Logik für Informatiker
Logic for computer scientists

The logic of quantifiers

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Logical consequence for quantifiers

$$\forall x(\mathsf{Cube}(x) \rightarrow \mathsf{Small}(x))$$
$$\forall x\ \mathsf{Cube}(x)$$
$$\vdash$$
$$\forall x\ \mathsf{Small}(x)$$

$$\forall x\ \mathsf{Cube}(x)$$
$$\forall x\ \mathsf{Small}(x)$$
$$\vdash$$
$$\forall x(\mathsf{Cube}(x) \land \mathsf{Small}(x))$$

However: ignoring quantifiers does not work!

$$\exists x(\mathsf{Cube}(x) \rightarrow \mathsf{Small}(x))$$
$$\exists x\ \mathsf{Cube}(x)$$
$$\vdash$$
$$\exists x\ \mathsf{Small}(x)$$

$$\exists x\ \mathsf{Cube}(x)$$
$$\exists x\ \mathsf{Small}(x)$$
$$\vdash$$
$$\exists x(\mathsf{Cube}(x) \land \mathsf{Small}(x))$$

Tautologies do not distribute over quantifiers

$$\exists x\ Cube(x) \lor \exists x\ \neg Cube(x)$$

is a logical truth, but

$$\forall x\ Cube(x) \lor \forall x\ \neg Cube(x)$$

is not. By contrast,

$$\forall x\ Cube(x) \lor \neg \forall x\ Cube(x)$$

is a tautology.

Truth-functional form

Replace all top-level quantified sub-formulas (i.e. those not ocurring below another quantifier) by propositional letters.

Replace multiple occurrences of the same sub-formula by the same propositional letter.

A quantified sentence of FOL is said to be a tautology iff its truth-functional form is a tautology.

$$\forall x\ Cube(x) \vee \neg\forall x\ Cube(x)$$

becomes

$$A \vee \neg A$$

Truth functional form — examples

FO sentence	t.f. form
$\forall x\, \mathrm{Cube}(x) \lor \neg \forall x\, \mathrm{Cube}(x)$	$A \lor \neg A$
$(\exists y\, \mathrm{Tet}(y) \land \forall z\, \mathrm{Small}(z)) \to \forall z\, \mathrm{Small}(z)$	$(A \land B) \to B$
$\forall x\, \mathrm{Cube}(x) \lor \exists y\, \mathrm{Tet}(y)$	$A \lor B$
$\forall x\, \mathrm{Cube}(x) \to \mathrm{Cube}(a)$	$A \to B$
$\forall x\, (\mathrm{Cube}(x) \lor \neg \mathrm{Cube}(x))$	A
$\forall x\, (\mathrm{Cube}(x) \to \mathrm{Small}(x)) \lor \exists x\, \mathrm{Dodec}(x)$	$A \lor B$

Examples of →-Elim

$\exists x(\mathrm{Cube}(x) \rightarrow \mathrm{Small}(x))$	A	No!
$\exists x\ \mathrm{Cube}(x)$	B	
$\exists x\ \mathrm{Small}(x)$	C	

$\exists x \mathrm{Cube}(x) \rightarrow \exists x\ \mathrm{Small}(x)$	$A \rightarrow B$	Yes!
$\exists x\ \mathrm{Cube}(x)$	A	
$\exists x\ \mathrm{Small}(x)$	B	

Tautologies and logical truths

Every tautology is a logical truth, but not vice versa.

Example: $\exists x\ Cube(x) \lor \exists x\ \neg Cube(x)$

is a logical truth, but not a tautology.

Similarly, every tautologically valid argument is a logically valid argument, but not vice versa.

$$\begin{array}{|l} \forall x\ \mathrm{Cube}(x) \\ \hline \exists x\ \mathrm{Cube}(x) \end{array}$$

is a logically valid argument, but not tautologically valid.

