

Logik für Informatiker

Logic for computer scientists

Model theory

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First-order structures

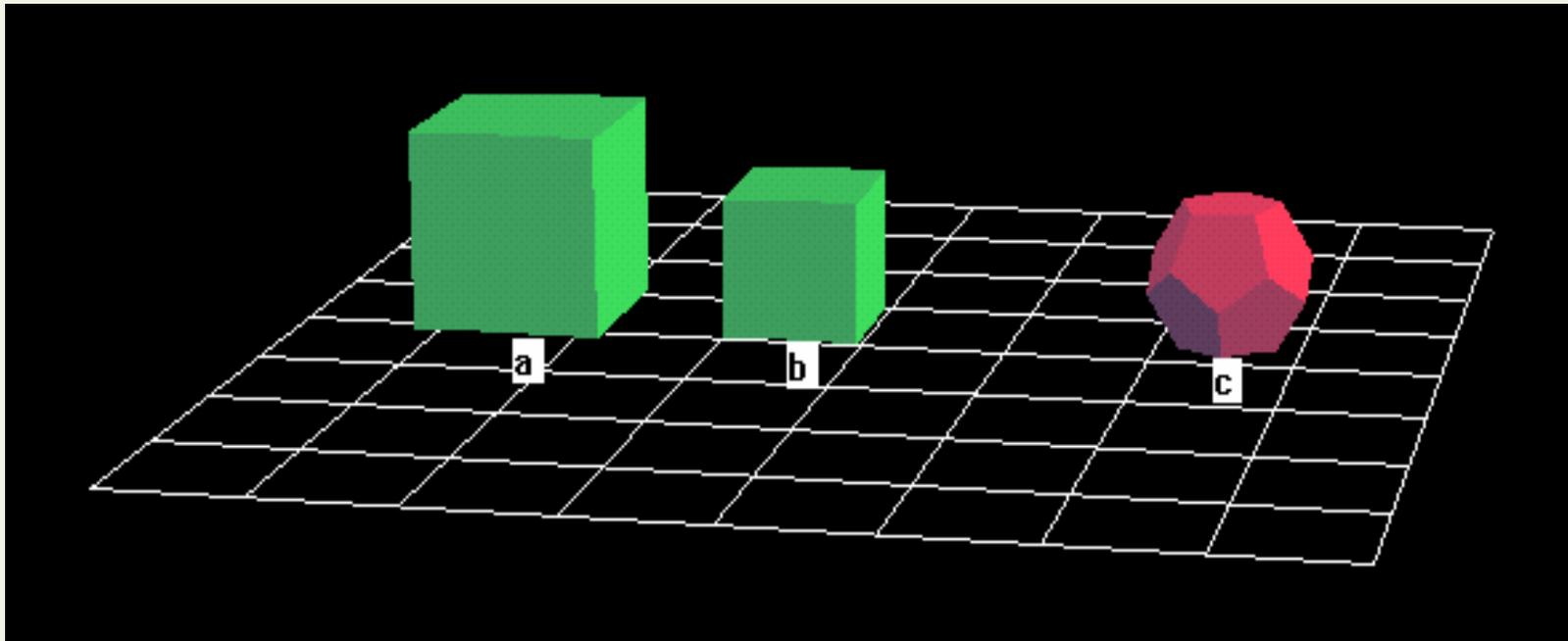
A first-order structure \mathfrak{M} consists of:

- a nonempty set $D^{\mathfrak{M}}$, the **domain of discourse**;
- for each n -ary predicate P of the language, a set $\mathfrak{M}(P)$ of n -tuples $\langle x_1, \dots, x_n \rangle$ of elements of $D^{\mathfrak{M}}$, called the **extension** of P .

The extension of the identity symbol $=$ must be $\{\langle x, x \rangle \mid x \in D^{\mathfrak{M}}\}$;

- for any name (individual constant) c of the language, an element $\mathfrak{M}(c)$ of $D^{\mathfrak{M}}$.

Example



An interpretation according to Tarski's World

Assume the language consists of the predicates *Cube*, *Dodec* and *Larger* and the names *a*, *b* and *c*.

