst. From varieties of algebras to covarietes of coalgebras. *Electronic Notes in Theoretical Computer Science* **44.1** (2001).

[6] J. Adámek and H.-E. Porst. On Varieties and Covarieties in a Category. To appear in *Math. Struct. in Computer Science.*

[7] J. Adámek and J. Rosický. Locally Presentable and Accessible Categories. *Cambridge University Press*, 1994. Cambridge.

[8] J. Adámek and L. Sousa. How accessible are categories of algebras? To appear in *J. Pure Appl. Algebra.*

[9] J. Adámek and V. Trnková. Automata and Algebras in Categories. *Kluwer Acad. Publ.*, 1990. Dordrecht

[10] M. Barr. Terminal coalgebras in well-founded set theory. *Theoretical Computer Science* **114** (1993), 299–315.

[11] P. Gabriel and F. Ulmer. Lokal präsentierbare Kategorien. *Lect. Notes Mathematics* **221**. Springer–Verlag, 1971. Berlin.

[12] H. P. Gumm. Elements of the general theory of coalgebras. *Preprint*, 1999

[13] H. P. Gumm and T. Schröder. Products of coalgebras. *Algebra Universalis* **46** (2001), 163–185.

[14] Y. Kawahara and M. Mori. A small final coalgebra theorem. *Theoretical Computer Science* **233** (2000), 129–145.

[15] S. MacLane. Categories for the working mathematician. *Springer–Verlag*, 1971. Berlin.

[16] M. Makkai and R. Paré. Accessible categories: the foundations of categorical modeltheory. *Contemporary Math.* **104**. Amer. Math. Soc. Providence.

[17] J. Power and H. Watanabe. An axiomatics for categories of coalgebras. *Electronic Notes in Comp. Sci.* **11** (1998).

[18] J.J.M.M. Rutten. Universal coalgebra: a theory of systems. *Theoretical Computer Science* **249** (2000), 3–80.