The Complexity of Satisfiability for Fragments of Hybrid Logic

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And now . . .

Hybrid logic in a nutshell

We're looking at the extension of standard modal logic with

- **nominals** $i, j, \ldots$

  name single states in models

- the **binder** $\downarrow$

  $\downarrow x.\varphi$ binds variable x dynamically to the current state;
  x in φ is treated as a nominal

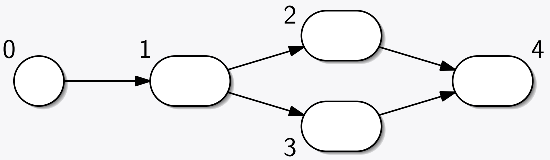
- the **satisfaction operator** $@_x$

  jumps to the state named by (the nominal or variable) x

Recap: modal logic

Modal logic, $\mathcal{ML}$: propositional logic plus $\Diamond, \Box$
speaks about relational structures, e.g.:

The frame $\mathcal{F}$



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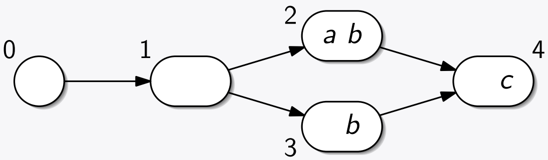

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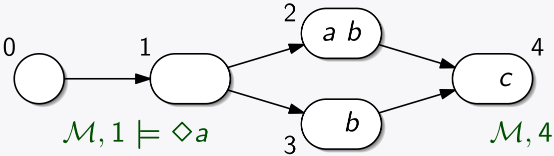

$\mathcal{M}, 1 \models \Diamond a$

$\mathcal{M}, 1 \not\models \Box a$

$\mathcal{M}, 1 \models \Box(\Diamond c \to b)$

$\mathcal{M}, 4 \models \Box\varphi$

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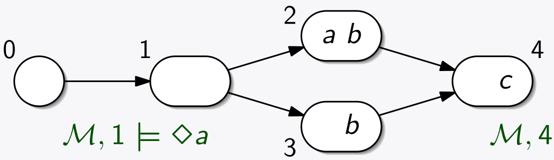


$\mathcal{M}, 1 \models \Diamond a$

$\mathcal{M}, 1 \not\models \Box a$

$\mathcal{M}, 1 \models \Box(\Diamond c \to b)$

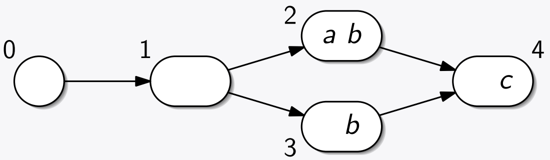
$\mathcal{M}, 4 \models \Box\varphi$

As in $\mathcal{FOL}$, we have $\Box\varphi \equiv \neg\Diamond\neg\varphi$.

Hybrid logic

Hybrid logic, $\mathcal{HL}$: $\mathcal{ML}$ plus nominals, @, $\downarrow$

The model $\mathcal{M}$

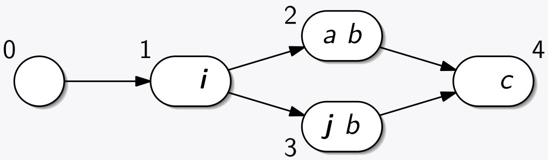


Hybrid logic

Hybrid logic, $\mathcal{HL}$: $\mathcal{ML}$ plus nominals, @, $\downarrow$

nominals name states:

The model $\mathcal{M}'$



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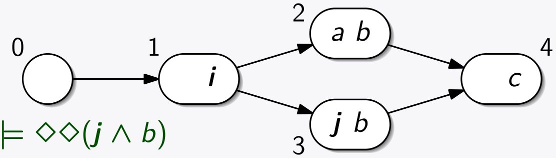


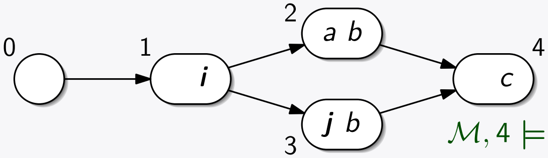
$\mathcal{M}, 0 \models \Diamond\Diamond(j \wedge b)$

Hybrid logic

Hybrid logic, $\mathcal{HL}$: $\mathcal{ML}$ plus nominals, $@, \downarrow$

$@_i$ jumps to the state named i:

The model $\mathcal{M}'$



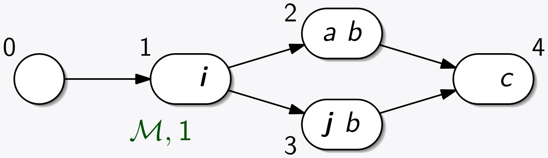
$\mathcal{M}, 4 \models @_i \Box\Diamond c$

