The Complexity of Satisfiability for Fragments of Hybrid Logic

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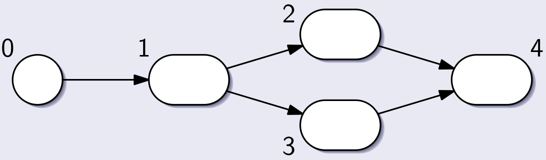
And now for . . .

What is hybrid logic?

Modal logic, $\mathcal{ML}$: propositional logic plus $\Diamond$, $\Box$
speaks about relational structures, e.g.:

The frame $\mathcal{F}$



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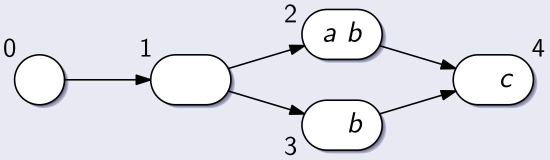

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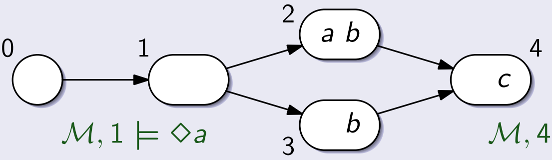

$\mathcal{M}, 1 \models \Diamond a$

$\mathcal{M}, 1 \not\models \Box a$

$\mathcal{M}, 4 \models \Box a$

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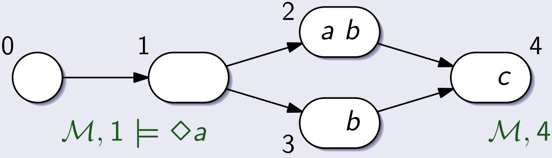

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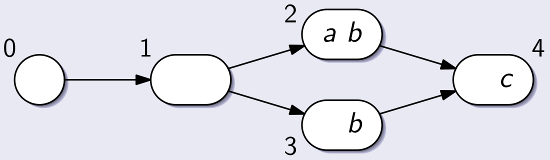
As in FOL, we have $\Box\varphi \equiv \neg\Diamond\neg\varphi$.

What is hybrid logic?

Hybrid logic, $\mathcal{HL}$: $\mathcal{ML}$ plus nominals, @, $\downarrow$

The model $\mathcal{M}$



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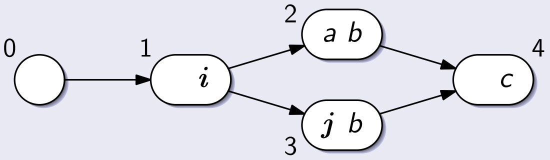

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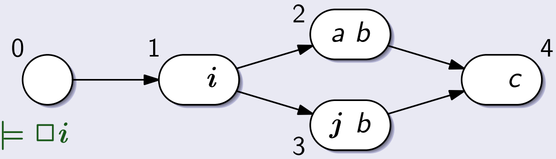

$\mathcal{M}, 0 \models \Box i$

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$@_i$ *jumps* to the state named i:

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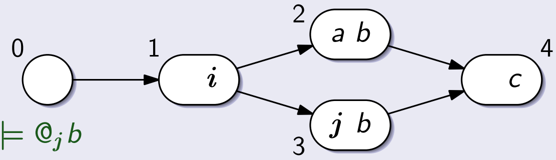

$\mathcal{M}, 0 \models @_j b$

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$\downarrow$ *binds* names to states:

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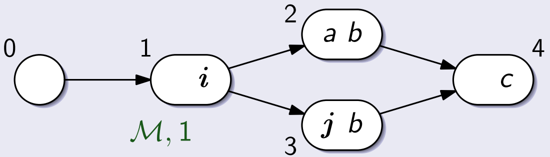

$\mathcal{M}, 1$

$\models$

$\downarrow x.\Diamond\Diamond\downarrow y.(c \wedge @_x\Box(\Diamond y \rightarrow b))$

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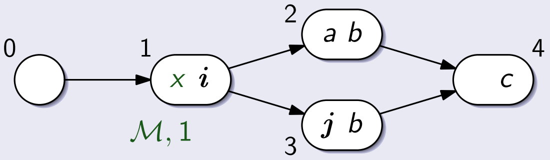

