Homework 4

Finite Automata on Infinite Words and Trees Winter semester, 2009-2010

Exercise 1 (10 points)**.** A *nondeterministic Müller automaton* is defined just like a normal (deterministic) Müller automaton except we allow several initial states and use an arbitrary transition relation instead of a transition function. Show that every language L which is recognized by some nondeterministic Müller automaton is also recognized by a nondeterministic Büchi automaton, and hence by McNaughton's theorem, by a standard deterministic Müller automaton.

Exercise 2 (30 points)**.** An idea for producing a simple determinization procedure for Büchi automata is to keep track both of the sets of reachable states, as in the standard power set construction, and also the set of states which are reached via final states since the last "reset" point. A run is accepting is we "reset" infinitely often. This construction can be seen as a much simpler version of Safra's construction, in which we have only one extra level, and a single child. Formally, given a Büchi automaton $\mathcal{A} = (Q, \Sigma, T, I, F)$, we would construct the deterministic Büchi automaton $\mathcal{A}' = (Q', \Sigma, T', I', F')$ where:

- $Q' = \{(X, Y, m) \,|\, Y \subseteq X \subseteq Q,\ m \in \{\text{marked, unmarked}\}\}$
- $T'((X, Y, m), a)$ is computed as follows:
  1. First, we set m to unmarked.
  2. Second, we add $X \cap F$ to the set Y.
  3. Third, we apply the standard power set construction to both X and the new Y.
  4. Fourth, if X and Y are now equal, we set $Y = \emptyset$ and m= marked.
- $I' = \{(I, \emptyset, \text{unmarked})\}$
- $F' = \{(X, \emptyset, \text{marked})\}$

You will show that this construction is sound, e.g. no "bad" strings are accepted, but not complete, e.g. some "good" strings may not be accepted.

1. Find a Büchi automaton $\mathcal{A}$ such that $\mathcal{L}(\mathcal{A}) \not\subseteq \mathcal{L}(\mathcal{A}')$.
2. Prove that $\mathcal{L}(\mathcal{A}') \subseteq \mathcal{L}(\mathcal{A})$.

Exercise 3 (30 points)**.** We now try to remedy the problem we identified with the approach from Exercise 2. The idea is to iterate the construction: we can now have several levels, but still only one child per level. An accepting run is one in which there is some level k which is always non-empty starting from some point in the run, and such that we "reset" to this level infinitely often. Formally, for a Büchi automaton $\mathcal{A} = (Q, \Sigma, T, I, F)$, we would construct the Rabin automaton $\mathcal{A}' = (Q', \Sigma, T', I', \Omega)$ where:

- Q' is the set of tuples $(X_1, X_2, \ldots, X_n, m)$ where:
  - $n = |Q| + 1$
  - $X_n \subseteq X_{n-1} \subseteq \ldots \subseteq X_2 \subseteq X_1 \subseteq Q$
  - if $X_{k+1} \neq \emptyset$, then X_{k+1} must be a proper subset of X_k
  - $m : \{1, \ldots, n\} \to \{\text{marked, unmarked}\}$ specifies which levels are marked with the special symbol
- $T'((X_1, \ldots, X_n, m), a)$ is computed as follows:
  1. First, we set $m(x) = \text{unmarked}$ for all $1 \leq i \leq n$.
  2. Second, we add $X_i \cap F$ to the set X_{i+1} for every $i < n$.
  3. Third, we apply the standard power set construction to all of the (possibly modified) sets X_i.
  4. Fourth, if some sets X_i and X_{i+1} are now equal, and i is the smallest value with this property, then we set $X_j = \emptyset$ for all $j \geq i + 1$ and $m(i) = \text{marked}$.
- $I' = \{(I, \emptyset, \ldots, \emptyset, m)\}$ where $m(k) = \text{unmarked}$ for every level k
- $\Omega = \{(E_1, F_1), \ldots, (E_n, F_n)\}$ where:
  - $E_i = \{(X_1, \ldots, X_n, m) \mid X_i = \emptyset\}$
  - $F_i = \{(X_1, \ldots, X_i, \emptyset, \ldots, \emptyset, m)\}$ where $m(i) = \text{marked}$

You will show that this construction is indeed complete, unlike the previous one, but unfortunately not sound.

1. Show that for the Büchi automaton $\mathcal{A} = \{\{q_0, q_1, q_2\}, \{a, b\}, \{(q_0, a, q_0), (q_0, b, q_0), (q_0, b, q_1), (q_0, a, q_2), (q_1, a, q_2), (q_2, b, q_2)\}, \{q_0\}, \{q_1, q_2\}\}$, we have $\mathcal{L}(\mathcal{A}') \nsubseteq \mathcal{L}(\mathcal{A})$. *Hint*: consider the string $(ba)^\omega$.

2. Prove that we always have $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{A}')$.

Exercise 4 (30 points)**.** Apply in a step-by-step manner Safra's procedure to the Büchi automaton in Figure 1. Show that the resulting Rabin automaton recognizes the same language as the original Büchi automaton.

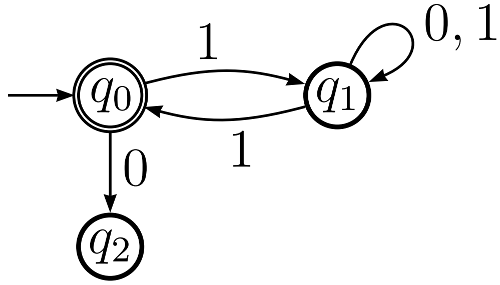

Figure 1: Automata for Exercise 4