Institutes: A simpler semantics for heterogeneous ontologies

Till Mossakowski\textsuperscript{1,2} and Oliver Kutz\textsuperscript{1}

1. Research Center on Spatial Cognition (SFB/TR 8), University of Bremen, Germany
2. DFKI GmbH Bremen, Germany
till.mossakowski@dfki.de okutz@informatik.uni-bremen.de

Abstract

We consider heterogeneous ontologies, i.e. ontologies whose specifications allow to blend different ontology languages such as propositional logic, OWL, first-order logic, Common Logic, or various extensions of these. In order to give such ontologies a precise formal semantics, we need a suitable notion of abstract logic. We here introduce a new framework, called institute theory, that extends abstract entailment and satisfaction systems with signatures while staying much simpler than the well-known framework of institutions. In particular, we abstain entirely from the usage of category theoretic machinery. The introduced framework is intended to give the logical and semantical underpinning for a forthcoming ISO standard for heterogeneous ontologies and ontology interoperability, OntoIOp.

Introduction

OWL is a popular language for ontologies. Yet, the restriction to a decidable description logic often hinders ontology designers from expressing knowledge that cannot (or can only in quite complicated ways) be expressed in a description logic. A practice to deal with this problem is to intersperse OWL ontologies with first-order axioms, e.g. in the case of bio-ontologies where mereological relations such as parthood are of great importance, though not definable in OWL. However, these remain informal annotations to inform the human designer, rather than first-class citizens of the ontology with formal semantics and impact on reasoning. One goal of this paper is to equip such heterogeneous ontologies with a precise semantics and proof theory.

A variety of languages is used for formalising ontologies. Some of these, as RDF, OBO and UML, can be seen more or less as fragments and notational variants of OWL, while others, like F-logic and Common Logic (CL), clearly go beyond the expressiveness of OWL.

In this paper, we face this diversity not by proposing yet another ontology language that would subsume all the others, but by accepting the diverse reality and formulating means (on a sound and formal semantic basis) to compare and integrate ontologies that are written in different formalisms. This view is a bit different from that of unifying languages such as OWL and CL, which are meant to be “universal” formalisms (for a certain domain/application field), into which everything else can be mapped and represented. While such “universal” formalisms are clearly important and helpful for reducing the diversity of formalisms, it is still a matter of fact that no single formalism will be the Esperanto that is used by everybody. It is therefore important to both accept the existing diversity of formalisms and to provide means of organising their coexistence in a way that enables formal interoperability among ontologies.

In this work, we lay the foundation for a distributed ontology language (DOL), which will allow users to use their own preferred ontology formalism while becoming interoperable with other formalisms (see [Kutz et al., 2011] for further details). The DOL language is in particular intended to be at the core of a new ISO standardisation effort on ontology interoperability called OntoIOp.\textsuperscript{1} At the heart of our approach is a graph of ontology languages and translations. This graph will enable users to

- relate ontologies that are written in different formalisms (e.g. prove that the OWL version of the foundational ontology DOLCE is logically entailed by the first-order version);
- re-use ontology modules even if they have been formulated in a different formalism;
- re-use ontology tools like theorem provers and module extractors along translations between formalisms.

Previous presentations of semantics of heterogeneous logical theories [Tarlecki, 2000; Diaconescu, 2002; Mossakowski and Tarlecki, 2009; Kutz, Mossakowski, and Lücke, 2010; Mossakowski and Kutz, 2011] relied heavily on the theory of institutions [Goguen and Burstall, 1992]. The central insight of the theory of institutions is that logical notions such as model, sentence, satisfaction and derivability should be indexed over signatures (vocabularies). In order to abstract from any specific form of signature, category theory is used. However, the use of category theory diminishes the set of

\textsuperscript{1}DOL is currently under standardisation as Working Draft ISO/WD 17347 in ISO/TC 37/SC 3 ‘Systems to manage terminology, knowledge and content’. See also http://ontolog.cim3.net/cgi-bin/wiki.pl?OntoIOp

In this work, we start with the signature-free approach, and then introduce signatures a posteriori, assuming that they form a preorder. While this approach covers only signature inclusions, not renamings, it is much simpler than the category-based approach of institutions. Moreover, since we add signatures a posteriori, our approach is more flexible than institutions, and not just a specialisation of these.

A comprehensive overview of the translational relationships between ontology languages has been given in [Mossakowski and Kutz, 2011]. We rely on this work here and present the resulting logic-translation graph in Figure 1, adapted to the notation and terminology used in this paper.

Indeed, this paper introduces a lot of terminology, some new, some known from the general field of abstract logic, in particular from institution theory. To provide a better overview, we have included at the end of the paper two tables. Table 1 shows notions and terminology common to both institution theory and the theory of institutes, introduced in this paper. The new terminology for institutes vis-à-vis those of institutions is summarised in Table 2.

### Consequence systems

The most basic ingredient of a logic is an ‘entailment’ relation between sentences. In the literature this is often also called a ‘consequence’ relation. The notion of (abstract) consequence relations has been formalised by Gentzen, Tarski and Scott [Gentzen, 1969; Scott, 1974; Avron, 1991].

**Definition 1** A Tarskian consequence relation (abbreviated: CR) \( S = (\text{Sen}, \vdash) \) on a set of sentences \( \text{Sen} \) is a binary relation \( \vdash \subseteq P(\text{Sen}) \times \text{Sen} \) between sets of sentences and sentences such that

1. reflexivity: for any \( \varphi \in \text{Sen} \), \( \{ \varphi \} \vdash \varphi \).
2. monotonicity: if \( \Gamma \vdash \varphi \) and \( \Gamma' \supseteq \Gamma \) then \( \Gamma' \vdash \varphi \).
3. transitivity: if \( \Gamma \vdash \varphi_i \), for \( i \in I \), and \( \Gamma \cup \{ \varphi_i \mid i \in I \} \vdash \psi \), then \( \Gamma \vdash \psi \).

