

## ON CORINGS AND COMODULES

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*To my friend and colleague Jiří Rosický on his 60th birthday*

ABSTRACT. It is shown that the categories of  $R$ -coalgebras for a commutative unital ring  $R$  and the category of  $A$ -corings for some  $R$ -algebra  $A$  as well as their respective categories of comodules are locally presentable.

## INTRODUCTION

The categories under consideration are defined as the categories of comonoids and comonoid-coactions in certain monoidally closed categories as follows:

- a) Given a commutative unital ring  $R$ 
  - the category  $\mathbf{Coalg}_R$  of  $R$ -coalgebras is the category of comonoids in  $(\mathbf{Mod}_R, - \otimes_R -, R)$ ,
  - the categories  $\mathbf{Comod}_A$  and  ${}_A\mathbf{Comod}$  of right resp. left  $A$ -comodules for an  $R$ -coalgebra  $A$  are the corresponding categories of right resp. left  $A$ -coactions on  $R$ -modules.
- b) Given an  $R$ -algebra  $A$ ,
  - the category  $\mathbf{Coring}_A$  of  $A$ -corings is the category of comonoids in  $({}_A\mathbf{Mod}_A, - \otimes_A -, A)$ , where  ${}_A\mathbf{Mod}_A$  denotes the category of  $A$ ,  $A$ -bimodules,
  - the categories  $\mathbf{Comod}_{\mathcal{C}}$  and  ${}_{\mathcal{C}}\mathbf{Comod}$  of right resp. left  $\mathcal{C}$ -comodules for an  $A$ -coring  $\mathcal{C}$  again are the respective categories of  $\mathcal{C}$ -coactions on left (right)  $A$ -modules.

Only scattered results are known about the structure of these categories: completeness of these categories is a rather trivial fact (see Fact 2 below), cocommutative coalgebras form a cartesian closed category ([3]),  $\mathbf{Comod}_A$  is locally presentable and comonadic over  $\mathbf{Mod}_R$  ([11]). A first systematic approach to completeness — limited, however, to the case where the rings involved are regular — was presented in [8] using the dualized construction of colimits in varieties.

In this note we will offer a unified approach to these and many new results by considering the categories in question as subcategories of certain categories of

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functor-coalgebras  $\mathbf{Coalg}F$ ; using methods from the theory of accessible categories (see [2], [8]) we will show first that these categories are complete and then, in a second step, that this also holds for their interesting subcategories  $\mathbf{Coalg}_R$ ,  $\mathbf{Coring}_A$ , and  $\mathbf{Comod}_A$ . In fact we will prove even more: all categories mentioned so far are locally presentable categories.

Local presentability of  $\mathbf{Coalg}_R$  generalizes Sweedler's so-called *Fundamental Theorem of Coalgebras* (see [10], [5]), which states that every coalgebra (over some field  $k$ ) is a directed colimit of coalgebras whose underlying vector space is finite dimensional, hence of finitely presentable coalgebras, since the following is easy to prove:

**Proposition 1.** *A  $k$ -coalgebra is finitely presentable iff its underlying  $k$ -vector space is of finite dimension.*

Note, however, that neither this proposition nor Sweedler's prove generalize to arbitrary rings.

### 1. THE CATEGORIES $\mathbf{Coalg}T_I$ AND $\mathbf{Coalg}M_A$

Let  $(\mathbf{C}, \otimes, I)$  be any of the monoidally closed categories  $(\mathbf{Mod}_R, - \otimes_R -, R)$  or  $({}_A\mathbf{Mod}_A, - \otimes_A -, A)$  mentioned in the introduction. We consider the following functors:

$$T_n: \begin{array}{ccc} \mathbf{C} & \longrightarrow & \mathbf{C} \\ C & \longmapsto & \otimes^n C \end{array}$$

$$T_I: \begin{array}{ccc} \mathbf{C} & \longrightarrow & \mathbf{C} \\ C & \longmapsto & T_2 C \times I = (C \otimes C) \times I \end{array}$$

$${}_A M: \begin{array}{ccc} \mathbf{C}' & \longrightarrow & \mathbf{C}' \\ C & \longmapsto & A \otimes C \end{array}$$

