

Qualitative Reasoning about Relative Direction of Oriented Points

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Abstract

An important issue in qualitative spatial reasoning is the representation of relative directions. In this paper we present simple geometric rules that enable reasoning about the relative direction between oriented points. This framework, the oriented point algebra $OPRA_m$, has a scalable granularity m . We develop a simple algorithm for computing the $OPRA_m$ composition tables and prove its correctness. Using a composition table, algebraic closure for a set of $OPRA_m$ statements is very useful to solve spatial navigation tasks. It turns out that scalable granularity is useful in these navigation tasks.

Keywords: Qualitative Spatial Reasoning, Constraint-based Reasoning, Qualitative Simulation

1. Introduction

The concept of *qualitative space* can be characterized by the following quotation from Galton [9]:

The divisions of qualitative space correspond to salient discontinuities in our apprehension of quantitative space.

If qualitative spatial divisions serve as knowledge representation in a reasoning system, deductive inferences can be realized as constraint-based reasoning [24]. An important issue in such qualitative spatial reasoning systems is the representation of relative direction [6], [1], [10]. Qualitative spatial constraint calculi typically store their spatial

knowledge in a composition table [24]. For a recent overview about qualitative spatial reasoning (QSR) we refer to Renz and Nebel [24].

A new qualitative spatial reasoning calculus about relative direction, the oriented point algebra \mathcal{OPRA}_m , which has a scalable granularity with parameter $m \in \mathbb{N}$, was presented in [16]. The motivation for this scalable granularity was that representing relatively fine distinctions was expected to be useful in more complex navigation tasks. It turned out to be difficult to analyze the reasoning rules for this calculus: the algorithm for computing composition tables presented in the original paper [16] contained gaps and errors. The correct and complete algorithm presented in [8] is quite lengthy and cumbersome: it is based on a complicated case distinction with dozens of cases (the paper is 29 pages long, 22 of which are devoted to the algorithm and its correctness). We give a very short (15 lines) algorithm that is both correct and simpler than the two existing algorithms. Moreover, it much better illustrates its goal, and we expect that it can be adapted and re-used for other similar calculi as well.

This paper is organized as follows: we will first give a short overview of the \mathcal{OPRA}_m calculus. We start this with a definition for a coarse type ($m = 2$), followed by the model for arbitrary $m \in \mathbb{N}$. Then we will present a new compact algorithm which performs \mathcal{OPRA}_m reasoning based on simple geometric rules, and prove its correctness. At the end we give an overview of several applications that use the \mathcal{OPRA}_m calculus for spatial navigation simulations and discuss the adequateness of specific choices for the granularity parameter m .

2. The oriented point algebra

Objects and locations can be represented as simple, featureless points. In contrast, the \mathcal{OPRA}_m calculus uses more complex basic entities: It is based on objects which are represented as oriented points. It is related to a calculus which is based on straight line segments (dipoles) [20]. Conceptually, the oriented points can be viewed as a transition from oriented line segments with concrete length to line segments with infinitely small length [18]. In this conceptualization the length of the objects no longer has any importance. Thus, only the orientation of the objects is modeled. An *o-point*, our term for an oriented point, is specified by a pair: a point and an orientation in the 2D-plane.

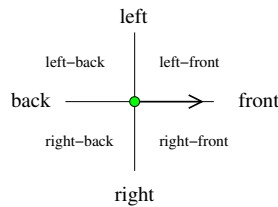


Figure 1: An oriented point and its qualitative spatial relative directions

2.1. Qualitative o-point relations

In a coarse representation a single o-point induces the sectors depicted in Fig. 1. “front,” “back,” “left,” and “right” are linear sectors; “left-front,” “right-front,” “left-back,” and “right-back” are quadrants. The position of the point itself is denoted as “same.” This qualitative granularity corresponds to Freksa’s single and double cross calculi [7, 25].

In $OPRA_2$, for the general case where the two points have different positions, we use the following relation symbols (the abbreviations lf, lb, rb, rf stand for “left-front,” “left-back,” “right-back,” and “right-front,” respectively):

$$\begin{array}{cccccccccccc} \text{front} & \text{lf} & \text{left} & \text{lb} & \text{back} & \text{rb} & \text{right} & \text{rf} & \text{front} & \text{lf} & \dots & \text{rf} \\ \text{front} & \text{front} & \text{front} & \text{front} & \text{front} & \text{front} & \text{front} & \text{front} & \text{lf} & \text{lf} & \dots & \text{rf} \end{array}$$

Here, a qualitative spatial relative direction relation between two o-points is represented by two pieces of information:

- the sector (seen from the first o-point) in which the second o-point lies (this determines the lower part of the relation symbol), and
- the sector (seen from the second o-point) in which the first o-point lies (this determines the upper part of the relation symbol).

The relations symbols are pairs of symbols which are written as stacked pairs. The sector name for the sector in which the second o-point position is located from the perspective of the first o-point is the lower part of the relation symbol. Conversely, the perspective from the second o-point generates the symbol put atop the first one. Altogether we obtain 8×8 base relations for the two points having different positions.

Then the configuration shown in Fig. 2 is expressed by the relation $A \overset{\text{lf}}{\underset{\text{rf}}{}} B$. If both points share the same position, the lower relation symbol part is the word “same” and the upper part denotes the direction of the second o-point with respect to the first one, as shown in Fig. 3.

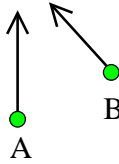


Figure 2: Qualitative spatial relation between two oriented points at different positions. The qualitative spatial relation depicted here is $A \overset{\text{lf}}{\underset{\text{rf}}{}} B$.

Altogether we obtain 72 different atomic relations (eight times eight general relations plus eight with the o-points at the same position). These relations are jointly exhaustive and pairwise disjoint (JEPD). The relation $\overset{\text{front}}{\underset{\text{same}}{}}$ is the identity relation.

2.2. Qualitative spatial reasoning

In order to apply constraint-based reasoning to a set of qualitative spatial relations, one typically starts with a jointly exhaustive and pairwise disjoint set of base relations

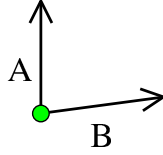


Figure 3: Qualitative spatial relation between two oriented points located at the same position. The qualitative spatial relation depicted here is $A \overset{rf}{\text{same}} B$.