Hybrid logic

Hybrid logic, $\mathcal{HL}$: $\mathcal{ML}$ plus nominals, @, $\downarrow$

$\downarrow$ binds names to states:

The model $\mathcal{M}'$



$$\mathcal{M}, 1 \models \downarrow x.\Diamond\Diamond\downarrow y.(c \wedge @_x \Box(\Diamond y \rightarrow b))$$

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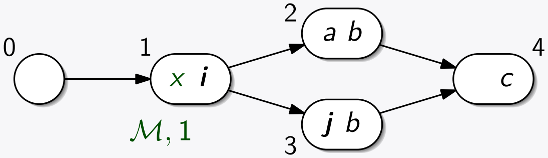

$\mathcal{M}, 1$

$\models$

$\downarrow x.\Diamond\Diamond\downarrow y.(c \land @_x\Box(\Diamond y \to b))$

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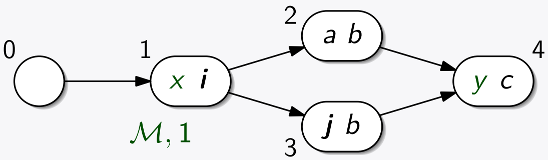

$\mathcal{M}, 1$

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$\downarrow x.\Diamond\Diamond\downarrow y.(c \wedge @_x \Box(\Diamond y \rightarrow b))$

The satisfiability problem for $\mathcal{HL}$

Definition

1. A formula φ is **satisfiable** if there is

   a model $\mathcal{M} = (W, R, V)$ based on a frame $\mathcal{F} = (W, R)$
   an assignment $g : \mathrm{SVAR} \rightarrow W$
   and a state $s \in W$

   such that $\mathcal{M}, g, s \models \varphi$

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Let $O \subseteq \{\Diamond\Box\downarrow@\}$.

2. $\mathcal{HL}(O)$ = set of all $\mathcal{HL}$-formulas with operators from O
3. $\mathrm{SAT}(O) = \{\varphi \in \mathcal{HL}(O) \mid \varphi \text{ is satisfiable}\}$

Complexity of satisfiability for $\mathcal{HL}$

Theorem

SAT($\Diamond\Box$)	is PSPACE-complete.	(Ladner '77)
SAT($\Diamond\Box@$)	is PSPACE-complete.	(Areces et al. '99)
SAT($\Diamond\Box\downarrow$)	is undecidable.	(Areces et al. '99)

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$\rightsquigarrow$

Tame $\downarrow$?

$\mathcal{HL}$ over restricted frame classes

$\mathfrak{F}$	condition on frames $(W, R) \in \mathfrak{F}$
all	—
trans	R is transitive
equiv	R is an equivalence relation
serial	every state has an R-successor
lin	R is a linear order (transitive, irreflexive, $\forall xy(xRy$ or $x = y$ or $yRx)$)
$\mathbb{N}$	$(W, R) = (\mathbb{N}, <)$
⋮	

Definition

$\mathfrak{F}$-SAT(O) =

$\{\varphi \in \mathcal{HL}(O) \mid \varphi$ is sat. in a model *based on a frame from* $\mathfrak{F}\}$

$\mathcal{HL}$ satisfiability over restricted frame classes

Theorem

trans-SAT($\Diamond\Box\downarrow$)	is NExpTime-complete.	(Mundhenk et al.
equiv-SAT($\Diamond\Box\downarrow$)	is NExpTime-complete.	“ '05)
trans-SAT($\Diamond\Box\downarrow@$)	is undecidable.	“
lin-SAT($\Diamond\Box\downarrow$)	is NP-complete.	(Areces et al. '00)
$\mathbb{N}$-SAT($\Diamond\Box\downarrow$)	is NP-complete.	“
lin-SAT($\Diamond\Box\downarrow@$)	is nonelementary.	(Franceschet et al.
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$\mathcal{HL}$ satisfiability over restricted frame classes

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$\rightsquigarrow$

Tame $\downarrow$ further?

Propositional fragments of $\mathcal{HL}$

$\rightsquigarrow$ **Restrict the set of *propositional* operators!** Why?

- **Propositional SAT** is tractable if $\not\to$[1] is disallowed (Lewis '79)
- **LTL-SAT** is tractable if $\not\to$ is disallowed (Bauland et al. '07)
- **SAT for $\mathcal{ML}(\Diamond\Box)$** is tractable if $\not\to$ and $\wedge$ are disallowed (Bauland et al. '06)
- **SAT for certain sub-Boolean description logics** is tractable (Baader et al. '98/05/08, Calvanese et al. '05–07)

[1] $x \not\to y \equiv \neg(x \to y) \equiv x \wedge \neg y$

Overall goal

Classify $\mathfrak{F}$-SAT(O, B) for decidability and complexity w.r.t.