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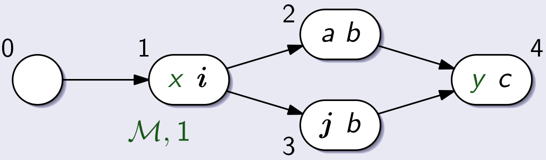

$$\mathcal{M}, 1 \models \downarrow x.\Diamond\Diamond\downarrow y.(c \wedge @_x \Box(\Diamond y \longrightarrow b))$$

The satisfiability problem for $\mathcal{HL}$

Definition

1. A formula φ is *satisfiable* if there is
   a model $\mathcal{M}$ based on a frame $\mathcal{F}$
   and a state s in $\mathcal{M}$
   such that $\mathcal{M}, s \models \varphi$[1].

[1] W.l.o.g. φ has no free state variables.

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Let $O \subseteq \{\Diamond, \downarrow, @\}$.

2. $\mathcal{HL}(O)$ = set of all $\mathcal{HL}$-formulae with operators from O
3. $\mathrm{SAT}(O) = \{\varphi \in \mathcal{HL}(O) \mid \varphi \text{ is satisfiable}\}$

[1] W.l.o.g. φ has no free state variables.

Complexity of satisfiability for $\mathcal{HL}$

Theorem

- SAT($\Diamond$) is PSPACE-complete. (Ladner '77)
- SAT($\Diamond$, @) is PSPACE-complete. (Areces et al. '99)
- SAT($\Diamond$, $\downarrow$) is coRE-complete. (Areces et al. '99)

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$\rightsquigarrow$

Tame $\downarrow$?

$\mathcal{HL}$ over restricted frame classes

$\mathfrak{F}$	
trans	accessibility relation is transitive
equiv	accessibility relation is an equivalence relation
serial	every state has a successor
$\vdots$	

Definition

$\mathfrak{F}$-SAT(O) =
$\{\varphi \in \mathcal{HL}(O) \mid \varphi$ is sat. in a model *based on a frame from* $\mathfrak{F}\}$

$\mathcal{HL}$ satisfiability over restricted frame classes

Theorem (Mundhenk et al. '05)

- trans-SAT($\Diamond, \downarrow$) is NEXPTIME-complete.
- equiv-SAT($\Diamond, \downarrow$) is NEXPTIME-complete.
- trans-SAT($\Diamond, \downarrow, @$) is CORE-complete.

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⇝

Tame $\downarrow$ further?

And now for . . .

Propositional fragments of $\mathcal{HL}$

Restrict the set of *propositional* operators!

- Why?

  Propositional SAT becomes tractable, e.g., without negation. (Lewis '79)

  SAT for $\mathcal{ML}$ or LTL becomes tractable for certain restrictions. (Bauland et al. '06/07)

  SAT for many sub-Boolean description logics is tractable. (Baader et al. '98/05/08, Calvanese et al. '05–07)

- 3 parameters:

$$\left.\begin{array}{l}\text{frame class } \mathfrak{F}\\ \text{set } O \text{ of modal/hybrid operators}\\ \text{set } B \text{ of Boolean operators}\end{array}\right\} \rightsquigarrow \mathfrak{F}\text{-SAT}(O, B)$$

Our goal

Classify $\mathfrak{F}$-SAT(O, B) for decidability and complexity w.r.t.

- *all B*
- O with $\{\Diamond, \downarrow\} \subseteq O \subseteq \{\Diamond, \Box, \downarrow, @\}$
- $\mathfrak{F} \in \{\text{all}, \text{trans}, \text{equiv}, \text{serial}\}$

- Find border between decidable and undecidable fragments
- Find tight complexity bounds