[11, 13]. By forming the power set, one obtains the general relations, with bottom, top, intersection, union and complement of relations defined in the set-theoretic way. Moreover, an identity base relation and a converse operation (\smile) on base relations must be provided; the latter naturally extends to general relations. Finally, if composition of base relations cannot be expressed using general relations (strong composition), this operation is approximated by a *weak composition* [22]:

$$b_1 \diamond b_2 = \{b \mid (b_1 \circ b_2) \cap b \neq \emptyset\}$$

where $b_1 \circ b_2$ is the usual set theoretic composition

$$R \circ S = \{(x, z) \mid \exists y. (x, y) \in R, (y, z) \in S\}$$

and a general relation (= set of base relations) is identified with its union. Composition is called *strong* if \diamond coincides with the set-theoretic composition \circ , otherwise it is called *weak*. For details we refer to [13, 22].

The composition of relations must be computed based on the semantics of the relations. The compositions are usually computed only for the atomic relations; this information is stored in a composition table. The composition of compound relations can be obtained as the union of the compositions of the corresponding atomic relations. The compositions of the atomic relations can be deduced directly from the geometric semantics of the relations (see Section 2.4).

O-point constraints are written as xRy where x, y are variables for o-points and R is an \mathcal{OPRA}_m relation. Given a set Θ of o-point constraints, an important reasoning problem is deciding whether Θ is *consistent*, i.e., whether there is an assignment of all variables of Θ with dipoles such that all constraints are satisfied (a *solution*). A partial method for determining the inconsistency of a set of constraints Θ is the *path-consistency method* [15], which computes the algebraic closure of Θ . This method iterates the following operation until a fixed point is reached:

$$\forall i, j, k : R_{ij} \leftarrow R_{ij} \cap (R_{ik} \diamond R_{kj})$$

where i, j, k are nodes and R_{ij} is the relation between i and j . The resulting set of constraints is equivalent to the original set, i.e. it has the same set of solutions. If the empty relation occurs while performing this operation, Θ is inconsistent, otherwise the resulting set is algebraically closed¹ [22]. Note that algebraic closure does not

¹which means that it is path-consistent in the case that the algebra has a strong composition

always imply consistency, and indeed, [8] show that this implication does not hold for the \mathcal{OPRA}_m calculus. Indeed, consistency in \mathcal{OPRA}_m has been shown to be NP-hard even for scenarios in base relations [27], while algebraic closure is a polynomial approximation of consistency.

2.3. O-point calculi with scalable granularity

The design principle for the coarse \mathcal{OPRA}_2 calculus described above can be generalized to calculi \mathcal{OPRA}_m with arbitrary $m \in \mathbb{N}$. Then an angular resolution of $\frac{2\pi}{2m}$ is used for the representation (a similar scheme for absolute direction instead of relative direction was designed by Renz and Mitra [23]). The granularity used for the introduction of the \mathcal{OPRA}_m calculus in the previous section is $m = 2$, the corresponding \mathcal{OPRA}_m version is then called \mathcal{OPRA}_2 .

To formally specify the o-point relations we use a two-dimensional continuous space, in particular \mathbb{R}^2 . Every o-point S in the plane is an ordered pair of a point \mathbf{p}_S (represented by its Cartesian coordinates x and y , with $x, y \in \mathbb{R}$) and an orientation ϕ_S .

$$S = (\mathbf{p}_S, \phi_S), \quad \mathbf{p}_S = (x_S, y_S)$$

We distinguish the relative locations and directions of the two o-points A and B expressed by a calculus \mathcal{OPRA}_m according to the following scheme. For A, B with $\mathbf{p}_A \neq \mathbf{p}_B$, we define

$$\varphi_{AB} := \text{atan2}(y_B - y_A, x_B - x_A)$$

where $\text{atan2}(y, x)$ is the angle between the positive x -axis and the point (x, y) , normalized to the interval $] - \pi, \pi]$. By the properties of atan2 , we get

$$\varphi_{BA} = \varphi_{AB} + \pi$$

modulo normalization to $] - \pi, \pi]$. In the sequel, we will normalize all angles to this interval, reflecting the cyclic order of the directions. Hence, e.g. $-\pi$ stands for π . Moreover, in case that $\alpha > \beta$, the open interval $] \alpha, \beta[$ will stand for $] \alpha, \pi] \cup] - \pi, \beta[$. For example, $] \frac{\pi}{2}, -\frac{\pi}{2}[$ stands for $] \frac{\pi}{2}, \pi] \cup] - \pi, -\frac{\pi}{2}[$.

Similarly, we enumerate directions by using the $4m$ elements of the cyclic group \mathcal{Z}_{4m} . Each element of the cyclic group is interpreted as a range of angles as follows:

$$[i]_m = \begin{cases}]2\pi \frac{i-1}{4m}, 2\pi \frac{i+1}{4m}[& \text{if } i \text{ is odd} \\ \{2\pi \frac{i}{4m}\} & \text{if } i \text{ is even} \end{cases}$$

Conversely, for each angle α , there is a unique element $i \in \mathcal{Z}_{4m}$ with $\alpha \in [i]_m$.² If $\mathbf{p}_A \neq \mathbf{p}_B$, the relation $A \mathop{m}\mathcal{L}_i^j B$ ($i, j \in \mathcal{Z}_{4m}$) reads like this: given a granularity m , the relative position of B with respect to A is described by i and the relative position of

²The unary operation $[\cdot]$ taking integers to certain intervals must not be confused with the standard binary interval-building operations $[\cdot, \cdot],] \cdot, \cdot], [\cdot, \cdot [$ and $] \cdot, \cdot [$ on the reals.

A with respect to B is described by j . Formally, it represents the set of configurations satisfying

$$\varphi_{AB} - \phi_A \in [i]_m \quad \text{and} \quad \varphi_{BA} - \phi_B \in [j]_m.$$

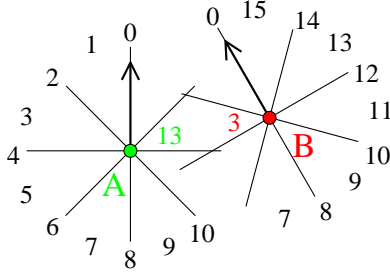


Figure 4: Two o-points in relation $A \ 4L_{13}^3 \ B$

Using this notation, a simple manipulation of the parameters yields the converse operation $({}_mL_j^i)^\smile = {}_mL_i^j$.