- *all* sets B of Boolean operators
- modal/hybrid operators O with $O \subseteq \{\Diamond\Box\downarrow@\}$
- $\mathfrak{F} = \underbrace{\text{all}, \text{trans}, \text{equiv}, \text{serial},}_{\text{allow cycles}} \ \underbrace{\text{lin}, \mathbb{N}}_{\text{acyclic}}$

- Locate border between decidable and undecidable fragments
- Establish tight complexity bounds

And now . . .

Scope of the results

We classified $\mathfrak{F}$-SAT(O, B) for decidability and complexity w.r.t.

- *almost all* sets B of Boolean operators
- modal/hybrid operators O with $\{\Diamond\downarrow\} \subseteq O \subseteq \{\Diamond\Box\downarrow@\}$
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Post's lattice

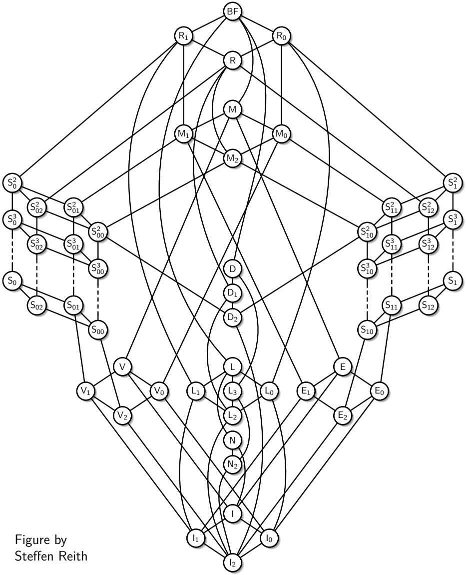
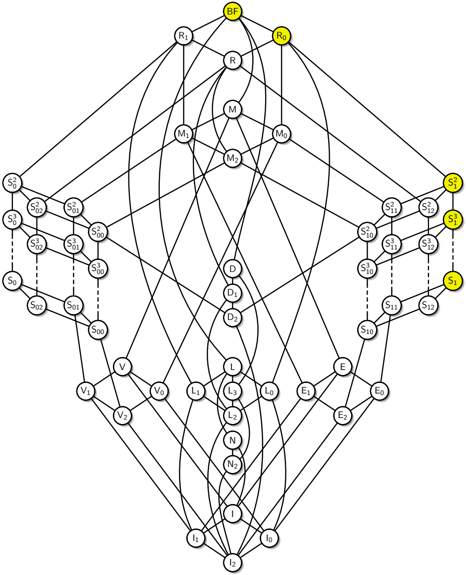


Figure by
Steffen Reith

Established 1941
by Emil Post

Satisfiability of propositional fragments in the literature



Theorem
(H. R. Lewis 1979)

$\mathrm{SAT}(\emptyset, B)$ is:

- NP-complete
- in P

S_1: $x \not\rightarrow y$

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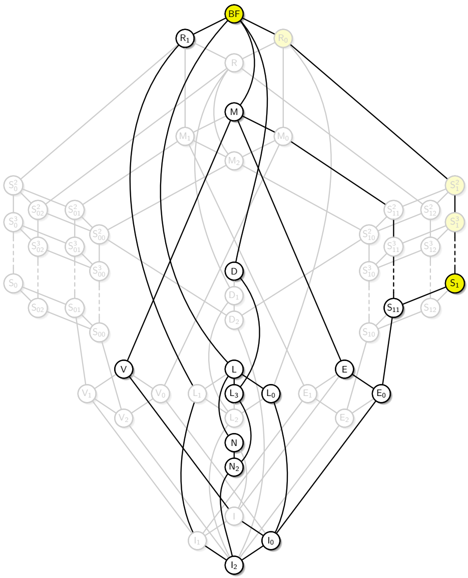

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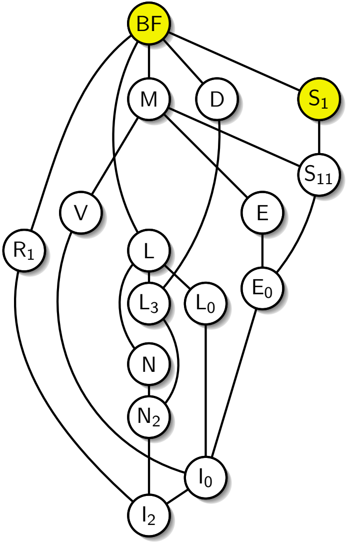

