Logik für Informatiker
Logic for computer scientists

Boolean Connectives and Formal Proofs

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Formal proofs in Fitch

P	
Q	
R	
S_1	Justification 1
...	
...	
S_n	Justification n
S	Justification n+1

Fitch rule: Identity introduction

Identity Introduction (= Intro):

$\triangleright \mid \mathsf{n} = \mathsf{n}$

Fitch rule: Identity elimination

Identity Elimination (= Elim):

$$
\begin{array}{l|l}
 & P(n) \\
 & \vdots \\
 & n = m \\
 & \vdots \\
\triangleright & P(m)
\end{array}
$$

Fitch rule: Reiteration

Reiteration (Reit):

$$
\begin{array}{c|l}
 & P \\
 & \vdots \\
\triangleright & P
\end{array}
$$

Example proof in fitch

Properties of predicates in Tarski's world

| Larger(a, b)
| Larger(b, c)
|—
| Larger(a, c)

| RightOf(b, c)
|—
| LeftOf(c, b)

Such arguments can be proved in Fitch using the special rule **Ana Con**.

This rule is only valid for reasoning about Tarski's world!

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Showing invalidity using counterexamples

| Al Gore is a politician
| Hardly any politicians are honest
|—
| Al Gore is dishonest

Imagine a situation where there are 10,000 politicians, and that Al Gore is the only honest one of the lot. In such circumstances both premises would be true but the conclusion would be false.
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Are the following arguments valid?

$$
\begin{array}{|l}
\text{Small}(a) \\
\text{Larger}(b, a) \\
\hline
\text{Large}(b)
\end{array}
$$

$$
\begin{array}{|l}
\text{Small}(a) \\
\text{Larger}(a, b) \\
\hline
\text{Large}(b)
\end{array}
$$

Negation — Truth table

P	$\neg$P
TRUE	FALSE
FALSE	TRUE

The Henkin-Hintikka game

"Checkmate!"

The Henkin-Hintikka game

Is a sentence true in a given world?

- Players: *you* and the *computer* (Tarski's world)
- You claim that a sentence is true (or false), Tarski's world will claim the opposite
- In each round, the sentence is *reduced* to a simpler one
- When an *atomic sentence* is reached, its truth can be directly inspected in the given world

You have a *winning strategy* exactly in those cases where your claim is *correct*.

Negation — Game rule

Form	Your commitment	Player to move	Goal
$\neg P$	either	—	Replace $\neg P$ by P and switch commitment

Conjunction — Truth table

P	Q	$P \wedge Q$
TRUE	TRUE	TRUE
TRUE	FALSE	FALSE
FALSE	TRUE	FALSE
FALSE	FALSE	FALSE

Conjunction — Game rule

Form	Your commitment	Player to move	Goal
$P \land Q$	TRUE	Tarski's World	Choose one of P, Q that is false.
	FALSE	you	

Disjunction — Truth table

P	Q	$P \vee Q$
TRUE	TRUE	TRUE
TRUE	FALSE	TRUE
FALSE	TRUE	TRUE
FALSE	FALSE	FALSE

Disjunction — Game rule

Form	Your commitment	Player to move	Goal
$P \vee Q$	TRUE	you	Choose one of P, Q that is true.
	FALSE	Tarski's World	

Formalisation

- Sometimes, natural language double negation means logical single negation
- The English expression *and* sometimes suggests a temporal ordering; the FOL expression $\wedge$ never does.
- The English expressions *but*, *however*, *yet*, *nonetheless*, and *moreover* are all stylistic variants of *and*.
- Natural language disjunction can mean *invlusive-or* ($\vee$) or *exclusive-or*: $A\ xor\ B \Leftrightarrow (A \vee B) \wedge (\neg A \vee \neg B)$

Logical necessity

A sentence is

- *logically necessary*, or *logically valid*, if it is true in all circumstances (worlds),
- *logically possible*, if it is true in some circumstances (worlds),
- *logically impossible*, if it is true in no circumstances (worlds).