A CR is said to be compact when for each \( E \vdash \varphi \) there exists a finite subset \( E_0 \subseteq E \) such that \( E_0 \vdash \varphi \).

While this signature-free treatment enjoys simplicity and is wide-spread in the literature, many concepts and definitions found in logics, e.g. the notion of a conservative extension, involve the vocabulary or signature used in sentences. Signatures can be extended with new symbols; abstractly, this leads to an ordering relation on signatures.

**Definition 2** A consequence system (short CS) \((\Sigma, \leq, \text{Sen}, \vdash)\) consists of

- a preordered class \((\Sigma, \leq)\) of signatures,
- a CR \( S = (\text{Sen}, \vdash) \),
- a function \( \text{sig} : \text{Sen} \rightarrow \Sigma \), giving the (minimal) signature of a sentence.

A CS is **sig-complete**, if \((\text{Sen}, \vdash)\) has arbitrary joins (and therefore also arbitrary meets). In many practical examples, signatures will be structured sets, and unions will provide joins. Of course, if some size restriction is employed, this will also have an effect on the existence of joins (e.g. with a restriction to finite signatures, only finite joins will exist).

The set of sentences over a signature \( \Sigma \) is defined as

\[
\text{Sen}(\Sigma) := \{ \varphi \in \text{Sen} | \text{sig}(\varphi) \leq \Sigma \}
\]

With this, we get

\[
\Sigma_1 \leq \Sigma_2 \Rightarrow \text{Sen}(\Sigma_1) \subseteq \text{Sen}(\Sigma_2)
\]

A **theory** is a set \( T \subseteq \text{Sen} \) of sentences, such that its signature defined by the supremum

\[
\text{sig}(T) := \bigvee_{\varphi \in T} \text{sig}(\varphi)
\]

exists. It is immediate that \( T_1 \subseteq T_2 \) implies \( \text{sig}(T_1) \leq \text{sig}(T_2) \). A **closed** theory is one that is closed under \( \vdash \).

**Example 3** We have the following standard examples of consequence systems, where the consequence relation is defined as in the literature, and \( \text{sig} \) extracts the set of symbols from a sentence.

- Classical propositional logic, intuitionistic propositional logic, modal propositional logics \( K, S_4, S_5 \) etc. Here, \( \text{Sig} \) is the class of all sets (elements of which are propositional symbols), ordered by inclusion. Although all joins exist, this is not a complete lattice (there is no top element). Alternately, only finite sets can be considered; in this case, \( \text{Sig} \) does not have all joins, and theories may involve only finitely many propositional symbols.
- A third option is to let \( \text{Sig} \) be the complete lattice of subsets of a fixed countable set.
- **ACC** and other description logics. Here, signatures are triples, consisting of a set of concept names, a set of role names, and a set of individual names, ordered by pointwise inclusion. Again, we have different options regarding the size of signatures.
- Classical first-order logic. Here, signatures are pairs consisting of a set of function symbols with arities, and a set of predicate symbols with arities.
- Many-sorted first-order logic. Here, signatures are many-sorted first-order signatures, consisting of sorts and typed function and predicate symbols.

Fix an arbitrary consequence systems.

**Definition 4** Given theories \( \Gamma_1 \subseteq \Gamma_2 \), \( \Gamma_2 \) is said to be a conservative extension of \( \Gamma_1 \), if for all \( \varphi \in \text{Sen}(\text{sig}(\Gamma_1)) \),

\[
\Gamma_2 \vdash \varphi \Rightarrow \Gamma_1 \vdash \varphi.
\]

3Notice that it may still fail to be a complete lattice in the standard sense due to lack of antisymmetry, or because it is a proper class.
Morphisms

Heterogeneous ontologies consisting of parts written in different logics only make sense if the involved logics are somehow related to each other. Therefore, we study morphisms of CRs.

**Definition 5** Given two CRs \( S_1 = (\text{Sen}_1, \models^1) \) and \( S_2 = (\text{Sen}_2, \models^2) \), a morphism of consequence relations \( \alpha : S_1 \rightarrow S_2 \). CR-morphism for short, is a function \( \alpha : \text{Sen}_1 \rightarrow \text{Sen}_2 \) such that

\[
\Gamma \models^1 \varphi \quad \text{implies} \quad \alpha(\Gamma) \models^2 \alpha(\varphi)
\]

If also the converse implication holds, the CR morphism is said to be conservative.

**Definition 6** Given two consequence systems \( S_1 = (\text{Sig}^1, \leq^1, \text{Sen}^1, \models^1) \) and \( S_2 = (\text{Sig}^2, \leq^2, \text{Sen}^2, \models^2) \), a consequence system comorphism (CS-comorphism) is a map \( \rho = (\Phi, \alpha) : S_1 \rightarrow S_2 \), consisting of

- a monotone map \( \Phi : (\text{Sig}^1, \leq^1) \rightarrow (\text{Sig}^2, \leq^2) \), and
- a CR-morphism \( \alpha : (\text{Sen}_1, \models^1) \rightarrow (\text{Sen}_2, \models^2) \),

such that

\[
\text{sig}^2(\alpha(\varphi_1)) \leq \Phi(\text{sig}^1(\varphi_1))
\]

for any sentence \( \varphi_1 \in \text{Sen}_1 \).