If $\mathbf{p}_A = \mathbf{p}_B$, the relation $A \ {}_mL_i \ B$ represents the set of configurations satisfying

$$\phi_B - \phi_A \in [i]_m.$$

Hence the relation for two identical o-points $A = B$ for arbitrary $m \in \mathbb{N}$ is $A \ {}_mL_0 \ B$. Using this notation a simple manipulation of the parameters yields the converse operation $({}_mL_i)^\smile = {}_mL(4m - i)$. The composition tables for the atomic relations of the $OPRA_m$ calculi can be computed using a small set of simple formulas detailed in the following subsection.

Fig. 4 shows an example for granularity $m = 4$. For $m = 2$ the previously-used symbolic names now get numeric counterparts, e.g. $\text{front}_{\text{front}}$ becomes ${}_2L_0^0$.

It should be mentioned that the passage from $OPRA_1$ to $OPRA_m$ ($m \geq 2$) is a qualitative jump: while $OPRA_1$ relations are preserved by all orientation-preserving affine bijections, for $m \geq 2$, $OPRA_m$ relations are only preserved by all angle-preserving affine bijections, see [18].

Proposition 1. *Composition in $OPRA_m$ is weak.*

Proof. By the left picture in Fig. 5, the configuration $A_1L_0^0B$, $B_1L_1^2C$ and $A_1L_3^3C$ is realizable. However, given A and C as in the right picture of Fig. 5, we have $A_1L_3^3C$, but we cannot find B with $A_1L_0^0B$ and $B_1L_1^2C$: by $A_1L_0^0B$, B 's carrier line is the same as A 's, and the two o-points face each other. But then, $B_1L_1^2C$ is not possible, since B would have to be located straight behind C (sector “back”).

The argument easily generalizes to $OPRA_m$ by considering $A_mL_0^0B$, $B_mL_1^{2m}C$ and $A_mL_{4m-1}^{4m-1}C$. \square

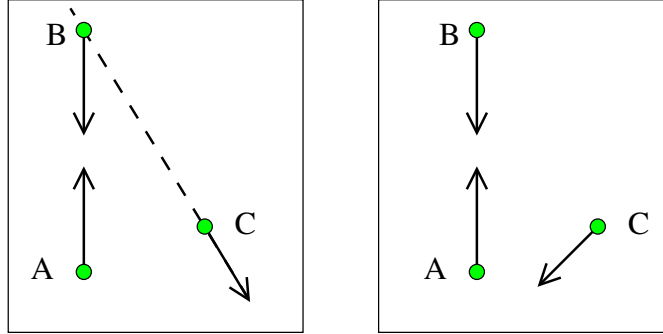


Figure 5: Composition in \mathcal{OPRA} is weak

2.4. Simple geometric rules for reasoning in \mathcal{OPRA}_m

The composition table can be viewed as a list (set) of all relation triples $Ar_{ab}B$, $Br_{bc}C$, $Cr_{ca}A$ for which r_{ab} , r_{bc} , and r_{ca} are consistent (A , B , and C being arbitrary o-points on the \mathbb{R}^2 plane). In the literature, there are two algorithms for computing the composition table: [16] presents a fairly simple algorithm, which, however, is incomplete (the relations ${}_m\angle i$ are not covered) and contains errors, and [8] provide a correct algorithm, which however is based on a complicated case distinction with dozens of cases according to whether the three involved relations involve even or odd numbers. We give an algorithm that is both correct and simpler than the two existing algorithms. In particular, the algorithm treats the cases distinguished in [8] in a uniform way; the case distinction only appears (in a much simpler form, since only two instead of six numbers are tested for evenness) in Proposition 2 below.

The first ingredient of the algorithm is a detection of complete turns. Recall that we work in the cyclic group \mathcal{Z}_{4m} , and the input parameters (i , j , k etc.) are understood to be in this group, as well as arithmetics performed within the algorithms. Accordingly, we also conveniently use -1 as synonym for $4m - 1$ etc. We define (see Fig. 6):

$$\text{turn}_m(i, j, k) \text{ iff } i + j + k \in \begin{cases} \{-1, 0, 1\}, & \text{if both } i \text{ and } j \text{ are odd} \\ \{0\}, & \text{otherwise} \end{cases}$$

This definition determines complete turns in the following sense:

- Proposition 2.**
1. $\text{turn}_m(i, j, k)$ iff $\exists \alpha \in [i]_m, \beta \in [j]_m, \gamma \in [k]_m . \alpha + \beta + \gamma = 0$
 2. $\text{turn}_m(i, j, k)$ implies that for any choice of one of the three angles in its interval, a suitable choice for the other two exists such that all three add up to 0.

Recall that angles are normalized into $] - \pi, \pi]$.

Proof. We prove the first statement by a case distinction.

Case 1: both i and j are even. This means that $[i]_m = \{2\pi \frac{i}{4m}\}$ and $[j]_m =$

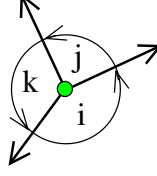


Figure 6: A complete turn $turn_m(i, j, k)$.

$\{2\pi \frac{j}{4m}\}$. Hence,

$$\begin{aligned}
 & \exists \alpha \in [i]_m, \beta \in [j]_m, \gamma \in [k]_m \cdot \alpha + \beta + \gamma = 0 \\
 \text{iff } & \exists \gamma \in [k]_m \cdot 2\pi \frac{i+j}{4m} + \gamma = 0 \\
 \text{iff } & i + j + k = 0 \\
 \text{iff } & turn_m(i, j, k)
 \end{aligned}$$

Case 2: i is odd and j is even. This means that $[i]_m =]2\pi \frac{i-1}{4m}, 2\pi \frac{i+1}{4m}[$ and $[j]_m = \{2\pi \frac{j}{4m}\}$. Hence,

$$\begin{aligned}
 & \exists \alpha \in [i]_m, \beta \in [j]_m, \gamma \in [k]_m \cdot \alpha + \beta + \gamma = 0 \\
 \text{iff } & \exists \gamma \in [k]_m \cdot -2\pi \frac{j}{4m} - \gamma \in]2\pi \frac{i-1}{4m}, 2\pi \frac{i+1}{4m}[\\
 \text{iff } & \exists \gamma \in [k]_m \cdot \gamma \in]2\pi \frac{-i-j-1}{4m}, 2\pi \frac{-i-j+1}{4m}[\\
 \text{iff } & \exists \gamma \in [k]_m \cdot \gamma \in [-i-j]_m \\
 \text{iff } & k = -i - j \\
 \text{iff } & turn_m(i, j, k)
 \end{aligned}$$

Case 3: i is even and j is odd: analogous to case 2.