- $D^{\mathfrak{M}} = \{Cube1, Cube2, Dodec1\}$;
- $\mathfrak{M}(Cube) = \{Cube1, Cube2\}$;
- $\mathfrak{M}(Dodec) = \{Dodec1\}$;
- $\mathfrak{M}(Larger) = \{(Cube1, Cube2), (Cube1, Dodec1)\}$;
- $\mathfrak{M}(=) = \{(Cube1, Cube1), (Cube2, Cube2), (Dodec1, Dodec1)\}$;
- $\mathfrak{M}(a) = Cube1$; $\mathfrak{M}(b) = Cube2$; $\mathfrak{M}(c) = Dodec1$.

An interpretation not conformant with Tarski's World

- $D^{\mathfrak{M}} = \{Cube1, Cube2, Dodec1\};$
- $\mathfrak{M}(Cube) = \{Dodec1, Cube2\};$
- $\mathfrak{M}(Dodec) = \emptyset;$
- $\mathfrak{M}(Larger) = \{(Cube1, Cube1), (Dodec1, Cube2)\};$
- $\mathfrak{M}(=) =$
 $\{(Cube1, Cube1), (Cube2, Cube2), (Dodec1, Dodec1)\};$
- $\mathfrak{M}(a) = Cube1; \mathfrak{M}(b) = Cube2; \mathfrak{M}(c) = Dodec1.$

Variable assignments

A **variable assignment** in \mathfrak{M} is a (possibly partial) function g defined on a set of variables and taking values in $D^{\mathfrak{M}}$.

Given a well-formed formula P , we say that the variable assignment g is **appropriate** for P if all the free variables of P are in the domain of g , that is, if g assigns objects to each free variable of P .

Examples

$D^{\mathfrak{M}} = \{Cube1, Cube2, Dodec1\}$

g_1 assigns *Cube1*, *Cube2*, *Dodec1* to the variables x , y , z , respectively.

g_1 is appropriate for $Between(x, y, z) \wedge \exists u(Large(u))$, but not for $Larger(x, v)$.

g_2 is the empty assignment.

g_2 is only appropriate for well-formed formulas without free variables (that is, for sentences).

Variants of variable assignments

If g is a variable assignment, $g[v/b]$ is the variable assignment

- whose domain is that of g plus the variable v , and
- which assigns the same values as g , except that
- it assigns b to the variable v .

$$[t]_g^{\mathfrak{M}}$$

$[t]_g^{\mathfrak{M}}$ is

- $\mathfrak{M}(t)$ if t is an individual constant, and
- $g(t)$ if t is a variable.

Satisfaction (A. Tarski)

1. $\mathfrak{M} \models R(t_1, \dots, t_n)[g]$ iff $\langle [t_1]_{\mathfrak{M}}^g, \dots, [t_n]_{\mathfrak{M}}^g \rangle \in \mathfrak{M}(R)$;
2. $\mathfrak{M} \models \neg P[g]$ iff it is not the case that $\mathfrak{M} \models P[g]$;
3. $\mathfrak{M} \models P \wedge Q[g]$ iff both $\mathfrak{M} \models P[g]$ and $\mathfrak{M} \models Q[g]$;
4. $\mathfrak{M} \models P \vee Q[g]$ iff $\mathfrak{M} \models P[g]$ or $\mathfrak{M} \models Q[g]$ or both;
5. $\mathfrak{M} \models P \rightarrow Q[g]$ iff not $\mathfrak{M} \models P[g]$ or $\mathfrak{M} \models Q[g]$ or both;
6. $\mathfrak{M} \models P \leftrightarrow Q[g]$ iff ($\mathfrak{M} \models P[g]$ iff $\mathfrak{M} \models Q[g]$);
7. $\mathfrak{M} \models \forall x P[g]$ iff for every $d \in D^{\mathfrak{M}}$, $\mathfrak{M} \models P[g[x/d]]$;
8. $\mathfrak{M} \models \exists x P[g]$ iff for some $d \in D^{\mathfrak{M}}$, $\mathfrak{M} \models P[g[x/d]]$.

Satisfaction, cont'd

Additionally,

- never $\mathfrak{M} \models \perp[g]$;
- always $\mathfrak{M} \models \top[g]$.