A CS morphism is said to be conservative if all the involved CR-morphisms are.

**Definition 7** A heterogeneous consequence environment \( E \) is given by a preorder \( (L^E, \leq^E) \), such that \( L^E \) is a set of CSs, and for each \( S_1 \leq^E S_2 \), there is a CS-comorphism \( \rho_{S_1,S_2} : S_1 \rightarrow S_2 \), subject to the following requirements:

- \( \rho_{S_1,S_1} \) is the identity;
- for \( S_1 \leq E S_2 \leq E S_3 \), \( \rho_{S_1,S_3} = \rho_{S_2,S_3} \circ \rho_{S_1,S_2} \).

We assume the notation \( \rho_{S_1,S_2} = (\Phi_{S_1,S_2}, \alpha_{S_1,S_2}) \).

**Example 8** Consider the heterogeneous consequence environment OWL-FOL consisting of the CSs for OWL and FOL and the standard comorphism from OWL to FOL.

Cycles in heterogeneous consequence environments can only be used to represent one and the same logic differently (consider e.g. propositional logic with different sets of base connectives).

**Proposition 9** In a heterogeneous consequence environment \( E \), if \( S_1 \equiv S_2 \), then \( S_1 \) and \( S_2 \) are isomorphic.

**Proof.** \( \rho_{S_2,S_1} \circ \rho_{S_1,S_2} = \rho_{S_1,S_1} = \text{id} \) and \( \rho_{S_1,S_2} \circ \rho_{S_2,S_1} = \rho_{S_2,S_2} = \text{id} \). \( \square \)
We now come to our central construction: the flattening of a heterogeneous consequence environment into a single consequence system. This construction combines all involved signatures and sentences by disjoint union, that is, each signature and sentence is marked with the CR that it stems from. Interaction between CRs is given by the comorphisms (see the example below).

**Definition 10** Given a heterogeneous consequence environment \( E \), define the Grothendieck consequence system (Grothendieck CS) \( E^\# = (Sig, \leq, Sen, \vdash) \) as follows:

- \( Sig = \bigcup_{S \in E} Sig^S \); let \( \Phi^S : Sig^S \rightarrow Sig \) denote the inclusion;

- \( \leq \) is the least preorder generated by

  \[
  \Sigma_1 \leq \Sigma_2 \text{ in } S \Rightarrow \Phi^S(\Sigma_1) \leq \Phi^S(\Sigma_2)
  \]

- \( Sen = \bigcup_{S \in E} Sen^S \); let \( \alpha^S : Sen^S \rightarrow Sen \) denote the inclusion;

- \( sig(\alpha^S(\varphi)) := \Phi^S(sig^S(\varphi)) \), which is obviously well-defined;

- \( \vdash \) is the least CR relation on \( Sen \) satisfying

  \[
  (1) \quad \Gamma \vdash \varphi \text{ in } S \Rightarrow \alpha^S(\Gamma) \vdash \alpha^S(\varphi)
  
  (2) \quad \alpha^S(\alpha_S^S(\varphi)) \vdash \alpha^S(\varphi) \text{ for } \Sigma_1 \leq \Sigma_2
  
  (3) \quad \alpha^S(\varphi) \vdash \alpha^S(\alpha_S^S(\varphi)) \text{ for } \Sigma_1 \equiv \Sigma_2
  \]

Here, \( S_1 \equiv S_2 \) if \( S_1 \leq S_2 \) and \( S_1 \geq S_2 \). Note that in case \( \langle L^E, \leq_E \rangle \) has arbitrary joins, \( \vdash \) can be defined to be the least relation satisfying the above two conditions and then is always a CR relation.

Let us spell out what the Grothendieck CS is for the case of \( OWL\text{-}FOL \). Basically, signatures and sentences of OWL and FOL are put side by side (disjoint union), and we have embeddings \( \Phi^{OWL} \) resp. \( \Phi^{FOL} \) of the OWL resp. FOL signatures into \( OWL\text{-}FOL \) signatures, and likewise \( \alpha^{OWL} \), \( \alpha^{FOL} \) for sentences. The interaction is as follows: each OWL signature is less than its translation to FOL. This implies also some interaction at the level of sentences. Namely, consider OWL signature \( \Sigma \) and its translation to FOL \( \Phi(\Sigma) \). Then in \( OWL\text{-}FOL \), this signature is \( \Sigma = \Phi^{FOL}(\Phi(\Sigma)) \). Now \( Sen(\Sigma) \) contains not only the FOL sentences for \( \Phi(\Sigma) \), but also the OWL sentences for \( \Sigma \). Moreover, in \( Sen(\Sigma) \), at the level of entailment, OWL sentences (over \( \Sigma \)) are implied by their FOL translations (over \( \Phi(\Sigma) \)).

This means that we can freely mix OWL and FOL sentences in one theory, while still keeping track of their origin, which is important when e.g. feeding the OWL sentences into an OWL reasoner.

While Def. 10 generates the preorder \( \leq \) on signatures by mixing intra-CR and inter-CR steps, actually one inter-CR step followed by one intra-CR step suffices:

**Lemma 11** In \( E^\# \), \( \Sigma_1 \leq \Sigma_2 \) iff \( \Sigma_1 = \Phi^{R_1}(\Sigma'_1) \), \( \Sigma_2 = \Phi^{R_2}(\Sigma'_2) \), \( R_1 \leq R_2 \) and \( \Phi^{R_1,R_2}(\Sigma'_1) \leq \Sigma'_2 \).