Case 4: both i and j are odd. This means that $[i]_m =]2\pi \frac{i-1}{4m}, 2\pi \frac{i+1}{4m}[$ and $[j]_m =]2\pi \frac{j-1}{4m}, 2\pi \frac{j+1}{4m}[$. Hence,

$$\begin{aligned}
 & \exists \alpha \in [i]_m, \beta \in [j]_m, \gamma \in [k]_m \cdot \alpha + \beta + \gamma = 0 \\
 \text{iff } & \exists \gamma \in [k]_m \cdot \gamma \in]2\pi \frac{-i-j-2}{4m}, 2\pi \frac{-i-j+2}{4m}[\\
 \text{iff } & \exists \gamma \in [k]_m \cdot \gamma \in [-i-j-1]_m \cup [-i-j]_m \cup [-i-j+1]_m \\
 \text{iff } & k \in \{-i-j-1, -i-j, -i-j+1\} \\
 \text{iff } & turn_m(i, j, k)
 \end{aligned}$$

The second statement is straightforward when inspecting the proof above. \square

Next, we turn to triangles. In a triangle, the sum of angles is always π . This can be expressed via a turn, when adding another $-\pi$ (i.e. $-2m$ in the abstract representation), leading to $turn_m(i, j, k - 2m)$. Moreover, all angles have the same sign (expressed as $sign_m(i) = sign_m(j) = sign_m(k)$). We include the degenerate case where two angles are 0 and the remaining one is π (this corresponds to three points on a line), but

we exclude the case of three angles being π (this is not geometrically realizable). This leads to the following definitions (see also Fig. 7):

$$\text{sign}_m(i) = \begin{cases} 0, & \text{if } (i \bmod 4m = 0) \vee (i \bmod 4m = 2m) \\ 1, & \text{if } i \bmod 4m < 2m \\ -1, & \text{otherwise} \end{cases}$$

$$\text{triangle}_m(i, j, k) \text{ iff } \text{turn}_m(i, j, k - 2m) \wedge (i, j, k) \neq (2m, 2m, 2m) \wedge \text{sign}_m(i) = \text{sign}_m(j) = \text{sign}_m(k)$$

Here, the angle π also has sign 0, which corresponds to geometric intuition and to the fact that the choice between $-\pi$ and π to represent this angle is rather arbitrary.

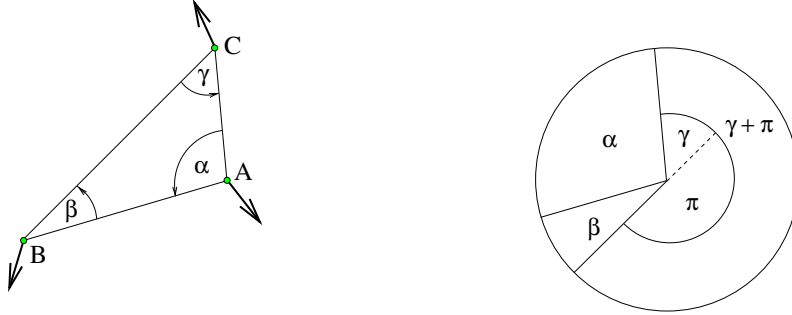


Figure 7: Triangles can be defined in terms of complete turns.

From the above discussion, it is then straightforward to see

Proposition 3.

$\text{triangle}_m(i, j, k)$ iff $\exists \alpha \in [i]_m, \beta \in [j]_m, \gamma \in [k]_m$. there exists a triangle with angles α, β, γ

Fig. 8 shows all possible triangles for $m = 1, 2, 3, 4$.

Algorithm 1 now gives the complete algorithm for computing OPRA_m compositions (actually, it is more a sequence of mathematical definitions, which however is directly implementable as a computer program). The ternary predicate opra computes the composition of OPRA_m relations, that is, $\text{opra}(R, S, T)$ holds if T belongs to the weak composition $R \diamond S$. opra is computed using a case distinction whether the relation is a same-relation or not; for three relations this yields eight cases. opra is defined in terms of the predicates triangle_m and turn_m and the function sign_m discussed above. Note that we have slightly rephrased the definition of $\text{turn}_m(i, j, k)$, the new version already taking care of our convention regarding the cyclic group \mathcal{Z}_{4m} and thus being directly implementable as a computer program using the usual integers instead of \mathcal{Z}_{4m} .

We illustrate algorithm 1 with an example. Fig. 9 shows a composition in OPRA_4 with $A \text{ } {}_4\mathcal{L}_{13}^3 B$ and $B \text{ } {}_4\mathcal{L}_{15}^9 C$ and $A \text{ } {}_4\mathcal{L}_{15}^7 C$. Let us check that indeed $\text{opra}({}_4\mathcal{L}_{13}^3, {}_4\mathcal{L}_{15}^9, {}_4\mathcal{L}_{15}^7)$ holds. Since we are in the last case, with $m = 4, i = 13, j = 3, k = 15, l = 9, s = 15$ and $t = 7$, we have to show $\exists 0 \leq u, v, w < 16. \text{turn}_4(u, 3, 15) \wedge$

