Correctness and Worst-Case Optimality of Pratt-Style Decision Procedures for Modal and Hybrid Logics

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TABLEAUX, 5 July 2011

And now . . .

Propositional Dynamic Logic (PDL)

- Expressive extension of modal logic [Fischer, Ladner 1977]
  - One diamond and box per program: $\langle\alpha\rangle$, $[\alpha]$
  - Complex Programs
- PDL-satisfiability is ExpTime-complete [Pratt 1979]
- Simple worst-case optimal decision procedure by Pratt [1979]
  - Elimination of Hintikka sets
  - Exploits Bounded Model Theorem
  - Best-case exponential in its pure form

Extensions of PDL

We add

- Nominals $x, y, \ldots$
- Difference modalities D, $\overline{\text{D}}$
- Converse actions a^{-}

(ExpTime-compl. follows from [de Giacomo 1995], [Areces et al. 2000])

We obtain

- First explicit decision procedure for PDL + these features
- Robustness of Pratt's original procedure
- Refactored proof of the Bounded Model Theorem
  - $\rightsquigarrow$ Transparent proofs, straightforward correctness result
  - $\rightsquigarrow$ Modular addition of expressivity

And now . . .

Basic notions

- **Formulas** in NNF, and **programs** (without tests)

$$s ::= p \mid \neg p \mid s \wedge s \mid s \vee s \mid \langle\alpha\rangle s \mid [\alpha]s$$
$$\alpha ::= a \mid \alpha\beta \mid \alpha + \beta \mid \alpha^*$$

- **Models** $\mathfrak{M}$
  - Nonempty set of states
  - Transition relations $\xrightarrow{a}_{\mathfrak{M}}$ between states, induce $\xrightarrow{\alpha}_{\mathfrak{M}}$
  - Valuation $\mathfrak{M}p$: set of states for every predicate p
- $\mathfrak{M}, w \models \langle\alpha\rangle s \iff \exists$ state v $\left(w \xrightarrow{\alpha}_{\mathfrak{M}} v \;\;\&\;\; \mathfrak{M}, v \models s\right)$

Syntactic representations of models

Hintikka set H

- Syntactic representation of a state in a model
- Downward-closed set of fmas without obvious contradictions
  - $\{p, \neg p\} \not\subseteq H$
  - $s \wedge t \in H \implies s \in H$ and $t \in H$
  - $[\alpha\beta]s \in H \implies [\alpha][\beta]s \in H$
  - $[\alpha^*]s \in H \implies [\alpha][\alpha^*]s \in H$ and $s \in H$
  - ...

Formula universe $\mathcal{F}$

non-empty, finite, small enough set of relevant formulas (Fischer-Ladner closure)

Hintikka system $\mathcal{S}$

non-empty, finite set of Hintikka sets

Demos

Induced transition relation on Hintikka systems $\mathcal{S}$

- $H \xrightarrow{a}_{\mathcal{S}} H' \iff \forall s\ (\text{if } [a]s \in H, \text{ then } s \in H')$
- $\xrightarrow{\alpha}_{\mathcal{S}}$ induced

Demo $\mathcal{D}$

- Hintikka system with
  (D◇) $\langle\alpha\rangle s \in H \in \mathcal{D} \implies \exists H' \in \mathcal{D}\ (H \xrightarrow{\alpha}_{\mathcal{D}} H' \ \&\ s \in H')$
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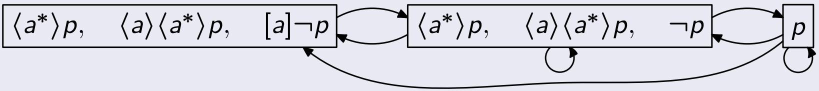
Example



From models to demos

Let $\mathfrak{M}, w$ be a model and state.

$H(w) = \{s \in \mathcal{F} \mid \mathfrak{M}, w \models s\}$ is a Hintikka set.

$\mathcal{S}(\mathfrak{M}) = \{H_w \mid w \text{ is a state}\}$

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Lemma

1. If $v \xrightarrow{\alpha}_{\mathfrak{M}} w$, then $H(v) \xrightarrow{\alpha}_{\mathcal{S}(\mathfrak{M})} H(w)$.
2. $\mathcal{S}(\mathfrak{M})$ is a demo.

