Logik für Informatiker
Logic for computer scientists

Quantifiers

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Motivating examples

$$\forall x\ Cube(x)$$ (“All objects are cubes.”)

$$\forall x\ (Cube(x) \rightarrow Large(x))$$ (“All cubes are large.”)

$$\forall x\ Large(x)$$ (“All objects are large.”)

$$\exists x \; Cube(x)$$

"There exists a cube."

$$\exists x \; (Cube(x) \wedge Large(x))$$

"There exists a large cube."

The four Aristotelian forms

All P's are Q's.	$\forall x(P(x) \rightarrow Q(x))$
Some P's are Q's.	$\exists x(P(x) \wedge Q(x))$
No P's are Q's.	$\forall x(P(x) \rightarrow \neg Q(x))$
Some P's are not Q's.	$\exists x(P(x) \wedge \neg Q(x))$

Note:

$\forall x(P(x) \rightarrow Q(x))$ does not imply that there are some $P's$.

$\exists x(P(x) \wedge Q(x))$ does not imply that not all $P's$ are $Q's$.

First-order signatures

A first-order signature consists of

- a set of predicate symbols with arities, like $Smaller^{(2)}, Dodec^{(1)}, Between^{(3)}, \leq^{(2)}$, including propositional symbols (nullary predicate symbols), like $A^{(0)}, B^{(0)}, C^{(0)}$, (written uppercase)
- its names or constants for individuals, like a, b, c, (written lowercase)
- its function symbols with arities, like $f^{(1)}, +^{(2)}, \times^{(2)}$.

Usually, arities are omitted.

In the book, the terminology "language" is used.

"Signature" is more precise, since it exactly describes the ingredients that are needed to generate a (first-order) language.

Terms

$t ::= a$	constant
$t ::= x$	variable
$\mid f^{(n)}(t_1, \ldots, t_n)$	application of function symbols to terms

Usually, arities are omitted.

Well-formed formulas

$F ::= p^{(n)}(t_1, \dots, t_n)$	application of predicate symbols
$\mid \bot$	contradiction
$\mid \neg F$	negation
$\mid (F_1 \wedge \dots \wedge F_n)$	conjunction
$\mid (F_1 \wedge \dots \vee F_n)$	disjunction
$\mid (F_1 \rightarrow F_2)$	implication
$\mid (F_1 \leftrightarrow F_2)$	equivalence
$\mid \forall \nu F$	universal quantification
$\mid \exists \nu F$	existential quantification

The variable ν is said to be bound in $\forall \nu F$ and $\exists \nu F$.

Parentheses

The outermost parenthese of a well-formed formula can be omitted:

$$Cube(x) \wedge Small(x)$$

In general, parentheses are important to determine the scope of a quantifier (see next slide).

Free and bound variables

An occurrence of a variable in a formula that is not bound is said to be free.

$\exists y\ LeftOf(x,y)$	x is free, y is bound
$(Cube(x) \land Small(x)) \rightarrow \exists y\ LeftOf(x,y)$	x is free, y is bound
$\exists x\ (Cube(x) \land Small(x))$	Both occurrences of x are bound
$\exists x\ Cube(x) \land Small(x)$	The first occurrence of x is bound, the second one is free

Sentences

A sentence is a well-formed formula without free variables.

$$\bot \qquad\qquad A \wedge B$$

$$Cube(a) \vee Tet(b)$$

$$\forall x\ (Cube(x) \rightarrow Large(x))$$

$$\forall x\ ((Cube(x) \wedge Small(x)) \rightarrow \exists y\ LeftOf(x, y))$$

Semantics of quantification

We need to fix some domain of discourse.

$\forall x\ S(x)$ is true iff for every object in the domain of discourse with name n, $S(n)$ is true.

$\exists x\ S(x)$ is true iff for some object in the domain of discourse with name n, $S(n)$ is true.

Not all objects need to have names — hence we assume that for objects, names $n_1, n_2, \ldots$ can be invented "on the fly".

The game rules

Form	Your commitment	Player to move	Goal
$P \vee Q$	TRUE FALSE	you Tarski's World	Choose one of P, Q that is true.
$P \wedge Q$	TRUE FALSE	Tarski's World you	Choose one of P, Q that is false.
$\exists x\, P(x)$	TRUE FALSE	you Tarski's World	Choose some ***b*** that satisfies the wff P(x).
$\forall x\, P(x)$	TRUE FALSE	Tarski's World you	Choose some ***b*** that does not satisfy P(x).

$\neg P$	either	—	Replace $\neg P$ by P and switch commitment.
$P \to Q$	either	—	Replace $P \to Q$ by $\neg P \vee Q$ and keep commitment.
$P \leftrightarrow Q$	either	—	Replace $P \leftrightarrow Q$ by $(P \to Q) \wedge (Q \to P)$ and keep commitment.