1	(0,0,2),(0,2,0), (1,1,1), (2,0,0), (3,3,3)
2	(0,0,4),(0,4,0), (1,1,1),(1,1,2),(1,1,3),(1,2,1),(1,3,1), (2,1,1), (3,1,1), (4,0,0), (5,7,7), (6,7,7), (7,5,7),(7,6,7),(7,7,5),(7,7,6),(7,7,7)
3	(0,0,6),(0,6,0), (1,1,3),(1,1,4),(1,1,5),(1,2,3),(1,3,1),(1,3,2),(1,3,3),(1,4,1),(1,5,1), (2,1,3),(2,2,2),(2,3,1), (3,1,1),(3,1,2),(3,1,3),(3,2,1),(3,3,1), (4,1,1), (5,1,1), (6,0,0), (7,11,11), (8,11,11), (9,9,11),(9,10,11),(9,11,9),(9,11,10),(9,11,11), (10,9,11),(10,10,10),(10,11,9), (11,7,11),(11,8,11),(11,9,9),(11,9,10),(11,9,11),(11,10,9),(11,11,7),(11,11,8), (11,11,9)
4	(0,0,8),(0,8,0), (1,1,5),(1,1,6),(1,1,7),(1,2,5),(1,3,3),(1,3,4),(1,3,5),(1,4,3),(1,5,1),(1,5,2),(1,5,3), (1,6,1),(1,7,1), (2,1,5),(2,2,4),(2,3,3),(2,4,2),(2,5,1), (3,1,3),(3,1,4),(3,1,5),(3,2,3),(3,3,1),(3,3,2),(3,3,3),(3,4,1),(3,5,1), (4,1,3),(4,2,2),(4,3,1), (5,1,1),(5,1,2),(5,1,3),(5,2,1),(5,3,1), (6,1,1), (7,1,1), (8,0,0), (9,15,15), (10,15,15), (11,13,15),(11,14,15),(11,15,13),(11,15,14),(11,15,15), (12,13,15),(12,14,14),(12,15,13), (13,11,15),(13,12,15),(13,13,13),(13,13,14),(13,13,15),(13,14,13),(13,15,11), (13,15,12),(13,15,13), (14,11,15),(14,12,14),(14,13,13),(14,14,12),(14,15,11), (15,9,15),(15,10,15),(15,11,13),(15,11,14),(15,11,15),(15,12,13),(15,13,11), (15,13,12),(15,13,13),(15,14,11),(15,15,9),(15,15,10),(15,15,11)

Figure 8: All possible triangles for $m = 1, 2, 3, 4$.

$turn_4(v, 1, 3) \wedge turn_4(w, 9, 9) \wedge triangle_4(u, v, w)$, noting that $-13 \equiv 3 \pmod{16}$, $-15 \equiv 1 \pmod{16}$ and $-7 \equiv 9 \pmod{16}$. Choosing e.g. $u = 13$, $v = 11$ and

Algorithm 1 Checking entries of the \mathcal{OPRA}_m composition table

$$\text{turn}_m(i, j, k) \text{ iff } |(i + j + k + 2m) \bmod 4m - 2m| \leq (i \bmod 2) \times (j \bmod 2)$$

$$\text{sign}_m(i) = \begin{cases} 0, & \text{if } (i \bmod 4m = 0) \vee (i \bmod 4m = 2m) \\ 1, & \text{if } i \bmod 4m < 2m \\ -1, & \text{otherwise} \end{cases}$$

$$\text{triangle}_m(i, j, k) \text{ iff} \\ \text{turn}_m(i, j, k - 2m) \wedge (i, j, k) \neq (2m, 2m, 2m) \wedge \text{sign}_m(i) = \text{sign}_m(j) = \text{sign}_m(k)$$

$$\text{opra}({}_m\angle i, {}_m\angle k, {}_m\angle s) \text{ iff } \text{turn}_m(i, k, -s)$$

$$\text{opra}({}_m\angle i, {}_m\angle k, {}_m\angle^t_s) \text{ iff false}$$

$$\text{opra}({}_m\angle i, {}_m\angle^l_k, {}_m\angle s) \text{ iff false}$$

$$\text{opra}({}_m\angle i, {}_m\angle^l_k, {}_m\angle^t_s) \text{ iff } l = t \wedge \text{turn}_m(i, k, -s)$$

$$\text{opra}({}_m\angle^j_i, {}_m\angle k, {}_m\angle s) \text{ iff false}$$

$$\text{opra}({}_m\angle^j_i, {}_m\angle k, {}_m\angle^t_s) \text{ iff } i = s \wedge \text{turn}_m(t, k, -j)$$

$$\text{opra}({}_m\angle^j_i, {}_m\angle^l_k, {}_m\angle s) \text{ iff } j = k \wedge \text{turn}_m(i, -l, -s)$$

$$\text{opra}({}_m\angle^j_i, {}_m\angle^l_k, {}_m\angle^t_s) \text{ iff}$$

$$\exists 0 \leq u, v, w < 4m. \text{turn}_m(u, -i, s) \wedge \text{turn}_m(v, -k, j) \wedge \text{turn}_m(w, -t, l) \wedge \text{triangle}_m(u, v, w)$$

$w = 15$ provides suitable witnesses for the existential quantification; this is illustrated in Fig. 10. (At the end of this section we discuss how to find witnesses efficiently.)

Theorem 4. *Algorithm 1 computes composition in \mathcal{OPRA}_m , that is, $\text{opra}(R, S, T)$ holds if T belongs to the weak composition $R \diamond S$*

Proof.

Case $\text{opra}({}_m\angle i, {}_m\angle k, {}_m\angle s)$. Since the points of all these o-points are the same, their direction must add up to a complete turn. More precisely, the configuration ${}_m\angle i, {}_m\angle k, {}_m\angle s$ is realizable iff there are o-points A, B and C with $\mathbf{p}_A = \mathbf{p}_B = \mathbf{p}_C$, $\phi_B - \phi_A \in [i]_m$, $\phi_C - \phi_B \in [k]_m$, and $\phi_C - \phi_A \in [s]_m$. Since for such A, B and C , $(\phi_B - \phi_A) + (\phi_C - \phi_B) - (\phi_C - \phi_A) = 0$ (i.e. we have a complete turn), by Proposition 2 this is in turn equivalent to $\text{turn}_m(i, k, -s)$. This is illustrated by Fig. 11 (but note that our argument is analytic and does not rely on pictures).

Cases $\text{opra}({}_m\angle i, {}_m\angle k, {}_m\angle^t_s)$, $\text{opra}({}_m\angle i, {}_m\angle^l_k, {}_m\angle s)$ and $\text{opra}({}_m\angle^j_i, {}_m\angle k, {}_m\angle s)$. Since sameness of points is transitive, these cases are not realizable.