$$M_A: \begin{array}{ccc} \mathbf{C}' & \longrightarrow & \mathbf{C}' \\ C & \longmapsto & C \otimes A \end{array}$$

where  $A$  is a monoid in  $(\mathbf{C}, \otimes, I)$  and  $\mathbf{C}' = \mathbf{C}$  in the commutative case and, for  $A$  non-commutative,  $\mathbf{C}' = A\text{-Mod}$  and  $\mathbf{Mod}\text{-}A$ , the categories of left and right  $A$ -modules respectively, with  $- \otimes -$  the obvious bifunctor  $\mathbf{C} \times \mathbf{C}' \rightarrow \mathbf{C}'$ .

Then  $\mathbf{Coalg}_R$  and  $\mathbf{Coring}_A$ , respectively, are the full subcategories of  $\mathbf{Coalg}T_I$  (w.r.t. the appropriately chosen  $(\mathbf{C}, \otimes, I)$  — see above) spanned by those  $T_I$ -coalgebras  $\mathbb{C} = (C, C \xrightarrow{\langle m, e \rangle} (C \otimes C) \times I)$  which make the following diagrams commute

$$(1) \quad \begin{array}{ccc} C & \xrightarrow{m} & C \otimes C \\ m \downarrow & & \downarrow m \otimes \text{id}_C \\ C \otimes C & \xrightarrow{\text{id}_C \otimes m} & C \otimes C \otimes C \end{array}$$

$$(2) \quad \begin{array}{ccc} C & & \\ m \downarrow & \searrow r_C & \\ C \otimes C & \xrightarrow{\text{id}_C \otimes e} & C \otimes I \end{array} \quad \text{and} \quad (3) \quad \begin{array}{ccc} C & \xrightarrow{m} & C \otimes C \\ & \searrow l_C & \downarrow e \otimes \text{id}_C \\ & & I \otimes C \end{array}$$

Similarly, the various categories of comodules are subcategories of  $\mathbf{Coalg}M_A$  and  $\mathbf{Coalg}_A M$ , respectively, defined by the obvious diagrammatic axioms.

We will need the following results, which are easy to prove (see [1], [8]).

**Fact 2.**

1. For each functor  $F: \mathbf{C} \rightarrow \mathbf{C}$  the category  $\mathbf{Coalg}F$  is cocomplete provided the category  $\mathbf{C}$  is so.
2. The categories of comonoids and comonoid-coactions are closed in their respective functor-categories under colimits.

Clearly  $M_A$  is accessible since it is a left adjoint. Also, if  $F$  is an accessible endofunctor on a category  $\mathbf{A}$  with biproducts, then  $F \times A = F + A$  is accessible for each object  $A$  in  $\mathbf{A}$ . Thus,  $T_I$  is accessible by the following fact:

**Lemma 3.** *Let  $(\mathbf{C}, - \otimes -, I)$  be a monoidally closed category and  $F: \mathbf{C} \rightarrow \mathbf{C}$  a finitary functor. Then  $\hat{F}: \mathbf{C} \rightarrow \mathbf{C}$  with  $\hat{F}(C) = C \otimes FC$  is finitary. In particular, the functor  $T_n: \mathbf{C} \rightarrow \mathbf{C}$  with  $T_n C = \otimes^n C$  preserves directed limits.*

**Proof.** If  $D: \mathbf{I} = (I, \leq) \rightarrow \mathbf{C}$  is a directed diagram in  $\mathbf{C}$  with colimit  $D_i \xrightarrow{d_i} C$  the colimit of the diagram  $\tilde{D}: \mathbf{I} \times \mathbf{I} \rightarrow \mathbf{C}$  with  $\tilde{D}(i, j) = D_i \otimes FD_j$  can be computed as  $D_i \otimes FD_j \xrightarrow{d_i \otimes Fd_j} C \otimes C$  since  $F$  and each  $X \otimes -$  and  $- \otimes Y$  preserve (directed) colimits. Finally, the diagram  $\hat{F} \circ D$  is a cofinal subdiagram of  $\tilde{D}$ .  $\square$

**Remark 4.** Since the monoidal categories under consideration are varieties also  $T_I$  preserves directed colimits (see also [8]). As a consequence of these observations we obtain that the underlying functors  $|-|$  of  $\mathbf{Coalg}T_I$  and  $\mathbf{Coalg}M_A$  into  $\mathbf{C}$  and  $\mathbf{C}'$ , respectively, have right adjoints and thus are comonadic (see [1]); their domains are also accessible by the following observation.