Tautologies and logical truths, cont'd

Propositional logic	First-order logic	General notion
Tautology	*FO validity*	*Logical Truth*
Tautological consequence	*FO consequence*	*Logical consequence*
Tautological equivalence	*FO equivalence*	*Logical equivalence*

Which ones are FO validities?

$$\forall x \; SameSize(x, x)$$

$$\forall x \; Cube(x) \rightarrow Cube(b)$$

$$(Cube(b) \land b = c) \rightarrow Cube(c)$$

$$(Small(b) \land SameSize(b, c)) \rightarrow Small(c)$$

Replacement method: Replace predicates by meaningless ones

$$\forall x \;\; Outgrabe(x, x)$$
$$\forall x \; Tove(x) \rightarrow Tove(b)$$
$$(Tove(b) \wedge b = c) \rightarrow Tove(c)$$
$$(Slithy(b) \wedge Outgrabe(b, c)) \rightarrow Slithy(c)$$

Is this a valid FO argument?

$$
\begin{array}{|l}
\forall x(\mathsf{Tet}(x) \rightarrow \mathsf{Large}(x)) \\
\neg \mathsf{Large}(b) \\
\hline
\neg \mathsf{Tet}(b)
\end{array}
$$

Replacement with nonsense predicates:

$$
\begin{array}{|l}
\forall x(\mathsf{Borogove}(x) \rightarrow \mathsf{Mimsy}(x)) \\
\neg \mathsf{Mimsy}(b) \\
\hline
\neg \mathsf{Borogove}(b)
\end{array}
$$

Is this a valid FO argument?

Replacement with a meaningless predicate:

$$
\begin{array}{|l}
\neg\exists x\ \mathsf{Larger}(x, a) \\
\neg\exists x\ \mathsf{Larger}(b, x) \\
\mathsf{Larger}(c, d) \\
\hline
\mathsf{Larger}(a, b)
\end{array}
\qquad
\begin{array}{|l}
\neg\exists x\ R(x, a) \\
\neg\exists x\ R(b, x) \\
R(c, d) \\
\hline
R(a, b)
\end{array}
$$

The method of counterexamples

In order to show that the argument

$$\begin{array}{|l} P_1 \\ \ldots \\ P_n \\ \hline Q \end{array}$$

is not valid, it suffices to give a counterexample, i.e. a world that makes the premises $P_1, \ldots, P_n$ true, but the conclusion Q false.

(For now, "world" is understood informally. Later on, we will formalize "world" as "first-order structure".)

A counterexample

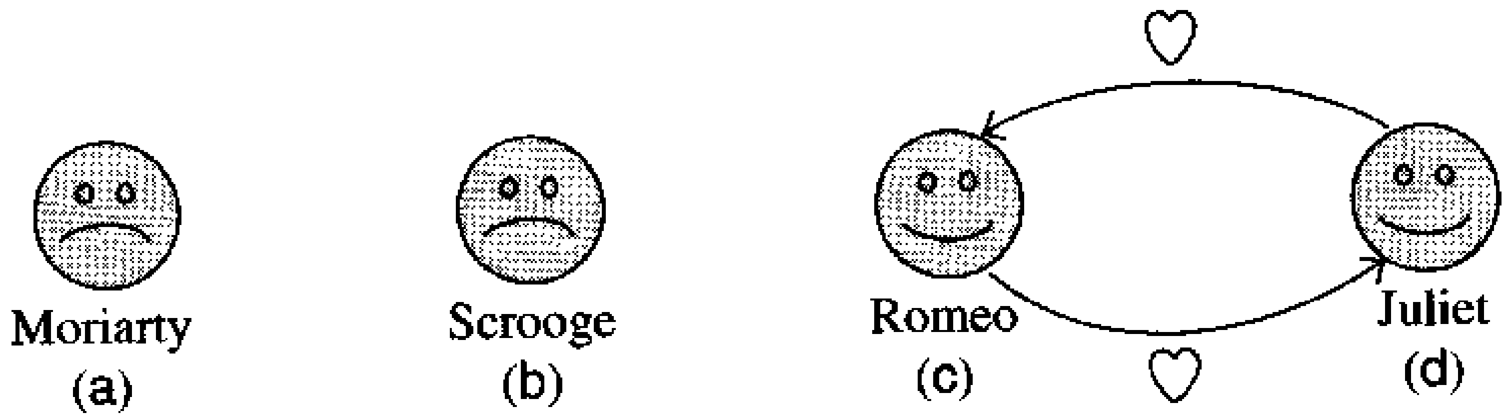

Figure 10.1: A first-order counterexample.