**Proof.** Let \( \leq \) be defined by the condition above. We show that \( \leq \Rightarrow \leq \).

“\( \leq \).” \( \Sigma \) satisfies the properties in Def. 10, and \( \leq \) is the least relation that does so.

“\( \supseteq \).” If \( \Sigma_1 \subseteq \Sigma_2 \), then \( \Sigma_1 = \Phi^{R_1}(\Sigma'_1) \subseteq \Phi^{R_2}(\Phi^{R_1,R_2}(\Sigma'_1)) \leq \Phi^{R_2}(\Sigma'_2) = \Sigma_2 \).

**Lemma 12** In \( E^\# \), \( \varphi \in Sen(\Phi^S(\Sigma)) \) iff \( \Sigma_1 \leq \Sigma \) and \( sig(\varphi) \leq \Sigma_1 \) in \( S_1 \) and \( \Phi_{S_1}(\Sigma_1) \leq \Sigma \).

**Proposition 13** Entailment of the individual CSs is faithfully reflected in the Grothendieck CS:

\[ \Gamma \vdash \varphi \in R \text{ iff } \alpha^R(\Gamma) \vdash \alpha^R(\varphi) \text{ in } E^\# \]

**Proof.** “\( \subseteq \):” immediate by Def. 10.

“\( \supseteq \):” If \( \alpha^R(\Gamma) \vdash \alpha^R(\varphi) \) in \( E^\# \), then there is a derivation

\[ \alpha^R(\Gamma), \alpha^R(\varphi_1), \ldots, \alpha^R(\varphi_n) = \alpha^R(\varphi), \]

where each step is obtained from previous steps by rules (1) to (3) of Def. 10. Now rule (1) keeps \( R \), while rule (2) and (3) keep or decrease it. Therefore, for each \( R_i, R \leq R_i \leq R \), i.e. all involved CRs belong to the same cycle. We thus can replace the proof by one that lives entirely in \( R \):

\[ \alpha^R(\Gamma), \alpha^R(\varphi_1), \ldots, \alpha^R(\varphi_n) = \alpha^R(\varphi), \]

where all applications of rule (1) in \( R_i \) are transfered via \( \rho_{R_i,R} \) to \( R \), while all applications of rule (2) can be replaced by reflexivity steps.

**Model Theory**

While consequence relations capture entailment, satisfaction systems \([Camielli et al., 2008]\), called ‘rooms’ in the terminology of [Goguen and Burstall, 1985], capture the Tarskian notion of satisfaction of a sentence in a model:

**Definition 14** A triple \( R = (S, M, \models) \) is called a satisfaction system, or room, if \( R \) consists of

- a set \( Sen \) of sentences,
- a class \( M \) of models, and
- a binary relation \( \models \subseteq M \times Sen \), called the satisfaction relation.

A theory \( \Gamma \subseteq Sen \) is satisfiable, if it has a model \( M \) (i.e., a model \( M \in M \) such that \( M \models \varphi \) for \( \varphi \in \Gamma \)). **Semantic entailment** is defined as usual: for a theory \( \Gamma \subseteq Sen \) and \( \varphi \in Sen \), we write \( \Gamma \models \varphi \), if all models satisfying all sentences in \( \Gamma \) also satisfy \( \varphi \).

The following result is folklore [Avron, 1991]:

**Proposition 15** Semantic entailment \( \models \) is a Tarskian consequence relation.

Tarskian semantics gains much of its importance from the fact that it allows definitions of CRs that are often easier to grasp and closer to intuitive concepts than definitions of CSs using proof rules. For example, we can equip the consequence relation \( CPL \) (classical propositional logic) with a semantics by taking valuations of the propositional variables into \( \{T, F\} \) as models.

Crucially, as in the case of CRs, we also provide a signature-indexed version:
A corridor is called model-expansive, if $\beta$ is a surjection.

**Definition 19** Given institutes $R_1 = (\Sigma_1, \leq_1, S_{e1}, M_1, \models)$ and $R_2 = (\Sigma_2, \leq_2, S_{e2}, M_2, \models)$, an institute comorphism $(\phi, \alpha, \beta) : R_1 \to R_2$ consists of

- a monotone map $\Phi : (\Sigma_1^1, \leq_1^1) \to (\Sigma_2^2, \leq_2^2)$, and
- a partial corridor $(\alpha, \beta) : (S_{e1}, M_1, \models) \to (S_{e2}, M_2, \models)$ such that

$$\phi^2(\alpha(\varphi_1)) \leq \Phi(\phi^1(\varphi_1)) \text{ for any sentence } \varphi_1 \in S_{e1}^1;$$

- for each signature $\Sigma$, $\beta$ restricts to a total function $\beta_\Sigma : Mod_2(\Phi(\Sigma)) \to Mod_1(\Sigma);$  

- model translation commutes with reduct, that is, given $\Sigma_1 \leq \Sigma_2$ in $R$ and a $\Phi(\Sigma_2)$-model $M$ in $R_2$,

$$\beta_\Sigma_2(M)|_{\Sigma_1} = \beta_{\Sigma_1}(M|_{\Phi(\Sigma_1)}).$$

An institute comorphism is called model-expansive, if all $\beta_\Sigma$ are surjective.

**Definition 20** A heterogeneous satisfaction environment $E$ is defined like a heterogeneous consequence environment (Def. 7), but using institutes $R$ and institute-comorphisms $\rho_{R_1, R_2} : R_1 \to R_2$ instead of CSs and CS-comorphisms.