Cases $\text{opra}({}_m\angle i, {}_m\angle^l_k, {}_m\angle^t_s)$, $\text{opra}({}_m\angle^j_i, {}_m\angle k, {}_m\angle^t_s)$ and $\text{opra}({}_m\angle^j_i, {}_m\angle^l_k, {}_m\angle s)$. We here only treat the case $\text{opra}({}_m\angle i, {}_m\angle^l_k, {}_m\angle^t_s)$; the other two cases being analogous. The configuration $A_m\angle iB, B_m\angle^l_kC, A_m\angle^t_sC$ is realizable iff

$$\left. \begin{array}{l} \text{there are o-points } A, B \text{ and } C \text{ with} \\ \mathbf{p}_A = \mathbf{p}_B, \phi_B - \phi_A \in [i]_m, \\ \phi_{BC} - \phi_B \in [k]_m, \phi_{CB} - \phi_C \in [l]_m, \\ \phi_{AC} - \phi_A \in [s]_m \text{ and } \phi_{CA} - \phi_C \in [t]_m. \end{array} \right\} (*)$$

We now show that $(*)$ is equivalent to

$$l = t \text{ and } \text{turn}_m(i, k, -s).$$

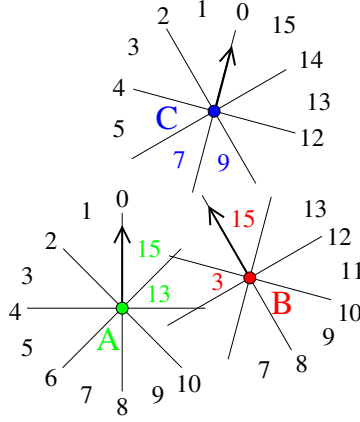


Figure 9: A composition of two \mathcal{OPR}_{A_4} relations $A \ 4L_{13}^3 \ B$ and $B \ 4L_{15}^9 \ C$, leading to $A \ 4L_{15}^7 \ C$.

Assume (*). By $\mathbf{p}_A = \mathbf{p}_B$, we have $\phi_{BC} = \phi_{AC}$ and $\phi_{CB} = \phi_{CA}$; from the latter, we also get $l = t$. Moreover, $(\phi_B - \phi_A) + (\phi_{BC} - \phi_B) - (\phi_{AC} - \phi_A) = 0$ is a turn, and by Proposition 2, we get $\text{turn}_m(i, k, -s)$. Conversely, assume $l = t$ and $\text{turn}_m(i, k, -s)$. By Proposition 2, there are angles α, β, γ with $\alpha \in [i]_m, \beta \in [k]_m$ and $\gamma \in [-s]_m$. Choose A arbitrarily. Then define B by $\mathbf{p}_B = \mathbf{p}_A$ and $\phi_B = \alpha - \phi_A$. Then choose \mathbf{p}_C on the half-line starting from \mathbf{p}_A and having angle β to B and $-\gamma$ to A . Finally, choose ϕ_C such that $\phi_{CA} - \phi_C = \phi_{CB} - \phi_C \in [t]_m = [l]_m$. This ensures the conditions of (*). This is illustrated by Fig. 12.

Case $\text{opra}(m\angle_i^j, m\angle_k^l, m\angle_s^t)$. We need to show that the existence of a configuration $A_m\angle_i^j B, B_m\angle_k^l C$ and $A_m\angle_s^t C$ is equivalent to

$$\left. \begin{array}{l} \exists 0 \leq u, v, w < 4m. \\ \text{turn}_m(u, -i, s) \wedge \text{turn}_m(v, -k, j) \wedge \text{turn}_m(w, -t, l) \\ \wedge \text{triangle}_m(u, v, w) \end{array} \right\} (**)$$

Given $A_m\angle_i^j B, B_m\angle_k^l C$ and $A_m\angle_s^t C$, let α, β and γ be the angles of the triangle $\mathbf{p}_A \mathbf{p}_B \mathbf{p}_C$, that is,

$$\begin{aligned} \alpha &= \phi_{AB} - \phi_{AC} \\ \beta &= \phi_{BC} - \phi_{BA} \\ \gamma &= \phi_{CA} - \phi_{CB} \end{aligned}$$

Let $u, v, w \in \mathcal{Z}_{4m}$ be such that $\alpha \in [u]_m, \beta \in [v]_m$ and $\gamma \in [w]_m$. By Proposition 3, $\text{triangle}_m(u, v, w)$. At the corners of the triangle $\mathbf{p}_A \mathbf{p}_B \mathbf{p}_C$, the following complete turns can be formed (see Fig. 13):

- at \mathbf{p}_A : $(\phi_{AB} - \phi_{AC}) - (\phi_{AB} - \phi_A) + (\phi_{AC} - \phi_A)$, corresponding to $\text{turn}_m(u, -i, s)$ by Proposition 2,
- at \mathbf{p}_B : $(\phi_{BC} - \phi_{BA}) - (\phi_{BC} - \phi_B) + (\phi_{BA} - \phi_B)$, corresponding to $\text{turn}_m(v, -k, j)$,

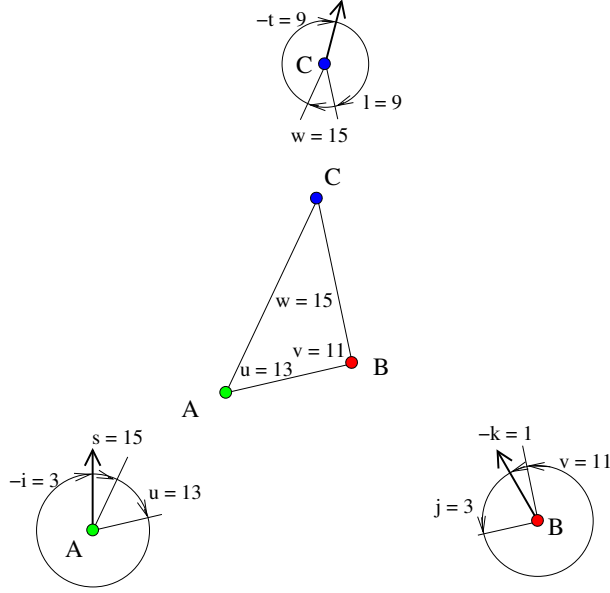


Figure 10: The composition in Fig. 9 is obtained by $turn_4(13, 3, 15)$, $turn_4(11, 1, 3)$, $turn_4(15, 9, 9)$ and $triangle_4(13, 11, 15)$. Note that the turns at A and C are drawn in clockwise direction (then 1 is the largest clockwise turn, and 15 the smallest one).

- at \mathbf{p}_C : $(\phi_{CA} - \phi_{CB}) - (\phi_{CA} - \phi_C) + (\phi_{CB} - \phi_C)$, corresponding to $turn_m(w, -t, l)$.