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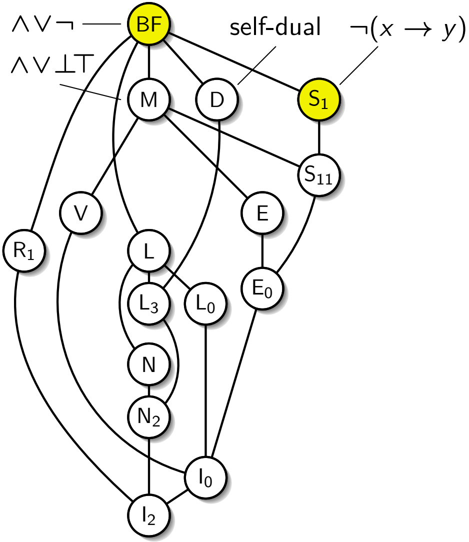


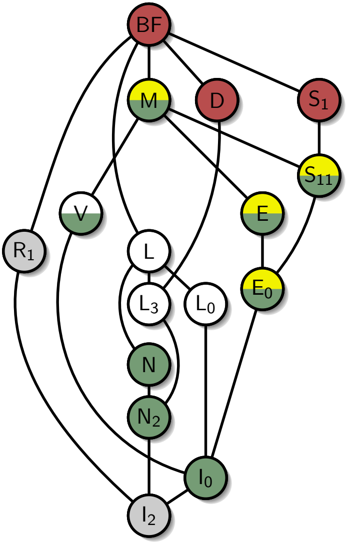
Theorem
(H. R. Lewis 1979)

SAT($\emptyset$, B) is:

- NP-complete (yellow)
- in P (white)

S_1: $x \not\rightarrow y$

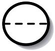
Results for all frames



Theorem 1

all-SAT(O, B) is:

- undecidable
- medium? (NP- or PSPACE-hard)
- low (L-compl. or below)
- trivial
- ?



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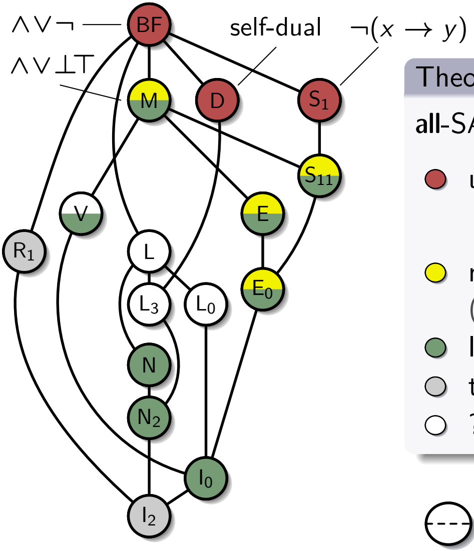

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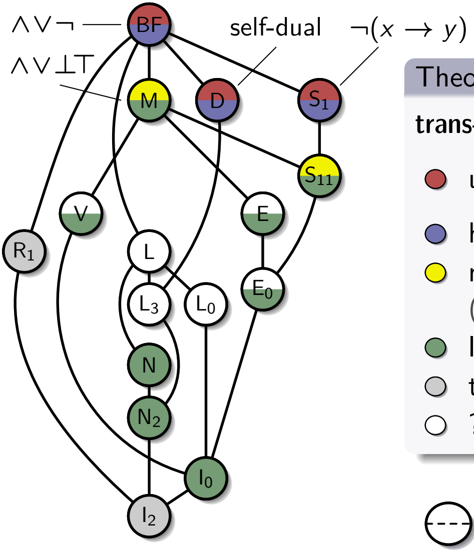
$\Box \in O$

$\Box \notin O$

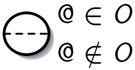
Results for transitive frames



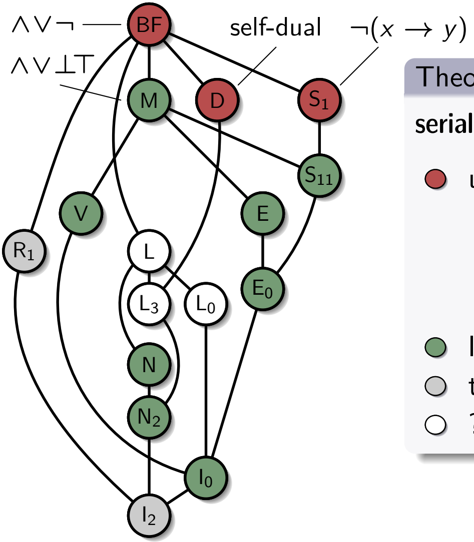
Theorem 2

trans-SAT(O, B) is:

- undecidable
- high (NExpTime-compl.)
- medium? (NP- or PSpace-hard)
- low (L-compl. or below)
- trivial
- ?