The axiomatic method

We have encountered arguments that are valid in Tarski's World but not FO valid.

Axiomatic method: bridge the gap between Tarski's World validity and FO validity by systematically expressing facts about the meanings of the predicates, and introduce them as *axioms*. Axioms restrict the possible interpretation of predicates.

Axioms may be used as premises within arguments/proofs.

The basic shape axioms

1. $\neg\exists x(Cube(x) \wedge Tet(x))$
2. $\neg\exists x(Tet(x) \wedge Dodec(x))$
3. $\neg\exists x(Dodec(x) \wedge Cube(x))$
4. $\forall x(Tet(x) \vee Dodec(x) \vee Cube(x))$

An argument using the shape axioms

$$\neg\exists x(\mathrm{Dodec}(x) \land \mathrm{Cube}(x))$$
$$\forall x(\mathrm{Tet}(x) \lor \mathrm{Dodec}(x) \lor \mathrm{Cube}(x))$$
$$\neg\exists x\ \mathrm{Tet}(x)$$
$$\vdash$$
$$\forall x(\mathrm{Cube}(x) \leftrightarrow \neg\mathrm{Dodec}(x))$$

$$\neg\exists x(P(x) \land Q(x))$$
$$\forall x(R(x) \lor P(x) \lor Q(x))$$
$$\neg\exists x\ R(x)$$
$$\vdash$$
$$\forall x(Q(x) \leftrightarrow \neg P(x))$$

SameShape introduction and elimination axioms

1. $\forall x \forall y((Cube(x) \land Cube(y)) \rightarrow SameShape(x, y))$
2. $\forall x \forall y((Dodec(x) \land Dodec(y)) \rightarrow SameShape(x, y))$
3. $\forall x \forall y((Tet(x) \land Tet(y)) \rightarrow SameShape(x, y))$

4. $\forall x \forall y((SameShape(x, y) \land Cube(x)) \rightarrow Cube(y))$
5. $\forall x \forall y((SameShape(x, y) \land Dodec(x)) \rightarrow Dodec(y))$
6. $\forall x \forall y((SameShape(x, y) \land Tet(x)) \rightarrow Tet(y))$

Peano's Axiomatization of the natural numbers

1. $\forall n\ \neg suc(n) = 0$
2. $\forall m \forall n\ suc(m) = suc(n) \rightarrow m = n$
3. $(\Phi(x/0) \wedge \forall n(\Phi(x/n) \rightarrow \Phi(x/suc(n)))) \rightarrow \forall n\ \Phi(x/n)$
   if Φ is a formula with a free variable x, and
   $\Phi(x/n)$ denotes the replacement of x with t within Φ

Other famous axiom systems

- Euclid's axiomatization of Geometry
- Zermelo-Fraenkel axiomatization of set theory
- axiomatizations in algebra: monoids, groups, rings, fields, vector spaces . . .
- Hoare's axiomatization of imperative programming with while-loops, if-then-else and assignment