**Example 21** Example 8 can be extended to a heterogeneous satisfaction environment by equipping OWL and FOL with models, as well as the standard comorphism from OWL to FOL with the obvious model translation.

**Proposition and Definition 22** Given a heterogeneous satisfaction environment $E$, the Grothendieck institute $E^# = (\Sigma, \leq, S_{e}, M, \models)$ is defined as follows:

- $\Sigma, \phi$ (for sentences), $\leq$ and $S_{e}$, as well as $\Phi^S$ and $\alpha^S$ are defined as in Def. 10 (the latter two are named $\Phi^R$ and $\alpha^R$ now due to the indexing over institutes instead of CSs);
- $\Phi = \bigcup_{R \in E} \Phi^R$; let $\beta^R : \mathcal{M}^R \to M$ denote the inclusion;
- $\phi(\beta^R(M)) = \Phi^R(\phi(M))$;

- reduct is given by reduct in the individual rooms and by the comorphisms. More precisely, if $\Sigma_1 \leq \Sigma_2$, then by Lemma 11, $\Sigma_1 = \Phi^R(\Sigma_1^1), \Sigma_2 = \Phi^R(\Sigma_2^2), R_1 \leq R_2$ and $\Phi_{R_1, R_2}(\Sigma_1^1) \leq \Sigma_2$, Given a $\Sigma_2$-model $M_2$, which must be of form $\beta^R(\Sigma_1^1)$, we define

$$M_2|_{\Sigma_2} = \beta^R(\rho_{R_1, R_2}(M_1|_{\rho_{R_1, R_2}(\Sigma_1^1)}))$$

- $\beta^R(\Sigma) \models \alpha^S(\varphi) \text{ iff } R_1 \leq R_2 \text{ and } \beta_{R_1, R_2}(\varphi) \text{ is defined and satisfies } \varphi \text{ in } R_1.$

We need to show that the properties of an institute are satisfied. Invariance of satisfaction under reduct follows from that for the individual institutes and the satisfaction condition for comorphisms, using Lemma 11. Compositionality of reduct follows from that for the individual institute, from the definition of $\rho_{R_1, R_3}$ (see Def. 7) and model translation commuting with reducts. For signature coherence, assume $M \models \varphi$. This means $M = \beta^R(\alpha^S(\varphi)), \varphi \models \alpha^R(\varphi')$,
Proposition 23 Logical consequence of the component logics is faithfully reflected in the Grothendieck construction:

\[ \Gamma \models \varphi \text{ in } \mathcal{R} \text{ iff } \alpha^{\mathcal{R}}(\Gamma) \models \alpha^{\mathcal{R}}(\varphi) \text{ in } E^\# \]

Proof. “\( \Leftarrow \)”: Let \( M \models \alpha^{\mathcal{R}}(\Gamma) \text{ in } E^\# \). By definition of \( \models \text{ in } E^\# \), there is some model \( M_1 \) with \( M = \beta^{\mathcal{R}_1}(M_1) \), \( \mathcal{R} \leq \mathcal{R}_1 \) and \( \beta^{\mathcal{R}_1}(M_1) \models \Gamma \) in \( \mathcal{R}_1 \). By assumption, \( \beta^{\mathcal{R}_1}(M_1) \models \varphi \). Hence, \( M \models \alpha^{\mathcal{R}}(\varphi) \).

“\( \Rightarrow \)”: Let \( M \models \Gamma \), then \( \beta^{\mathcal{R}}(M) \models \alpha^{\mathcal{R}}(\Gamma) \). By assumption, \( \beta^{\mathcal{R}}(M) \models \alpha^{\mathcal{R}}(\varphi) \), hence \( M \models \varphi \). \( \square \)

Proposition 24 Assume that \( \Gamma \) is satisfiable. Then

\[ \alpha^{\mathcal{R}_1}(\Gamma) \models \alpha^{\mathcal{R}_1}(\varphi) \Rightarrow \mathcal{R}_2 \leq \mathcal{R}_1 \]

Proof. Let \( M \) be a model of \( \Gamma \). Then \( \beta^{\mathcal{R}_1}(M) \models \alpha^{\mathcal{R}_1}(\Gamma) \). Hence \( \beta^{\mathcal{R}_1}(M) \models \alpha^{\mathcal{R}_1}(\varphi) \). By definition, this implies \( \mathcal{R}_2 \leq \mathcal{R}_1 \). \( \square \)

Linking proof theory and model theory

In this section, we link entailment and satisfaction at various levels: the signature-free level, the signature-indexed level and the heterogeneous level.

Definition 25 A logic room \((\text{Sen}, \mathcal{M}, \models, \vdash)\) consists of a room \((\text{Sen}, \mathcal{M}, \models)\) and an consequence relation \((\text{Sen}, \vdash)\). A logic room is sound, if

\[ \Gamma \models \varphi \text{ implies } \Gamma \models \varphi, \]

and is complete, if the converse holds. It is weakly complete, if \( \emptyset \models \varphi \) implies \( \emptyset \vdash \varphi \).

A logic corridor \((\alpha, \beta)\) is a corridor \((\alpha, \beta)\) such that \( \alpha \) simultaneously is a CR morphism.

Definition 26 A logique \( \mathcal{R} = (\text{Sig}, \leq, \text{Sen}, \mathcal{M}, \models, \vdash) \) is a signature-indexed logic room, i.e. consists of a CS \((\text{Sig}, \leq, \text{Sen}, \vdash)\) and an institute \((\text{Sig}, \leq, \text{Sen}, \models)\). The notions of soundness and completeness are inherited from logic rooms.