Results for serial frames

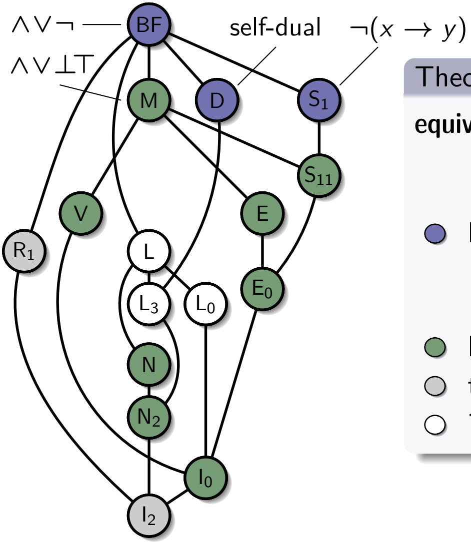


Theorem 3

serial-SAT(O, B) is:

- undecidable
- low (L-compl. or below)
- trivial
- ?

Results for frames with equivalence relations



Theorem 4

equiv-SAT(O, B) is:

- ● high (NExpTime-compl.)
- ● low (L-compl. or below)
- ● trivial
- ○ ?

Summary and lessons learnt

We have established . . .

- the **computational complexity** of SAT for all fragments of $\mathcal{HL}$
  - with *almost all* Boolean operators
  - with modal and hybrid operators $\{\Diamond\downarrow\} \subseteq O \subseteq \{\Diamond\Box\downarrow@\}$
  - over cyclic frame classes (all, trans, serial, equiv)
- a complexity **border** and interesting **dichotomy:**

undecidable (or very hard)	$\leftrightarrow$	tractable
self-dual op.s or $\not\rightarrow$		monotone op.s

And now . . .

Scope of the results

We classified $\mathfrak{F}$-SAT(O, B) for decidability and complexity w.r.t.

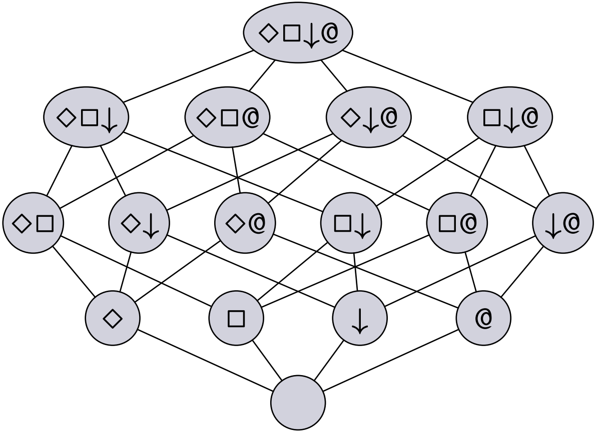
- *monotone* Boolean operators $\wedge \vee \bot \top$
- modal/hybrid operators O with $O \subseteq \{\Diamond \Box \downarrow @\}$
- $\mathfrak{F} = \text{lin}, \mathbb{N}$ (acyclic)

- **Why?**
  - $\mathcal{HL}$ over linear frames and $\mathbb{N}$ is an extension of LTL
  - M: largest clone with tractable results in the previous part

- **Observation**

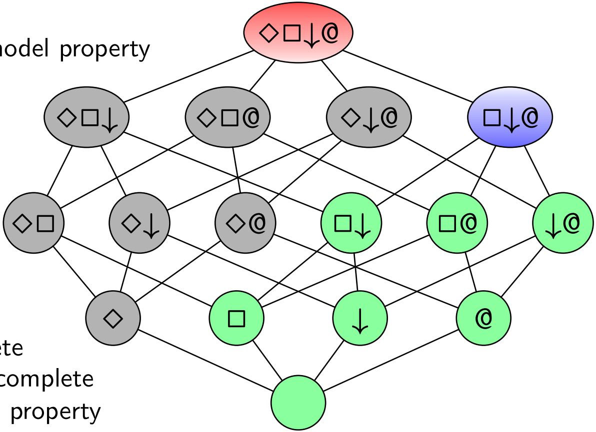
  with monotone operators, we can forgo propositional variables (replace them with $\top$)

Classification by *modal and hybrid* operators



Overview

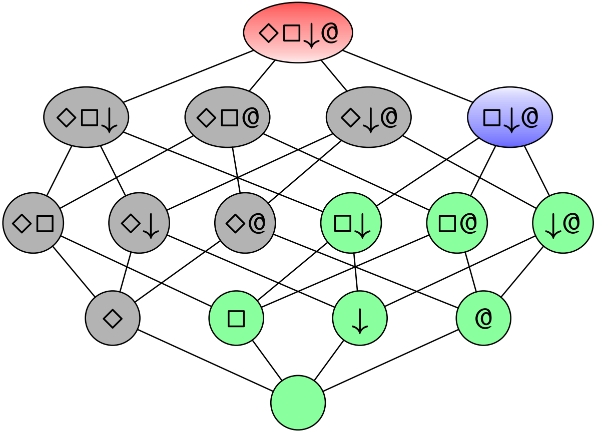
- lin: decidable, non-elementary
  $\mathbb{N}$: PSPACE-complete
- NP-complete
  quasi-polysize model property