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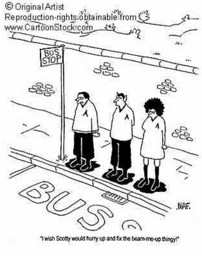
Logically possible

Logically and physically possible



Logically impossible

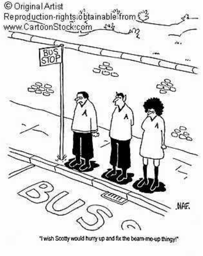
$P \wedge \neg P$ $\qquad a \neq a$

Logically necessary

$P \vee \neg P$ $\qquad a = a$

Logically possible

Logically and physically possible



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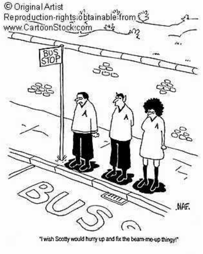


Logically impossible

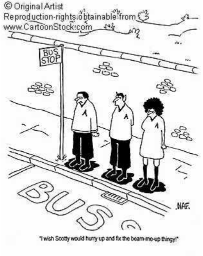
$P \land \neg P$ $a \neq a$

Logically necessary

$P \lor \neg P$ $a = a$

Logically possible

Logically and physically possible



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Logic, Boolean logic and Tarski's world

A sentence is

- *logically necessary*, or *logically valid*, if it is true in all circumstances (worlds),
- *TW-necessary*, if it is true in all worlds of Tarski's world,
- a *tautology*, if it is true in all valuations of the atomic sentences with $\{\mathrm{TRUE}, \mathrm{FALSE}\}$.

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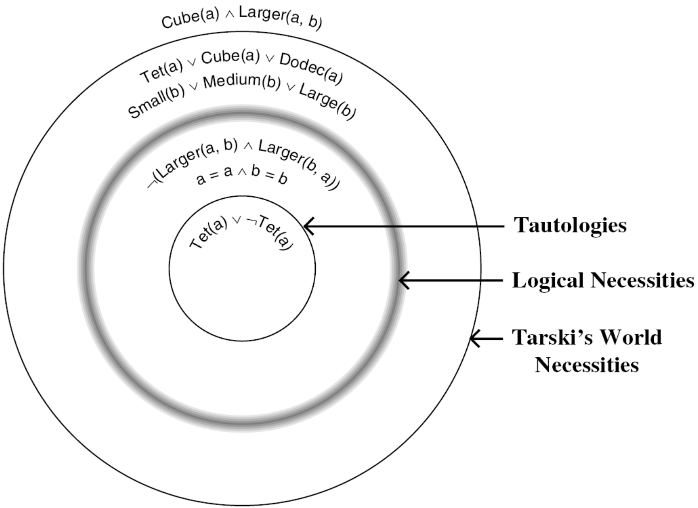
Cube(a) ∧ Larger(a, b)
Tet(a) ∨ Cube(a) ∨ Dodec(a)
Small(b) ∨ Medium(b) ∨ Large(b)
¬(Larger(a, b) ∧ Larger(b, a))
a = a ∧ b = b
Tet(a) ∨ ¬Tet(a)
Tautologies
Logical Necessities
Tarski's World
Necessities

The truth table method

- A sentence is a tautology if and only if it evaluates to TRUE in all rows of its complete truth table.
- Truth tables can be constructed with the program *Boole*.

Tautological equivalence and consequence

- Two sentences P and Q are *tautologically equivalent*, if they evaluate to the same truth value in all valuations (rows of the truth table).
- Q is a *tautological consequence* of $P_1, \ldots, P_n$ if and only if every row that assigns TRUE to each of $P1, \ldots P_n$ also assigns TRUE to Q.
- If Q is a tautological consequence of $P_1, \ldots P_n$, then Q is also a *logical consequence* of $P_1, \ldots, P_n$.
- Some logical consequences are not tautological ones.

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The **Con** rules in Fitch

- **Taut Con** proves all tautological consequences.
- **FO Con** proves all first-order consequences (like $a = c$ follows from $a = b \land b = c$).
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de Morgan's laws and double negation

$$\neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$$

$$\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$$

$$\neg\neg P \Leftrightarrow P$$

Note: $\neg$ binds stronger than $\wedge$ and $\vee$. Bracktes are needed to override this.

Negation normal form

- *Substitution of equivalents*: If P and Q are logically equivalent: $P \Leftrightarrow Q$ then the results of substituting one for the other in the context of a larger sentence are also logically equivalent: $S(P) \Leftrightarrow S(Q)$
- A sentence is in *negation normal form* (NNF) if all occurrences of $\neg$ apply directly to atomic sentences.
- Any sentence built from atomic sentences using just $\wedge$, $\vee$, and $\neg$ can be *put into negation normal form* by repeated application of the de Morgan laws and double negation.

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Distributive laws

For any sentences P, Q, and R:

- *Distribution of* $\wedge$ *over* $\vee$:

  $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R).$

- *Distribution of* $\vee$ *over* $\wedge$:

  $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R).$

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Conjunctive and disjunctive normal form

- A sentence is in *conjunctive normal form* (CNF) if it is a conjunction of one or more disjunctions of one or more literals.
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Disjunctive normal form

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- Some sentences are in both CNF and DNF.

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