Definition 27 Given logiques \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \), a logique comorphism \((\Phi, \alpha, \beta): \mathcal{R}_1 \rightarrow \mathcal{R}_2 \) is a comorphism of institutes such that \((\Phi, \alpha)\) simultaneously is a comorphism of consequence systems.

A logique comorphism is said to be conservative, if the comorphism of consequence systems is so. It is said to be model expansive, if the comorphism of institutes is so. A sublogique comorphism is a logique comorphism \((\Phi, \alpha, \beta): \mathcal{R}_1 \rightarrow \mathcal{R}_2 \) with \( \Phi \) injective and preorder-reflecting, \( \alpha \) injective and \( \beta \) bijective for each \( \Sigma \). In this case, \( \mathcal{R}_1 \) is said to be a sublogique of \( \mathcal{R}_2 \).

Definition 28 A heterogeneous logic environment \( E \) is defined like a heterogeneous satisfaction environment (Def. 20), but using logiques \( \mathcal{R} \) and logique-comorphisms \( \rho_{\mathcal{R}_1, \mathcal{R}_2}: \mathcal{R}_1 \rightarrow \mathcal{R}_2 \) instead of institutes and institute-comorphisms.

The Grothendieck construction for heterogeneous logic environments (called Grothendieck logique) is similar to that for heterogeneous satisfaction environments, but combining it with that for heterogeneous consequence environments.

Theorem 29 Soundness and completeness lift from a heterogeneous logic environment to homogeneous entailments (i.e. involving formulas of one logic only) on its Grothendieck logique.

Proof. Immediate by Props. 13 and 23. \( \square \)

Theorem 30 Soundness lifts from a heterogeneous logic environment to arbitrary entailments on its Grothendieck logique.

Proof. Let \( E \) be a heterogeneous logic environment. Assume that all logiques in \( E \) are sound. We show that the consequence relation defined in Def. 10 is sound. We consider the three rules:

1. If \( \Gamma \vdash \varphi \) in \( \mathcal{R} \), by soundness of \( \mathcal{R} \), \( \Gamma \models \varphi \) in \( \mathcal{R} \).

2. Let \( \mathcal{R}_1 \leq \mathcal{R}_2 \) and \( M \models \alpha^{\mathcal{R}_2}(\alpha^{\mathcal{R}_1}\models \varphi) \). By definition of \( \models \text{ in } E^\# \), there is some model \( M_3 \) in \( \mathcal{R}_3 \) with \( M = \beta^{\mathcal{R}_1}(M_3) \), \( \mathcal{R}_2 \leq \mathcal{R}_3 \) and \( \beta^{\mathcal{R}_1}(M_3) \models \alpha^{\mathcal{R}_1}\models \varphi \) in \( \mathcal{R}_2 \). By the satisfaction condition for \( \rho_{\mathcal{R}_1, \mathcal{R}_2}, \beta^{\mathcal{R}_1, \mathcal{R}_2}(\beta^{\mathcal{R}_1}\models \varphi) \) in \( \mathcal{R}_2 \). By definition of \( \models \), \( M \models \alpha^{\mathcal{R}_1}(\varphi) \). Thus, deriving \( \alpha^{\mathcal{R}_2}(\alpha^{\mathcal{R}_1}\models \varphi) \) is sound.

3. Let \( \mathcal{R}_1 \equiv \mathcal{R}_2 \). By Prop. 9, \( \rho_{\mathcal{R}_2, \mathcal{R}_1} \) is inverse to \( \rho_{\mathcal{R}_1, \mathcal{R}_2} \). Therefore, \( \alpha^{\mathcal{R}_1}(\varphi) \) semantically entails \( \alpha^{\mathcal{R}_1, \mathcal{R}_2}(\alpha^{\mathcal{R}_1, \mathcal{R}_2}(\varphi)) \). By 2. above, this semantically entails \( \alpha^{\mathcal{R}_2}(\alpha^{\mathcal{R}_1, \mathcal{R}_2}(\varphi)) \). Allogether, we have shown that deriving \( \alpha^{\mathcal{R}_1}(\varphi) \vdash \alpha^{\mathcal{R}_2}(\alpha^{\mathcal{R}_1, \mathcal{R}_2}(\varphi)) \) is sound.

The following example shows that in the definition of Grothendieck consequence system, we cannot just reverse rule (2).

Example 31 Consider the heterogeneous logic environment consisting of propositional logic \( PL \) and first-order logic \( FOL \) with the logique comorphism \( FOL \rightarrow PL \) that maps predicate symbols to propositional symbols (thereby forgetting arities), erases all terms and all quantifiers. A \( PL \) model is mapped to a \( FOL \) model by taking a singleton
universe, noting that a valuation of a propositional variable corresponds to the interpretation of a predicate over a singleton set. Note that this comorphism is neither conservative nor model-expansive. Consider the FOL sentence $\forall x. P(x) \lor Q(x)$. The comorphism maps it to the PL sentence $P \lor Q$. Now in the Grothendieck logique, we have

$$\alpha^{PL}(P \lor Q) \models \alpha^{FOL}(\forall x. P(x) \lor Q(x))$$

but

$$\alpha^{FOL}(\forall x. P(x) \lor Q(x)) \not\models \alpha^{PL}(P \lor Q)$$

The above example still does not rule out a converse of rule (2) for conservative and model expansive logique comorphisms. The following example does:

**Example 32** Consider the heterogeneous logic environment consisting of OWL, FOL and CL, and (conservative and model expansive) comorphisms from OWL to FOL and CL, capturing the standard translation. However, there are no comorphisms between FOL and CL. Take a FOL model $M$ satisfying an OWL sentence $\phi$. $M$ does not take the translation $\alpha_{\text{OWL},\text{CL}}(\phi)$ of $\phi$ to CL, just because there is no way to get a CL-model out of $M$. Of course, we can also switch the roles of FOL can CL.