- lin: NC^1-complete
  $\mathbb{N}$: LOGSPACE-complete
  canonical model property
- NC^1-complete
  canonical model property

The hard cases

- lin: decidable, non-elementary
  ℕ: PSPACE-complete



The hard cases

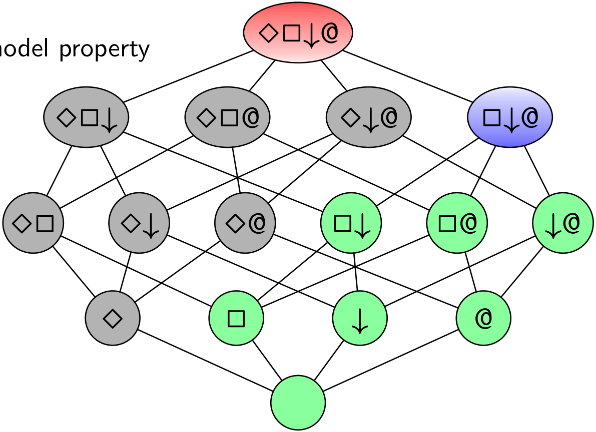
- **Nonelementary lower bound**:
  - Reduction from $\mathcal{FOL}$-SAT over $\mathbb{N}$ with one unary predicate P (Stockmeyer'74)
  - Encode P from an $\mathcal{FOL}(P, <)$-interpretation using alternations of dense and discrete intervals in lin
- **PSPACE-membership:**

  Reduction to SAT for $\mathcal{FOL}(<)$ over $\mathbb{N}$ (Ferrante, Rackoff '79)
- **PSPACE-hardness:**

  Straightforward encoding of QBF-SAT

The intermediate cases

- NP-complete
  quasi-polysize model property



The intermediate cases

- **Lower bound:**

  Straightforward reduction from 3-SAT

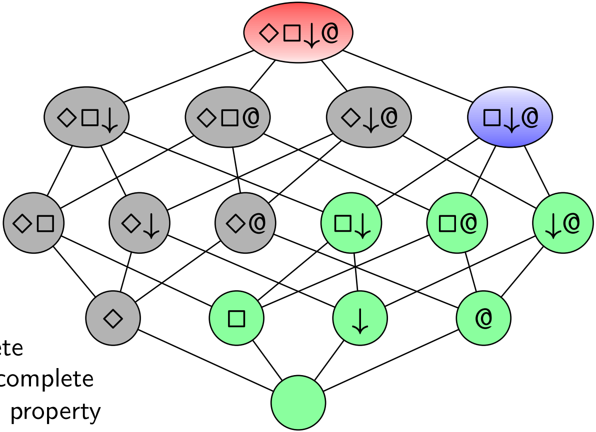
- **Upper bound:**

  Previous results or obvious consequences (Areces et al. '00)

- **Quasi-polysize model property:**

  If φ satisfiable,
  then φ has a model that can be represented polynomially

The easy cases



- lin: NC^1-complete
  ℕ: LogSpace-complete
  canonical model property
- NC^1-complete
  canonical model property

The easy cases

- **Canonical models**:

  Satisfiability is equivalent to satisfaction in a particular model

  $\rightsquigarrow$ Most cases reduce to propositional MSAT

  $\rightsquigarrow$ NC^1-completeness (Schnoor '07)

- **LogSpace-hardness:**

  Reduction from "Order between vertices"

- **LogSpace-membership:**

  Via *unique assignment and state of evaluation* obtained from the canonical model

Summary and lessons learnt

We have established ...

- the **computational complexity** of SAT for all fragments of $\mathcal{HL}$
  - with *monotone* Boolean operators $\wedge \vee \bot \top$
  - with modal and hybrid operators $O \subseteq \{\Diamond \Box \downarrow @\}$
  - over acyclic frame classes (lin, $\mathbb{N}$)

- **small-model properties**
  for all intermediate and easy cases

  $\rightsquigarrow$ upper bounds for other $\mathfrak{F} \subseteq$ lin — e.g., $\mathbb{Q}, \mathbb{R}$!

Interesting **observation:**

- Fragment $(\Diamond\Box\downarrow@)$ is **harder** over lin than over $\mathbb{N}$,
  but fragment $(\Box\downarrow@)$ is **easier** over lin than over $\mathbb{N}$

And now . . .