Post's lattice

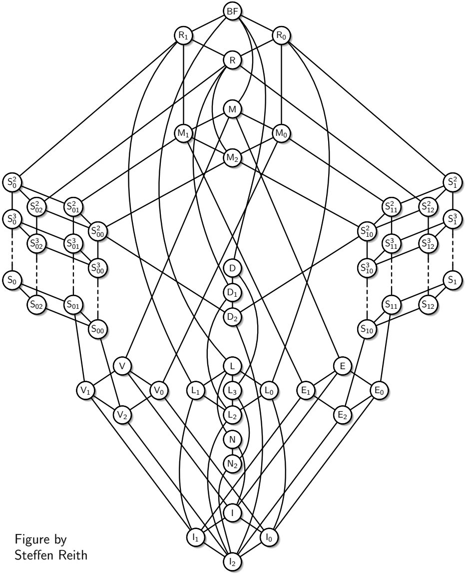
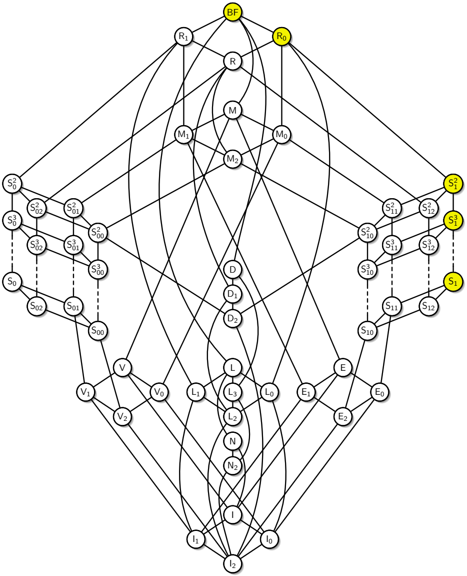


Figure by
Steffen Reith

Established 1941
by Emil Post

Satisfiability of propositional fragments in the literature



Theorem
(H. R. Lewis 1979)

SAT($\emptyset$, B) is:

- ● NP-complete
- ○ in P

S_1: $\neg(x \longrightarrow y)$

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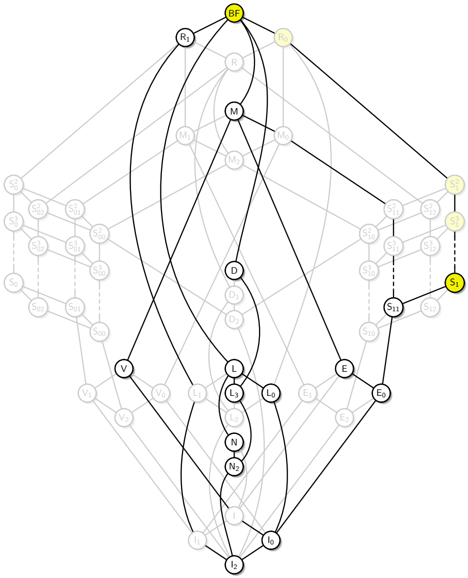

Theorem
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SAT($\emptyset$, B) is:

- NP-complete (yellow)
- in P (white)

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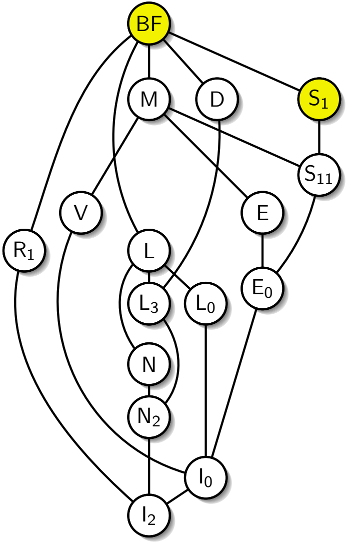
Satisfiability of propositional fragments in the literature



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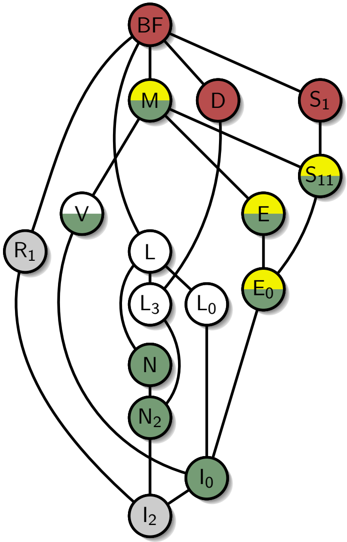
Results for all frames



Theorem 1

all-SAT(O, B) is:

- undecidable
- medium (NP- or PSPACE-hard)
- low (L-compl. or below)
- trivial
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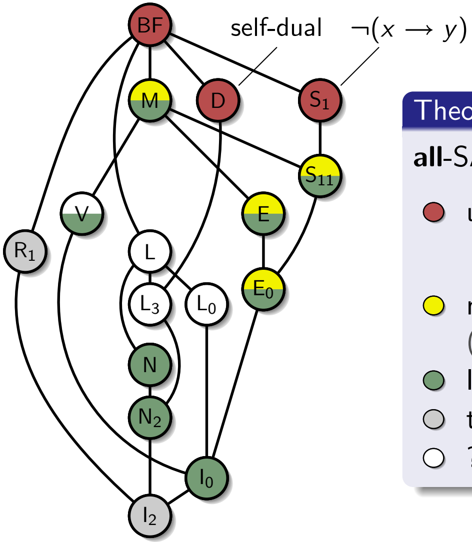