**Relation to Institution Theory**

We have developed a framework for heterogeneous logical theories based on preorders of signatures, and thus avoided the complexity of the theory of institutions that is based on category theory. A natural question is then how our framework relates to institution theory. We first recall the basic notions, and therefore (unlike in the rest of this paper), assume some acquaintance with the basic notions of category theory and refer to [Adámek, Herrlich, and Strecker, 1990] or [Mac Lane, 1998] for an introduction.

**Definition 33** An **institution** is a quadruple $I = (\text{Sign}, \text{Sen}, \text{Mod}, \models)$ consisting of the following:

- a category $\text{Sign}$ of signatures and signature morphisms,
- a functor $\text{Sen}: \text{Sign} \rightarrow \text{Set}$ giving, for each signature $\Sigma$, the set of sentences $\text{Sen}(\Sigma)$, and for each signature morphism $\Sigma: \Sigma' \rightarrow \Sigma''$, the sentence translation map $\text{Sen}(\sigma): \text{Sen}(\Sigma) \rightarrow \text{Sen}(\Sigma')$, where often $\text{Sen}(\sigma)(\varphi)$ is written as $\sigma(\varphi)$,
- a functor $\text{Mod}: \text{Sign}^{\text{op}} \rightarrow \text{Cat}$ giving, for each signature $\Sigma$, the category of models $\text{Mod}(\Sigma)$, and for each signature morphism $\Sigma: \Sigma' \rightarrow \Sigma''$, the reduct functor $\text{Mod}(\sigma): \text{Mod}(\Sigma') \rightarrow \text{Mod}(\Sigma)$, where often $\text{Mod}(\sigma)(M')$ is written as $M'_\sigma$, and $M'_\sigma$ is called the $\sigma$-reduct of $M'$, while $M'$ is called the $\sigma$-expansion of $M'_\sigma$,
- a satisfaction relation $\models_\Sigma \subseteq \text{Mod}(\Sigma) \times \text{Sen}(\Sigma)$ for each $\Sigma \in \text{Sign}$.

such that for each $\sigma: \Sigma \rightarrow \Sigma'$ in $\text{Sign}$ the following satisfaction condition holds:

$$(\ast) \quad M' \models_{\Sigma'} \sigma(\varphi) \iff M'|_\sigma \models_{\Sigma} \varphi$$

for each $M' \in \text{Mod}(\Sigma')$ and $\varphi \in \text{Sen}(\Sigma)$, expressing that truth is invariant under change of notation and context.\(^7\)

**Proposition 34** Every institute gives rise to an institution.

**Proof.** Each signature preorder can be construed as a signature category (a so-called thin category where each hom-set has at most one element). For a signature $\Sigma$, $\text{Sen}(\Sigma)$ and $\text{Mod}(\Sigma)$ have been already defined. Since $\Sigma_1 \subseteq \Sigma_2$ implies $\text{Sen}(\Sigma_1) \subseteq \text{Sen}(\Sigma_2)$, sentence translation along signature morphisms is just the identity. Model reduction is given by the institute, this is functorial by compositionality of reducts. Satisfaction needs to be restricted to pairs of models and sentences over the same signature.\(^8\) Invariance under reducts implies the satisfaction condition. \(\square\)

We also have a converse relationship: each inclusive institution [Goguen and Rosu, 2004] gives rise to an institute.

Institutions are translated via comorphisms:

**Definition 35 (Institution Comorphism)** Given two institutions $I_1$ and $I_2$ with $I_1 = (\text{Sign}_1, \text{Mod}_1, \text{Sen}_1, \models_1)$ and $I_2 = (\text{Sign}_2, \text{Mod}_2, \text{Sen}_2, \models_2)$, an institution comorphism from $I_1$ to $I_2$ consists of a functor $\Phi: \text{Sign}_1 \rightarrow \text{Sign}_2$, and natural transformations $\beta: \text{Mod}_2 \circ \Phi \Rightarrow \text{Mod}_1$ and $\alpha: \text{Sen}_1 \Rightarrow \text{Sen}_2$, such that $M' \models_2 \beta(\Sigma) \alpha(\varphi) \iff \beta(\Sigma')(M') \models_1 \varphi$.

holds, called the satisfaction condition.

**Proposition 36** Every institute comorphism gives rise to an institution comorphism.

**Proof.** Let $(\Phi, \alpha, \beta): R_1 \rightarrow R_2$ be an institute comorphism. $\Phi$ is a monotone map between preorders, hence a functor between the corresponding thin categories. The condition $\text{sig}^2(\alpha(\varphi_1)) \leq \Phi(\text{sig}^1(\varphi_1))$ ensures that the sentence translation $\alpha$ can be restricted to $\alpha_\Sigma: \text{Sen}_1(\Sigma) \rightarrow \text{Sen}_2(\Phi(\Sigma))$. Naturality of $\alpha$ follows since the $\alpha_\Sigma$ are restrictions of $\alpha$ and intra-institute sentence translation is just inclusion. $\beta$ can be restricted to $\beta_\Sigma$ by definition of institute comorphism. Naturality of $\beta$ follows since model translation commutes with reduct. \(\square\)

Analogously, we can relate consequence systems and entailment systems, and logics in the sense of [Meseguer, 1989] and logiques.