Outlook

- **Cyclic frame classes:** close gaps
  - Clones L, L_0, L_3 based on $\oplus$
  - Upper bounds for some clones below M with $O = \{\Diamond\Box\downarrow@\}$

- **Acyclic frame classes:**
  - Small-model property for the PSPACE-complete case?
  - Transport to strictly dense frame classes, e.g., $(\mathbb{Q},<)$
  - Other combinations of Boolean operators

- Systematise modal/hybrid operators and frame classes

- Consider multi-modal languages

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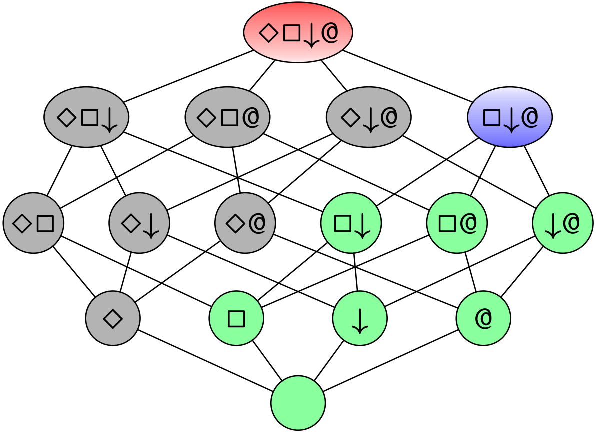
Thank you.

The hard cases

- lin: decidable, non-elementary
  $\mathbb{N}$: PSPACE-complete



A nonelementary lower bound

Theorem

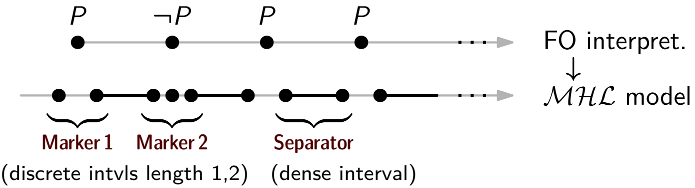
lin-MSAT($\Diamond\Box\downarrow @$) is decidable and nonelementary.

Proof sketch.

- Decidability from lin-SAT($\Diamond\Box\downarrow @$) (Franceschet et al. '03)
- Reduce from $\mathcal{FOL}$-SAT over $\mathbb{N}$ with predicates (Stockmeyer'74)
  - $<$ (natural "less-than" on $\mathbb{N}$)
  - P (one arbitrary unary predicate)
- Encode
  - $\mathcal{FOL}(P,<)$-interpretations over $\mathbb{N}$, using no propos. variables
  - formulas from $\mathcal{FOL}(P,<)$ as monotone formulas

Details of the encoding

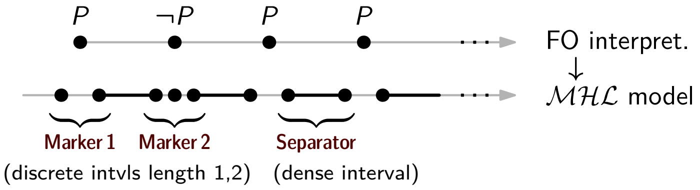
- Encode FO interpretations as sequences of intervals:



Use $\mathcal{MHL}(\Diamond\Box{\downarrow}@)$ to enforce this structure in a hybrid model

Details of the encoding

- Encode FO interpretations as sequences of intervals:



Use $\mathcal{MHL}(\Diamond\Box{\downarrow}@)$ to enforce this structure in a hybrid model

- Encoding of formulas (example):
  - $\forall x\big(Px \rightarrow \exists y(x < y \wedge \neg Py)\big)$ becomes

    $\Box_{\mathrm{m}}{\downarrow}x.\big(1(x) \rightarrow \Diamond_{\mathrm{m}}{\downarrow}y.2(y)\big)$; without implication:

    $\Box_{\mathrm{m}}{\downarrow}x.\big(2(x) \vee \Diamond_{\mathrm{m}}{\downarrow}y.2(y)\big)$
  - $\Diamond_{\mathrm{m}}\psi$ = "in some future state that starts a marker, ψ holds"

    $\Box_{\mathrm{m}}\psi$ = "all future states start no marker or satisfy ψ"

A PSPACE upper and lower bound

- Over $\mathbb{N}$, we can no longer use dense-discrete alternation to encode unary predicates.
- SAT for $\mathcal{FOL}(<)$ over $\mathbb{N}$ is PSPACE-complete
  (Ferrante, Rackoff '79)

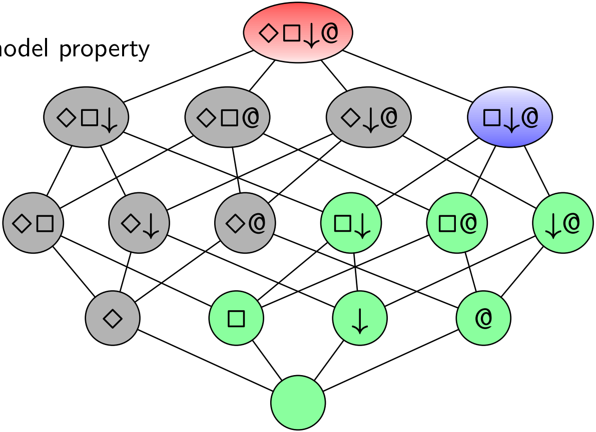
Theorem

$\mathbb{N}$-MSAT($\Diamond\Box\downarrow@$) is PSPACE-complete.