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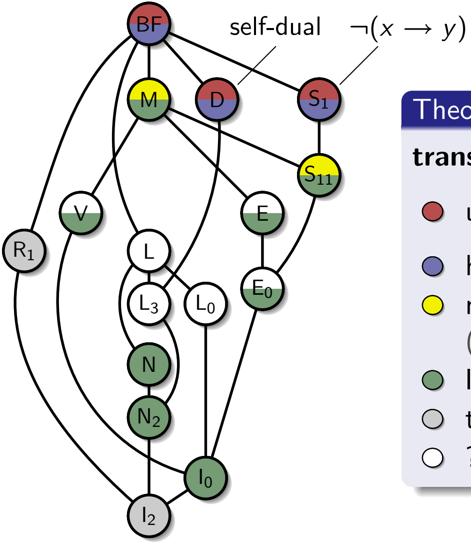
Results for transitive frames



Theorem 2

trans-SAT(O, B) is:

- undecidable
- high (NEXPTIME-compl.)
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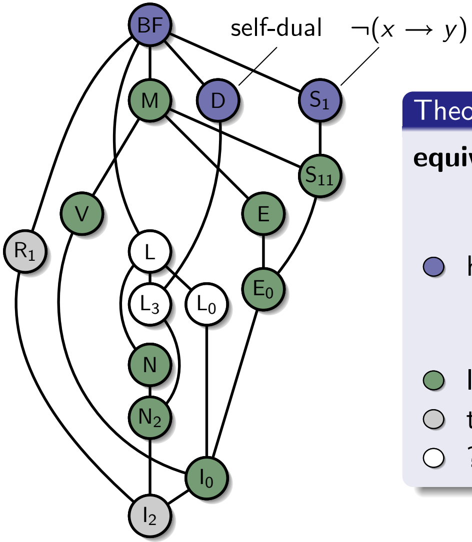
Results for frames with equivalence relations

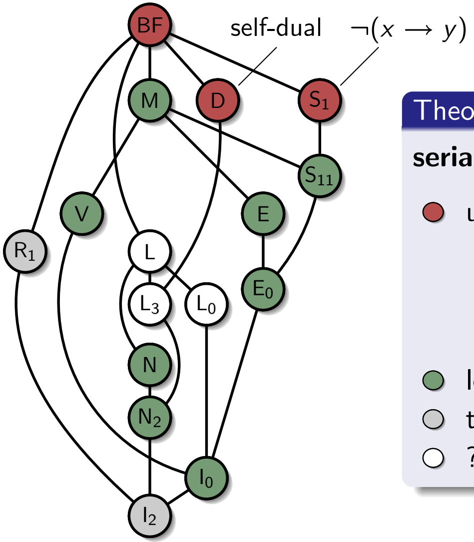


Theorem 3

equiv-SAT(O, B) is:

- high (NEXPTIME-compl.)
- low (L-compl. or below)
- trivial
- ?

Results for serial frames



Theorem 4

serial-SAT(O, B) is:

- undecidable
- low (L-compl. or below)
- trivial
- ?

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Conclusion

Classified $\mathfrak{F}$-SAT(O, B) for decidability and complexity w.r.t.

- almost all B
- most O with $\{\Diamond, \downarrow\} \subseteq O \subseteq \{\Diamond, \Box, \downarrow, @\}$
- $\mathfrak{F} \in \{\text{all}, \text{trans}, \text{equiv}, \text{serial}\}$

Open cases:

- Clones $\mathsf{L}, \mathsf{L}_0, \mathsf{L}_3$ based on $\oplus$
- Upper bounds for some clones below M with $O = \{\Diamond, \Box, \downarrow, @\}$

Perspectives

- Close gaps!
- Consider other frame classes (e.g., trees, linear)
- Consider other operators
- Systematise operator sets and frame classes *(Oh dear!)*
- Consider multi-modal languages

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Thank you.