\(^7\)Note, however, that non-monotonic formalisms can only indirectly be covered this way, but compare, e.g., Guerra, 2001.

\(^8\)This means that satisfaction information for non-matching model/sentence pairs is lost when moving from an institute to an institution. However, this loss of information is not as harsh as it looks: by signature coherence, we can always embed the signature of the sentence into that of the model.
Conclusion and Outlook

We have presented a semantics for heterogeneous ontologies and logical theories that avoid the technicalities of category theory, and is instead based on order theory. Future work will provide proof support for such heterogeneous theories via the Heterogeneous Tool Set (Hets) [Mossakowski, Maeder, and Lütthich, 2007], which is based on (Grothendieck) institutions. The bridge from institute theory to Hets will use Prop. 36, but we expect that Hets also needs to be adapted in order to capture the type of interaction between sentences of different logics exhibited in Examples 31 and 32. This type of interaction is not present in Grothendieck institutions.

An open question is the lifting of completeness from heterogeneous logic environments to the Grothendieck logique.

Some readers might find it unpleasant that sentences in the Grothendieck logique have a memory of their logical origin, such that an OWL sentence may be strictly weaker than its translation to FOL, see Example 32. An alternative satisfaction relation for the Grothendieck logique could make \( \varphi \) and \( \alpha_{R_1, R_2}(\varphi) \) logically equivalent at least for model-expansive \( \rho \). We have refrained from doing so here because the resulting satisfaction relation would have a quite complicated definition, and we feel that satisfaction of a sentence in a model should be a simple concept.

An important extension of the present work will be the generalisation to non-monotonic logics [Arieli and Avron, 2000], in order to capture e.g. the mixture of OWL with non-monotonic rules, or paraconsistent description logics.

Acknowledgements Work on this paper has been supported by the DFG-funded collaborative research center SFB/TR 8 Spatial Cognition (project II-[OntoSpace]).

References

Table 1: Notions common to Institutions and Institutes

<table>
<thead>
<tr>
<th>Definition</th>
<th>Description</th>
<th>Institutions / Institutes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Def. 1</td>
<td>Tarskian consequence relation</td>
<td>CR: Tarskian consequence relation</td>
</tr>
<tr>
<td>Page. 2</td>
<td>non-logical vocabulary</td>
<td>signature</td>
</tr>
<tr>
<td>Def. 5</td>
<td>consequence preserving translation</td>
<td>CR-morphism: morphism of consequence relations</td>
</tr>
<tr>
<td>Def. 14</td>
<td>Tarskian satisfaction of a sentence in a model</td>
<td>satisfaction system/room</td>
</tr>
<tr>
<td>Def. 18</td>
<td>satisfaction-invariant translation</td>
<td>(partial) corridor</td>
</tr>
<tr>
<td>Def. 25</td>
<td>consequence relation &amp; satisfaction</td>
<td>logic room</td>
</tr>
<tr>
<td>Page. 6</td>
<td>translation for this</td>
<td>logic corridor</td>
</tr>
</tbody>
</table>

Table 2: Institutions vs. Institutes: new notions for institutes and correspondents in institution theory

<table>
<thead>
<tr>
<th>Definition</th>
<th>Description</th>
<th>Institution Theory</th>
<th>Institute Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>Page. 2</td>
<td>translation between vocabulary</td>
<td>signature morphisms, forming a category</td>
<td>signature embedding, forming a preorder</td>
</tr>
<tr>
<td>Def. 2</td>
<td>signature-indexed consequence relation</td>
<td>entailment system</td>
<td>CS: consequence system</td>
</tr>
<tr>
<td>Page. 2</td>
<td>co-complete</td>
<td></td>
<td>sig-complete</td>
</tr>
<tr>
<td>Def. 6</td>
<td>proof-theoretic logic translation</td>
<td>entailment system comorphism</td>
<td>consequence system (CS-) comorphism</td>
</tr>
<tr>
<td>Def. 7</td>
<td>graph of consequence systems</td>
<td></td>
<td>heterogeneous consequence environment</td>
</tr>
<tr>
<td>Def. 10</td>
<td>flat combination of consequence systems</td>
<td></td>
<td>Grothendieck (CS) consequence system</td>
</tr>
<tr>
<td>Def. 16</td>
<td>signature-indexed-room</td>
<td>institution</td>
<td>institute</td>
</tr>
<tr>
<td>Def. 19</td>
<td>model-theoretic logic translation</td>
<td>institution comorphism</td>
<td>institute comorphism</td>
</tr>
<tr>
<td>Def. 20</td>
<td>graph of institutions</td>
<td>indexed institution</td>
<td>heterogeneous satisfaction environment</td>
</tr>
<tr>
<td>Def. 22</td>
<td>flat combination of institutions</td>
<td>Grothendieck institution</td>
<td>Grothendieck institute</td>
</tr>
<tr>
<td>Def. 26</td>
<td>signature-indexed logic room</td>
<td>logic</td>
<td>logique</td>
</tr>
<tr>
<td>Page. 6</td>
<td>logic translation</td>
<td>map of logics</td>
<td>logique comorphism</td>
</tr>
<tr>
<td>Def. 28</td>
<td>logic graph</td>
<td>indexed logic</td>
<td>heterogeneous logic environment</td>
</tr>
<tr>
<td>Page. 6</td>
<td>flat combination of logics</td>
<td>Grothendieck logic</td>
<td>Grothendieck logique</td>
</tr>
</tbody>
</table>