- Hardness via straightforward encoding of QBF-SAT
- Membership via reduction to SAT for $\mathcal{FOL}(<)$ over $\mathbb{N}$

The intermediate cases

- NP-complete
  quasi-polysize model property



NP-completeness

Theorem

$\Diamond \in O \subsetneq \{\Diamond\Box{\downarrow}@\} \quad \Rightarrow \quad$ lin- and $\mathbb{N}$-MSAT(O) are NP-complete.

- **Lower bound:** straightforward reduction from 3-SAT

  uses nominals: one per variable; 2 for "true" and "false"

- **Upper bound:**
  - lin- and $\mathbb{N}$-MSAT($\Diamond\Box@$): in NP (Areces et al. '00)
  - lin- and $\mathbb{N}$-MSAT($\Diamond\Box\downarrow$): obvious reduction to $\mathbb{N}$-MSAT($\Diamond\Box$)
  - lin- and $\mathbb{N}$-MSAT($\Diamond{\downarrow}@$):

    without $\Box$, $\downarrow$ binds state variables "existentially"

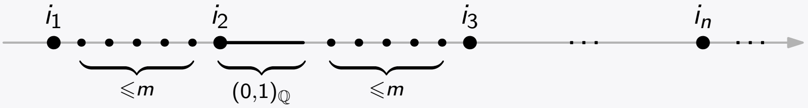
    $\rightsquigarrow$ replace with fresh nominals

    $\rightsquigarrow$ straightforward reduction to $\mathbb{N}$-MSAT($\Diamond@$)

A quasi-polysize model property (QPMP)

Theorem

Every $\varphi \in$ lin-MSAT($\Diamond\Box@$) of modal depth m has a model which,



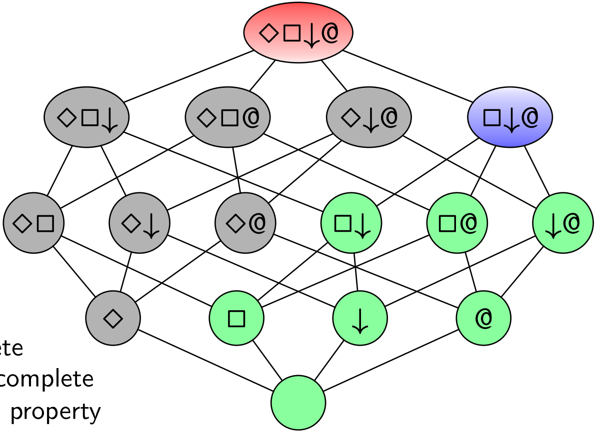
between two successive nominal states, has $\leqslant m$ further states, possibly preceded by one copy of the dense interval $(0,1)_{\mathbb{Q}}$.

Proof idea: States with distance $> m$ from nominal states satisfy the same modal formulas of modal depth $\leqslant m$

Gain:

- Such structures can be represented polynomially
- With little extra effort, QPMP yields NP upper bounds for SAT over lin, $\mathbb{N}$, $\mathbb{Q}$

The easy cases



- lin: NC^1-complete
  $\mathbb{N}$: LogSpace-complete
  canonical model property
- NC^1-complete
  canonical model property

A canonical model property

Theorem

(1) Every $\varphi \in$ lin-MSAT($\Box\downarrow@$) is satisfiable

- in a one-state structure
- under an assignment g that maps all SVARs to the only state.

(2) Every $\varphi \in \mathbb{N}$-MSAT($\Box\downarrow@$) is satisfiable

- in $(\mathbb{N}, <)$
- under an assignment g that maps all SVARs to 0.

Main observation:
without $\Diamond$, we cannot control the order of two states

Consequence:
With (1), we can reduce lin-MSAT($\Box\downarrow@$) to propositional MSAT
$\rightsquigarrow$ NC^1-completeness (Schnoor '07)

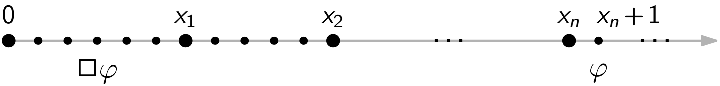
A LogSpace result over $\mathbb{N}$

Theorem

$\mathbb{N}$-MSAT($\Box\downarrow@$) is LogSpace-complete.

Proof sketch.

- Lower bound: reduction from "Order between vertices"
- Upper bound:
  - Despite $\Box$, every subformula has a **unique assignment and state of evaluation (UASE)**



  - Use UASEs to replace all SVARs with 0 or 1; relevant information can be computed on-the-fly in LogSpace
  - Evaluate remaining propositional formula (in NC^1)