A structure \mathfrak{M} satisfies a sentence P ,

$$\mathfrak{M} \models P,$$

if $\mathfrak{M} \models P[g_\emptyset]$ for the empty assignment g_\emptyset .

Example

$$D^{\mathfrak{M}} = \{a, b, c\}$$

$$\mathfrak{M}(\text{likes}) = \{\langle a, a \rangle, \langle a, b \rangle, \langle c, a \rangle\}$$

$$\mathfrak{M} \models \exists x \exists y (\text{Likes}(x, y) \wedge \neg \text{Likes}(y, y))$$

$$\mathfrak{M} \models \neg \forall x \exists y (\text{Likes}(x, y) \wedge \neg \text{Likes}(y, y))$$

Satisfaction invariance

Proposition Let \mathfrak{M}_1 and \mathfrak{M}_2 be structures which have the same domain and assign the same interpretations to the predicates and constant symbols in P . Let g_1 and g_2 be variable assignments that assign the same objects to the free variables in P . Then

$$\mathfrak{M}_1 \models P[g_1] \text{ iff } \mathfrak{M}_2 \models P[g_2]$$

First-order validity and consequence

A sentence P is a **first-order consequence** of a set \mathcal{T} of sentences if and only if every structure that satisfies all the sentences in \mathcal{T} also satisfies P .

A sentence P is a **first-order validity** if and only if every structure satisfies P .

A set \mathcal{T} of sentences is called **first-order satisfiable**, if there is a structure satisfies each sentence in \mathcal{T} .

Soundness of \mathcal{F} for FOL

Theorem If $\mathcal{T} \vdash S$, then S is a first-order consequence of \mathcal{T} .

Proof: By induction over the derivation, we show that any sentence occurring in a proof is a first-order consequence of the assumption in force in that step.

An assumption is **in force** in a step if it is an assumption of the current subproof or an assumption of a higher-level proof.

Completeness of the shape axioms

The basic shape axioms

1. $\neg\exists x(Cube(x) \wedge Tet(x))$
2. $\neg\exists x(Tet(x) \wedge Dodec(x))$
3. $\neg\exists x(Dodec(x) \wedge Cube(x))$
4. $\forall x(Tet(x) \vee Dodec(x) \vee Cube(x))$

SameShape introduction and elimination axioms

1. $\forall x \forall y ((Cube(x) \wedge Cube(y)) \rightarrow SameShape(x, y))$
2. $\forall x \forall y ((Dodec(x) \wedge Dodec(y)) \rightarrow SameShape(x, y))$
3. $\forall x \forall y ((Tet(x) \wedge Tet(y)) \rightarrow SameShape(x, y))$
4. $\forall x \forall y ((SameShape(x, y) \wedge Cube(x)) \rightarrow Cube(y))$
5. $\forall x \forall y ((SameShape(x, y) \wedge Dodec(x)) \rightarrow Dodec(y))$
6. $\forall x \forall y ((SameShape(x, y) \wedge Tet(x)) \rightarrow Tet(y))$

Completeness of the shape axioms

Two structures are **isomorphic** if there is a bijection between their domains, which is compatible with extensions of predicate and interpretation of constants.

Assume the language *Cube, Tet, Dodec, SameShape*

Lemma For any structure satisfying the shape axioms, there is an isomorphic Tarski's world structure.

Theorem Let S be a sentence.

If S is a Tarski's world logical consequence of \mathcal{T} , then S is a first-order consequence of \mathcal{T} plus the shape axioms.

Skolemization

The sentence

$$\forall x \exists y \text{Neighbor}(x, y)$$

is logically equivalent to the second-order sentence

$$\exists f \forall x \text{Neighbor}(x, f(x))$$

In first-order logic, we have the **Skolem normal form**

$$\forall x \text{Neighbor}(x, f(x))$$

Theorem about Skolem normal form

Theorem

A sentence $S \equiv \forall x \exists y P(x, y)$ is satisfiable iff its Skolem normal form $\forall x P(x, f(x))$ is.

Every structure satisfying the Skolem normal form also satisfies S . Moreover, every structure satisfying S can be turned into one satisfying the Skolem normal form. This is done by interpreting f by a function which picks out, for any object b in the domain, some object c such that they satisfy $P(x, y)$.