This shows (**). Conversely, assume (**). By $triangle_m(u, v, w)$ and Proposition 3, we can choose \mathbf{p}_A , \mathbf{p}_B and \mathbf{p}_C such that

$$\begin{aligned}\phi_{AB} - \phi_{AC} &\in [u]_m \\ \phi_{BC} - \phi_{BA} &\in [v]_m \\ \phi_{CA} - \phi_{CB} &\in [w]_m\end{aligned}$$

Since $turn_m(u, -i, s)$, by Proposition 2, we can find α_A , β_A and γ_A such that $\alpha_A + \beta_A + \gamma_A = 0$ and $\alpha_A \in [-i]_m$, $\beta_A \in [s]_m$ and $\gamma_A \in [u]_m$. Proposition 2(2) implies that it is possible to choose $\gamma_A = \phi_{AB} - \phi_{AC}$ (note that the latter angle is also in $[u]_m$). Put $\phi_A := \phi_{AB} + \alpha_A$, then $\phi_{AB} - \phi_A = -\alpha_A \in [i]_m$, and $\phi_{AC} - \phi_A = (\phi_{AB} - \phi_A) - (\phi_{AB} - \phi_{AC}) = -\alpha_A - \gamma_A = \beta_A \in [s]_m$. ϕ_B and ϕ_C can be chosen similarly, fulfilling the constraints given by j and k resp. l and t . \square

Using Algorithm 1, a composition table for $OPRA_m$ can be computed by enumerating all possible triples and only keeping those for which the predicate *opra* holds.

The run time of the predicate *opra* is $O(m^3)$, since the algorithm contains an existential quantification over the variables u , v , and w , ranging from 0 to $4m-1$. However, the existential quantification can be replaced by a constant number of case distinctions: e.g. we look for u such that $turn_m(u, -i, s)$. But since $u - i + s$ must add up to -1 ,

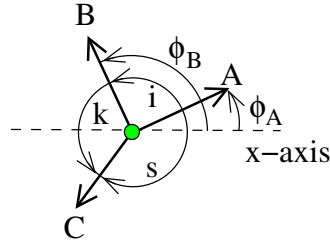


Figure 11: Case $p_A = p_B = p_C$: one complete turn at $p_A = p_B = p_C$.

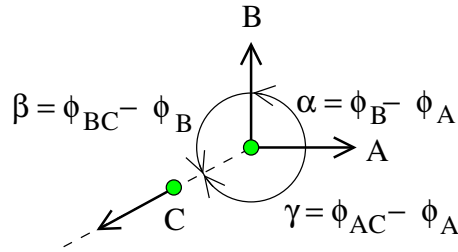


Figure 12: Case $p_A = p_B \neq p_C$: one complete turn at $p_A = p_B$.

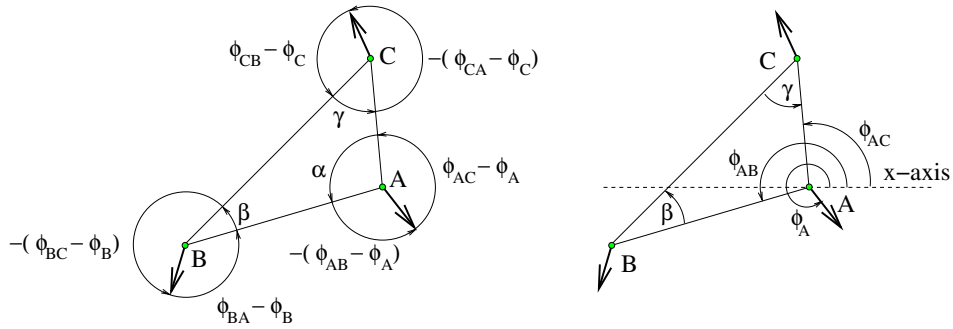


Figure 13: Case p_A, p_B and p_C all distinct: three complete turns. The right picture illustrates the definitions of ϕ_{AB} and ϕ_{AC} , as well as the geometric interpretation of ϕ_A .

0 or 1, it is clear that u must be taken from the set $\{i - s - 1, i - s, i - s + 1\}$. As a result, we get an improved run time that is constant. This holds only when assuming a register machine with arithmetic operations executed in constant time. For a Turing machine with binary representations of numbers, basic arithmetic operations take time $\log m$. Then the run time is $O(\log m)$.

The computation of the composition of two relations needs to enumerate all possible third relations, and then check each triple in constant time. Since there are $(4m)^2 + 4m$ relations, this takes $O(m^2)$ time, which is the same time as in [8]. Again, for Turing machines, this is multiplied by a factor of $\log m$, hence we get an overall running time of $O(m^2 \log m)$. Of course, the same remark applies to the algorithm of [8].

A Haskell version of the *opra* algorithm (also implementing the above optimization of the existential quantification) can be downloaded from http://quail.rsis.e.anu.edu.au/uploads/Opra_comp.hs.

The $OPRA_m$ calculus can be used to express many other qualitative position calculi [4].

3. Applications of the $OPRA_m$ calculus

Spatial knowledge expressed in $OPRA_m$ can be used for deductive reasoning based on constraint propagation (algebraic closure), resulting in the generation of useful indirect knowledge from partial observations in a spatial scenario. Several researchers have developed applications using the $OPRA_m$ calculus. We will give a short overview and then make some concluding comments about the first $OPRA_m$ calculus applications in our concluding section which follows.

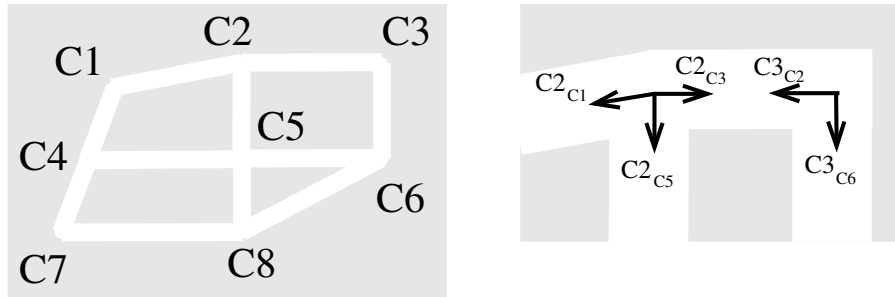


Figure 14: Street networks with unique crossing names (detail with o-points to the right including the two relations $C2_{C3}^{\text{front}} C3_{C2}$, and $C2_{C3}^{\text{right}} C2_{C5}$).