Recall that for functors  $F, G: \mathbf{K} \rightarrow \mathbf{L}$  the *inserter of  $F$  and  $G$*  is the full subcategory  $\mathbf{Ins}(F, G)$  of the comma category  $F \downarrow G$  spanned by all arrows  $FK \rightarrow GK$  ([2, 2.71]). Since  $\mathbf{Coalg}F = \mathbf{Ins}(\text{id}_{\mathbf{C}}, F)$  it follows from [2, 2.72] and the remark above that the categories  $\mathbf{Coalg}T_I$  and  $\mathbf{Coalg}M_A$  are accessible. Since any co-complete accessible category is locally presentable, we obtain

**Proposition 5.** *The categories  $\mathbf{Coalg}T_I$  and  $\mathbf{Coalg}M_A$  are locally presentable.*

**Remark 6.** There is no reason to assume that limits in these categories are respected by their obvious underlying functors  $|-|$  into  $\mathbf{Mod}_R$ . Consult [8] or [11] for how to possibly describe these limits.

## 2. THE CATEGORIES $\mathbf{Coalg}_R$ AND $\mathbf{Comod}_A$

The defining axioms for  $R$ -coalgebras, i.e., the commutativity of the diagrams (1), (2), and (3) above can be interpreted as follows:

Denote by  $\varphi$  and  $\psi$  the natural transformations

$$\begin{aligned} \varphi & : | - | \longrightarrow T_3 \circ | - | \\ \varphi_{\mathbb{C}} & = C \xrightarrow{m} C \otimes C \xrightarrow{m \otimes \text{id}_C} C \otimes C \otimes C \end{aligned}$$

and

$$\begin{aligned} \psi & : | - | \longrightarrow T_3 \circ | - | \\ \psi_{\mathbb{C}} & = C \xrightarrow{m} C \otimes C \xrightarrow{\text{id}_C \otimes m} C \otimes C \otimes C \end{aligned}$$

(Naturality of  $\varphi$  and  $\psi$  is a consequence of functoriality of  $- \otimes -$  and the definition of coalgebra homomorphism.)

**Lemma 7.**  $\mathbb{C} = (C, \langle m, e \rangle)$  satisfies (1) iff  $\varphi_{\mathbb{C}} = \psi_{\mathbb{C}}$ .

Similarly,

$$\begin{aligned} \varrho & : | - | \longrightarrow | - | \otimes R \\ \varrho_{\mathbb{C}} & = C \xrightarrow{m} C \otimes C \xrightarrow{\text{id}_C \otimes e} C \otimes R \end{aligned}$$

and

$$\begin{aligned} \lambda & : | - | \longrightarrow R \otimes | - | \\ \lambda_{\mathbb{C}} & = C \xrightarrow{m} C \otimes C \xrightarrow{e \otimes \text{id}_C} R \otimes C \end{aligned}$$

are natural transformations and the following obviously hold

**Lemma 8.**

1.  $\mathbb{C} = (C, \langle m, e \rangle)$  satisfies (2) iff  $\varrho_{\mathbb{C}} = r_{|\mathbb{C}|}$ .
2.  $\mathbb{C} = (C, \langle m, e \rangle)$  satisfies (3) iff  $\lambda_{\mathbb{C}} = l_{|\mathbb{C}|}$ .

Recall now that (see [2, 2.76]), for accessible functors  $F^t, G^t: \mathbf{K} \longrightarrow \mathbf{L}_t$  and families of natural transformations  $\mu^t, \nu^t: F^t \longrightarrow G^t$  ( $t \in T$ ) the equifier of  $(\mu^t)$  and  $(\nu^t)$  is the full subcategory  $\mathbf{Eq}(\mu^t, \nu^t)$  of  $\mathbf{K}$  spanned by all  $K$  in  $\mathbf{K}$  with  $\mu_K^t = \nu_K^t$  for all  $t \in T$ , and that this subcategory is accessible.

