Logik für Informatiker
Logic for computer scientists

Proof rules for quantifiers

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WiSe 2005

Universal Elimination (∀ Elim)

$$
\begin{array}{l|l}
 & \forall x\, S(x) \\
 & \vdots \\
\triangleright & S(c)
\end{array}
$$

Existential Introduction ($\exists$ Intro)

$$
\begin{array}{l|l}
 & S(c) \\
 & \vdots \\
\triangleright & \exists x\, S(x)
\end{array}
$$

Example: $\forall$-Elim and $\exists$-Intro

$\forall x[\mathrm{Cube}(x) \rightarrow \mathrm{Large}(x)]$

$\forall x[\mathrm{Large}(x) \rightarrow \mathrm{LeftOf}(x, b)]$

$\mathrm{Cube}(d)$

$\exists x[\mathrm{Large}(x) \wedge \mathrm{LeftOf}(x, b)]$

Existential Elimination (∃ Elim):

$\exists x\, S(x)$

⋮

  $\boxed{c}\ S(c)$

  ⋮

  Q

▷ Q

Where c does not occur outside the subproof where it is introduced.

Example: $\exists$-Elim

$$\forall x[\mathrm{Cube}(x) \longrightarrow \mathrm{Large}(x)]$$
$$\forall x[\mathrm{Large}(x) \longrightarrow \mathrm{LeftOf}(x, b)]$$
$$\exists x\ \mathrm{Cube}(x)$$

$$\exists x[\mathrm{Large}(x) \wedge \mathrm{LeftOf}(x, b)]$$

General Conditional Proof ($\forall$ Intro):

$$
\begin{array}{l|l}
 & \quad \boxed{c}\ P(c) \\
 & \quad \vdots \\
 & \quad Q(c) \\
\triangleright & \forall x\,(P(x) \rightarrow Q(x))
\end{array}
$$

Where c does not occur outside the subproof where it is introduced.

Universal Introduction (∀ Intro):

| c
|---
| ⋮
| P(c)

▷ $\forall x\, P(x)$

Where c does not occur outside the subproof where it is introduced.

Example: $\forall$-Intro

$$\exists y[\mathrm{Girl}(y) \land \forall x(\mathrm{Boy}(x) \rightarrow \mathrm{Likes}(x, y))]$$

$$\forall x[\mathrm{Boy}(x) \rightarrow \exists y(\mathrm{Girl}(y) \land \mathrm{Likes}(x, y))]$$

Example: de Morgan's Law

$$\frac{\neg\forall x\, P(x)}{\exists x\, \neg P(x)}$$

(is not valid in intuitionistic logic, only in classical logic)

Example: The Barber Paradox

$$
\begin{array}{|l}
\exists z\ \exists x\ [ManOf(x,z) \land \forall y\ (ManOf(y,z) \rightarrow \\
\qquad\qquad (Shave(x,y) \leftrightarrow \neg Shave(y,y)))] \\
\hline
\bot
\end{array}
$$

Identity Introduction (= Intro)

$$\triangleright \;\Big|\; \mathsf{n} = \mathsf{n}$$

Identity Elimination (= Elim)

$$
\begin{array}{l|l}
 & P(n) \\
 & \vdots \\
 & n = m \\
 & \vdots \\
\triangleright & P(m)
\end{array}
$$

Reflexivity, symmetry and transitivity

$$\vdash \forall x\ x = x$$

$$\vdash \forall x\ \forall y\ x = y \rightarrow y = x$$

$$\vdash \forall x\ \forall y\ \forall z\ (x = y \wedge y = z) \rightarrow x = z$$