In a simple application by Lücke et al. [14] for benchmarking purposes between different spatial calculi, a spatial agent (a simulated robot, cognitive simulation of a biological system etc.) explores a spatial scenario. The agent collects local observations and wants to generate survey knowledge. Fig. 14 shows a spatial environment consisting of a street network. The notation $C2_{C1}$ refers to the o-point at position $C2$ with an intrinsic orientation towards point $C1$. In this street network some streets continue straight after a crossing and some streets meet with orthogonal angles. These features are typical of real-world street networks and can be directly represented in $OPRA_2$ expressions about o-points that constitute relative directions of o-points located at crossings and pointing to neighboring (e.g. visible) crossings. For example

two relations corresponding to local observations referring to the street network part depicted in Fig. 14 are: $C2_{C3}^{\text{front}} C3_{C2}$ and $C2_{C3}^{\text{right}} C2_{C5}$. Spatial reasoning in this spatial agent simulation uses constraint propagation (e.g., algebraic closure computation) to derive indirect constraints between the relative location of streets which are further apart from local observations between neighboring streets. The resulting survey knowledge can be used for several tasks including navigation tasks. The details of this scenario can be found in Lücke et al. [14].

A related application developed by Wallgrün [26] uses qualitative spatial reasoning with $OPRA_2$ to determine the correct graph structure from a sequence of local observations collected by a simulated robot while moving through an environment consisting of hallways. These hallway networks are analogous to the street networks of Lücke et al., but the local observations are modeled in a more complex, realistic way. The identity of a crossing revisited after a cyclic path is not given but has to be inferred, which makes navigation much more challenging. Since there are many ambiguities left, the task is to track the multiple geometrically possible topologies of the network during an incremental observation. Thus, the goal of Wallgrün’s qualitative mapping algorithm is to process the history of observations and determine all route graph hypotheses which can be considered valid explanations so far. This consistency checking can be based on qualitative spatial reasoning about positions. The local relative observations are modeled based on $OPRA_2$ expressions about o-points in a similar way as in the street network described above. Composition-based $OPRA_2$ reasoning is the key part of the spatial reasoning. In this application the search space is significantly reduced and the solution quality improved by composition-based reasoning. Wallgrün [26] concludes that relative direction information provided with $OPRA_2$ is only slightly inferior to absolute direction information provided with the cardinal direction calculus [12] but has the advantage of being more accessible. Wallgrün also applied this approach to real sensor data collected with a mobile robot, confirming the positive results from the simulation. Wallgrün’s application shows that using a composition table, algebraic closure is very useful for solving spatial navigation tasks even if algebraic closure is not sufficient to decide the global consistency of constraint networks.

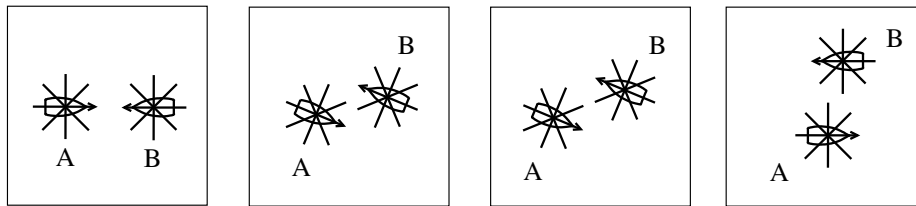


Figure 15: Representation of vessel navigation with conceptual neighborhood in $OPRA_4$

A comprehensive simulation which uses the $OPRA_4$ calculus for an important subtask was built by Dylla et al. [2]. Their system called SailAway simulates the behavior of different vessels following declarative (written) navigation rules for collision avoidance. This system can be used to verify whether a given set of rules leads to stable avoidance between potentially colliding vessels. The different vessel cate-

gories that determine their right of way priorities are represented in an ontology. The movement of the vessels is described by a method called conceptual neighborhood-based reasoning (CNH reasoning). CNH reasoning describes whether two spatial configurations of objects can be transformed into each other by small changes [5], [9]. A CNH transformation can be an object movement in a short period of time. Fig. 15 shows a CNH transition diagram which represents the relative trajectories of two rule following vessels. The depicted sequence between two vessels A and B is: $A \text{ } {}_4\angle_0^0 B \rightarrow A \text{ } {}_4\angle_1^1 B \rightarrow A \text{ } {}_4\angle_2^2 B \rightarrow A \text{ } {}_4\angle_3^3 B$. Based on this qualitative representation of trajectories, CNH reasoning is used as a simple, abstract model of the navigation of the potentially colliding vessels in the SailAway simulator [2]³.

These three applications make use of qualitative spatial reasoning with $OPRA_2$ or $OPRA_4$ in simulated spatial agent scenarios. The granularity $m = 2$ can model straight continuation and right angles which are important for representing idealized street networks. The granularities $m = 2$ and $m = 4$ also correspond to earlier work on computational models of linguistic projective expressions (left, right, in front, behind) by Moratz et al. [17] [21]. The applications presented in this section could benefit from additional qualitative relative distance knowledge. The TPCC calculus presented by Moratz and Ragni [19] is a first step towards this direction. However, in contrast to our new results for the $OPRA_m$ calculus, the TPCC calculus only has a complex, manually derived and therefore unreliable composition table.

4. Summary and conclusion

A calculus for representing and reasoning about qualitative relative direction information was recently introduced in [16] by one of the authors of this paper. In this calculus, oriented points serve as the basic entities since they are the simplest spatial entities that have an intrinsic orientation. Sets of base relations can have adjustable granularity levels in this calculus.

However, the previous work had difficulties in finding an adequate method to derive the composition table for this calculus, which is central to constraint-based reasoning. In this paper we provided new simple geometric rules for computing the calculus's composition based on triples of oriented points.

We gave a short overview about three applications that are based on oriented points and their relative position represented as $OPRA_m$ relations with granularity $m = 2$, or $m = 4$ which seem to be suited for linguistically inspired spatial expressions. These applications show that using a composition table, algebraic closure is very useful for solving spatial navigation tasks even if algebraic closure is not sufficient to decide the global consistency of constraint networks.

As future work we plan to augment our relative orientation calculus with additional qualitative relative distance knowledge.

³An earlier version of qualitative navigation simulation by Dylla and Moratz can be found in [3].

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