**Theorem 9.** *The categories  $\mathbf{Coalg}_R$  and  $\mathbf{Coring}_A$  are locally presentable categories.*

**Proof.** By Lemmas 7 and 8 the category of comonoids in  $\mathbb{C}$  is the equifier of the three pairs  $(\varphi, \psi), (\lambda, l_{|-|}), (\varrho, r_{|-|})$  of natural transformations. Since all categories and functors under consideration are accessible, it is accessible as well. Moreover the categories under consideration are closed under colimits in their respective  $\mathbf{Coalg}T_I$  by Fact 2 and hence cocomplete. Now the same argument used in the proof of Proposition 5 gives the result.  $\square$

In a completely analogous way one obtains

**Theorem 10.** *The categories  $\mathbf{Comod}_A, {}_A\mathbf{Comod}, \mathbf{Comod}_C$  and  ${}_C\mathbf{Comod}$  are locally presentable categories and therefore have all limits.*

We now get, as simple corollaries,

**Proposition 11.**

1.  $\mathbf{Coalg}_I$  is coreflective in  $\mathbf{Coalg}T_I$ .
2.  $\mathbf{Comod}_A$  is coreflective in  $\mathbf{Coalg}M_A$ .

**Proof.** The proof is the same in both cases: the embedding of the respective subcategory preserves colimits and both subcategories, being locally presentable, are co-wellpowered and have a generator. Now apply the (dual of the) Special Adjoint Functor Theorem.  $\square$

**Theorem 12.**

1.  $\mathbf{Coalg}_R$  is comonadic over  $\mathbf{Mod}_R$ ,
2.  $\mathbf{Coring}_A$  is comonadic over  ${}_A\mathbf{Mod}_A$ ,
3.  $\mathbf{Comod}_A$  is comonadic over  $\mathbf{Mod}_R$ , and
4.  $\mathbf{Comod}_C$  is comonadic over  $\mathbf{Mod}-A$ .

**Proof.** The respective underlying functors have right adjoints by Remark 4 and Proposition 11. They also create split equalizers by Remark 6 and because all of these categories are closed in their respective categories of functor-coalgebras w.r.t. subobjects carried by split monos (see [8] or Fact 17 below).  $\square$

**Remark 13.** The existence of cofree comodules certainly can be obtained directly. The argument given in [6] generalizes to our somewhat more general situation. See also [11]. Note also that the existence of a cofree coalgebra is known (see [3] for the cocommutative case with an argument similar or ours, and [10] with an explicit construction via the tensor algebra for the case of a field, which however generalizes to rings).

Generalizing a result in [1] we might reformulate the statement of the last theorem as follows

**Theorem 14.** *All categories  $\mathbf{Coalg}_R$ ,  $\mathbf{Coring}_A$ ,  $\mathbf{Comod}_A$ , and  $\mathbf{Comod}_C$  respectively, are covarieties.*

**Remark 15.** Obviously, the the above results can be extended to the categories of cocommutative coalgebras.

**Problem 16.** It is not clear that the kernel of a morphism in the categories  $\mathbf{Coalg}_R$  or  $\mathbf{Comod}_A$  is a subobject in the sense of say [4], i.e., whether its carrier map is injective. It has been shown in [8] that each injective homomorphism is a strong monomorphism, but it is clear that the converse doesn't hold: the categories under consideration, being locally presentable, carry an (epi, strong mono)-factorization structure (see [2]) but don't allow for image-factorizations of morphisms (see [8]). It thus would be interesting to characterize the injective homomorphism in these categories categorically and to describe the strong monomorphisms explicitly. If  $\mathbf{Ker}f$  could be shown to be a subcoalgebra of  $f$ 's domain, it would be an easy consequence to prove that it is the largest subcoalgebra contained in the  $\mathbf{Mod}_R$ -kernel of  $f$ . This is the case, if the ring  $R$  is regular (see [8]).