Proof: (1) Induction on α. (2) Show (D$\Diamond$) via 1. $\square$

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Demo existence lemma

If $s \in \mathcal{F}$ satisfiable, then there is a demo $\mathcal{D}$ over $\mathcal{F}$ containing s.

From demos to models

Let $\mathcal{S}$ be a Hintikka system.

$\mathfrak{M}(\mathcal{S})$ consists of:

States : $\mathcal{S}$

$$\xrightarrow{a}_{\mathfrak{M}(\mathcal{S})} = \xrightarrow{a}_{\mathcal{S}}$$

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Demo satisfaction lemma

If $\mathcal{D}$ is a demo, then $\mathfrak{M}(\mathcal{D}), H \models H$ for all $H \in \mathcal{D}$.

Proof: Simple induction on the formulas in H. □

Satisfiability and the Bounded Model Theorem

Remember: demo existence and satisfaction

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Theorem

1. $s \in \mathcal{F}$ is satisfiable iff there is a demo $\mathcal{D}$ over $\mathcal{F}$ containing s.
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Satisfiability test: compute maximal demo, search for s

Pratt's decision procedure in our notation

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Theorem

If $\mathcal{S} \overset{\mathsf{p}}{\rightsquigarrow} \mathcal{S}'$ and $\mathcal{S}$ contains a demo, then $\mathcal{S}'$ is the max. such demo.

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Input: formula s

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2. $\mathcal{H} = \{H \mid H \text{ is a Hintikka set with } H \subseteq \mathcal{F}\}$
3. Compute $\mathcal{D}$ with $\mathcal{H} \overset{\mathsf{p}}{\leadsto} \mathcal{D}$.
4. s is satisfiable iff $s \in H$ for some $H \in \mathcal{D}$.

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Worst-case optimal:

- $|\mathcal{F}| = O(|s|)$
- $|\mathcal{H}| = 2^{O(|s|)}$
- Each pruning step is linear in $|\mathcal{H}|$.
- There can be at most $|\mathcal{H}|$ pruning steps.

And now . . .

Demos for PDL with nominals

Nominals = predicates true at exactly one state

$\mathcal{S}$ is **nominally coherent (nc):**
Every nominal $x \in \mathcal{F}$ occurs in exactly one $H \in \mathcal{S}$

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$\rightsquigarrow$ Revised decision procedure:
Guess maximal nc set of Hintikka sets and apply pruning

The decision procedure with nominals

Input: formula s

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2. $\mathcal{H} = \{H \mid H \text{ is a Hintikka set with } H \subseteq \mathcal{F}\}$
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4. Compute $\mathcal{D}$ with $\mathcal{H}' \overset{\mathrm{p}}{\rightsquigarrow} \mathcal{D}$.
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Determinise guessing step:

- For every nominal $x \in \mathcal{F}$
  - guess one $H \in \mathcal{H}$ with $x \in H$
  - discard all other H' with $x \in H'$
- Number of binary guesses: poly($|s|$)

$\leadsto$ Shallow nondeterministic computation tree with $2^{O(|s|)}$ nodes

Demos for PDL with difference modalities

$\mathrm{D}s \;\hat{=}\;$ "s is true in some other state"

$\overline{\mathrm{D}}s \;\hat{=}\;$ "s is true in all other states"

Extend demo conditions

(DD) If $\mathrm{D}s \in H \in \mathcal{D}$, then $\exists H' \in \mathcal{D}\ (H' \neq H \ \&\ s \in H')$.

(D$\overline{\mathrm{D}}$) If $\overline{\mathrm{D}}s \in H \in \mathcal{D}$, then $\forall H' \in \mathcal{D}\ (H' \neq H \Rightarrow s \in H')$.

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Difficulties

1. With D, $\mathcal{S}(\mathfrak{M})$ does no longer have to be a demo.
2. With $\overline{\mathrm{D}}$, demos are again not closed under union.