Multiple quantifiers

$$\forall x \exists y \; Likes(x, y)$$

is very different from

$$\exists x \forall y \; Likes(x, y)$$

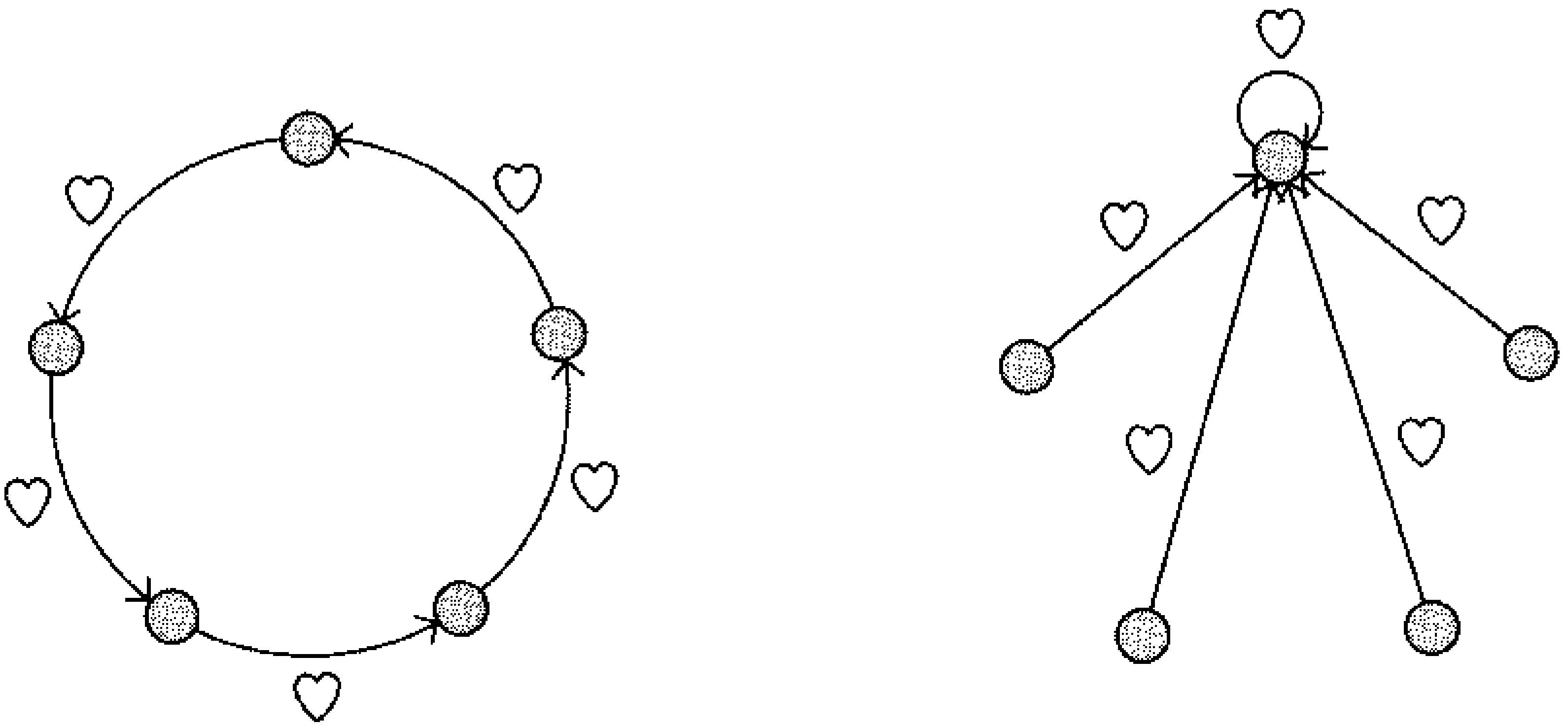

Figure 11.1: A circumstance in which $\forall x\, \exists y\, \mathsf{Likes}(x, y)$ holds versus one in which $\exists y\, \forall x\, \mathsf{Likes}(x, y)$ holds. It makes a big difference to someone!

Arguments involving multiple quantifiers

$$\begin{array}{|l} \exists y[\mathrm{Girl}(y) \wedge \forall x(\mathrm{Boy}(x) \rightarrow \mathrm{Likes}(x,y))] \\ \hline \forall x[\mathrm{Boy}(x) \rightarrow \exists y(\mathrm{Girl}(y) \wedge \mathrm{Likes}(x,y))] \end{array}$$

$$\begin{array}{|l} \forall x[\mathrm{Boy}(x) \rightarrow \exists y(\mathrm{Girl}(y) \wedge \mathrm{Likes}(x,y))] \\ \hline \exists y[\mathrm{Girl}(y) \wedge \forall x(\mathrm{Boy}(x) \rightarrow \mathrm{Likes}(x,y))] \end{array}$$