## 3. PURITY

It is easy to see that  $\mathbf{Coalg}_R$ ,  $\mathbf{Coring}_A$ , and  $\mathbf{Comod}_A$  are closed in their respective categories  $\mathbf{Coalg}T_I$  and  $\mathbf{Coalg}M_A$  w.r.t. subobjects whose underlying embedding in  $\mathbf{Mod}_R$  splits (see [8]). In fact, the proof of this statement given in [8] shows more:

Given a homomorphism  $m: \mathbb{C} \rightarrow \mathbb{D}$  in  $\mathbf{Coalg}T_I$  and  $\mathbf{Coalg}M_A$ , respectively, then  $\mathbb{C}$  is a comonoid and an  $A$ -coaction, respectively, provided that  $\mathbb{D}$  is and  $m \otimes m \otimes m$  and  $m \otimes \text{id}$  are monomorphisms in  $\mathbf{C}$ . In the commutative case this clearly holds, provided that  $m$  is a pure homomorphism. We thus have

**Fact 17.**  *$\mathbf{Coalg}_R$  and  $\mathbf{Comod}_A$  are closed in  $\mathbf{Coalg}T_R$  and  $\mathbf{Coalg}M_A$ , respectively, under subobjects carried by pure  $R$ -linear maps.*

The categorical concept of  $\lambda$ -purity ( $\lambda$  a regular cardinal) as presented, e.g., in [2] generalizes the notion of a pure module homomorphism in the sense that an  $\aleph_0$ -pure morphism in  $\mathbf{Mod}_R$  is simply a pure homomorphism, provided  $R$  is a *PID*. We do not know whether this fact has appeared in print elsewhere but believe it must be well known: an argument would be a straightforward generalization of the proof given for [9, 61.11], considering finitely generated submodules instead of single generated ones.

Also in the non-commutative case the notion of  $\aleph_0$ -purity can be exploited: Since the functor  $C \otimes -$  is left adjoint it preserves (directed) colimits and finitely presentable objects, hence  $\aleph_0$ -pure morphisms by [2, 2.38] which are (regular) monomorphisms. Thus the closure-statement of Fact 17 holds also in the non-commutative case.

Since the underlying functors  $\mathbf{Coalg}F \rightarrow \mathbf{C}$  ( $F = T_I$  or  $F = M_A$ ) are left adjoints and  $\mathbf{Coalg}F$  is a  $\lambda$ -presentable category for some  $\lambda$  they preserve  $\lambda$ -pure morphisms by the same argument as above, so that we can deduce

**Proposition 18.** *Each of the categories  $\mathbf{Coalg}_R$ ,  $\mathbf{Coring}_A$ ,  $\mathbf{Comod}_A$ , and  $\mathbf{Comod}_C$  is closed in its respective category of functor coalgebras under  $\lambda$ -pure subobjects for a suitable  $\lambda$ .*

**Proof.** Let  $\mathbf{Coalg}F$  be  $\lambda$ -presentable and  $\mathbb{C}$  a  $\lambda$ -pure subobject of  $\mathbb{D}$ ,  $\mathbb{D}$  in the subcategory under consideration. Then, in  $\mathbf{C}$ , the embedding  $C \hookrightarrow D$  is  $\lambda$ -pure, thus  $\aleph_0$ -pure. Now the claim follows from the above observations.  $\square$

**Remark 19.** The proposition above allows for an alternative proof of Theorem 9 and Theorem 10. Accessibility of our subcategories is in view of [2, 2.36] an immediate consequence of Proposition 18 since they are clearly closed under colimits (see [8]).

Let us finally relate  $\aleph_0$ -purity in  ${}_A\mathbf{Mod}_A$  with purity in  ${}_A\mathbf{Mod}$  and  $\mathbf{Mod}_A$ .

**Proposition 20.** *If  $f$  is an  $\aleph_0$ -pure morphism in  ${}_A\mathbf{Mod}_A$ , then  $f$  is pure in  ${}_A\mathbf{Mod}$  and in  $\mathbf{Mod}_A$ .*

**Proof.** By [2, 2.30]  $f$  is a directed colimit of split monomorphisms in  ${}_A\mathbf{Mod}_A$ , hence it is a directed colimit of split monomorphisms in each of the categories  $\mathbf{Mod}_A$  and  ${}_A\mathbf{Mod}$  as well and therefore pure in both categories again by [2, 2.30].  $\square$

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