Ensuring that $\mathcal{S}(\mathfrak{M})$ is a demo

Difficulty 1: with D, $\mathcal{S}(\mathfrak{M})$ does no longer have to be a demo.

Example

$\mathcal{F} = \{p, \mathrm{D}p\}, \quad \mathfrak{M} = \overset{p}{\underset{v}{\bullet}} \quad \overset{p}{\underset{w}{\bullet}}$

$\mathfrak{M}, v \models \mathrm{D}p \quad \Rightarrow \mathrm{D}p \in H(v) = H(w)$

But $\mathcal{S}(\mathfrak{M}) = \{H(v)\} \quad \Rightarrow$ (DD) violated

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- Introduce auxiliary nominal $x(\mathrm{D}s)$ for every $\mathrm{D}s \in \mathcal{F}$
- $x(\mathrm{D}s)$ denotes a state satisfying s if one exists
- Then all other states satisfy $\mathrm{D}s$.

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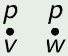
$\mathcal{F} = \{p, \mathrm{D}p\}$, $\quad \mathfrak{M} =$ 

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Nice model $\mathfrak{M}$

Whenever s is satisfiable in $\mathfrak{M}$, then so is $s \wedge x(\mathrm{D}s)$.

$\leadsto$ $\mathcal{S}(\mathfrak{M})$ is a demo

Pruning with difference modalities

Pruning step $\mathcal{S} \xrightarrow{\text{p}} \mathcal{S}'$: delete one $H \in \mathcal{S}$ violating (D$\Diamond$) *or* (DD).

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Difficulty 2: with $\overline{\mathrm{D}}$, demos are again not closed under union.

Solution

- Guess maximal $\mathcal{H}' \subseteq \mathcal{H}$ that is nc *and* satisfies (D$\overline{\mathrm{D}}$).
- Determinisation similar to nominals

Tests and converse actions . . .

- . . . require minor changes to proof machinery for model-demo correspondence
- . . . do not affect the decision procedures

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We have obtained

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- Transparent proofs of BMT and correctness
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We are missing

- Average-case efficiency
- Efficient implementation for the hybrid language

Related work

- Basis: [Pratt 1979] = [Fischer, Ladner 1979] + pruning
- Variants:
  - [Harel 1984], [Kozen, Tiuryn 1990], [Harel et al. 2000]
    simultaneous or non-standard induction;
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- Complexity results without explicit decision procedure:
  [Passy, Tinchev 1991], [de Giacomo 1995], [Areces et al. 2000]

Related work

- Tableau construction instead of $\mathcal{H}$
  - for PDL: [Pratt 1980]
  - for PDL^{-}: [Goré, Widmann 2009+10]
    more practical and implemented
  - for HL with $\mathsf{D}, \overline{\mathsf{D}}$: [Kaminski, Smolka 2010]
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  - for HL with D, $\overline{\mathrm{D}}$: [Kaminski, Smolka 2010] Hintikka sets replaced by clauses and support NExpTime with nominals
- Notions related to demos: Hintikka structures for CTL in [Emerson, Halpern 1985] Richer: explicit transition relation, multiset of Hintikka sets

Future work

- Extension to hybrid μ-calculus and/or graded modalities
- Towards implementation:
  Interleave tableau construction and guessing for nominals

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Thank you.

Guessing for the difference modality

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For every $\overline{\text{D}}s$, 3 cases for its occurrence in a max. demo $\mathcal{D} \subseteq \mathcal{H}$.

1. $\overline{\text{D}}s$ not in $\mathcal{D}$.
   $\rightsquigarrow$ discard all Hintikka sets containing $\overline{\text{D}}s$
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2. All Hintikka sets in $\mathcal{D}$ contain s.
   $\leadsto$ discard all Hintikka sets not containing s
3. $\mathcal{D}$ contains H, H' with $\overline{\text{D}}s \in H$ and $s \in H'$, w.l.o.g. $s \notin H$.
   $\leadsto$ $s \notin H$; no $H' \neq H$ contains $\overline{\text{D}}s$
   $\leadsto$ choose one H containing $\overline{\text{D}}s$ and not s;
   discard all other Hintikka sets containing $\overline{\text{D}}s$ or not s