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From syllogism to common sense . . .

Exercise Sheet 5: Propositional Logic

To be discussed on 15 December 2011

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1. Supply the missing details for the proofs of soundness (Slide 61) and completeness (Lemma on Slide 64):

a) Show that the six basic rules preserve the consequence relation. For example, for ($\wedge$2), show: if $X \models \alpha \wedge \beta$, then $X \models \alpha, \beta$.

(This is almost trivial for (IS), (MR), ($\wedge$1), ($\wedge$2), and not difficult for ($\neg$1), ($\neg$2).)

b) Show properties $\mathtt{C}^+$ and $\mathtt{C}^-$.

2. Show that every set can be totally ordered. That is, show the following proposition:

For every set M, there is an irreflexive, transitive, and connex relation $<$ on M.
($<$ is a connex relation on M if, for all $a, b \in M$, it holds that $a < b$ or $b < a$ or $a = b$.)

a) As a warm-up, show the proposition for finite M via induction on n.

b) For infinite M, proceed according to the following schedule, using the propositional compactness theorem.

- For every pair $(a, b) \in M \times M$, introduce a new propositional variable p_{ab} that represents the statement $a < b$.
- Construct an infinite set X of propositional formulas that expresses irreflexivity, transitivity and connexity of $<$.
  For example, reflexivity of $<$ would be expressed by $\{p_{aa} \mid a \in M\}$.
- Explain how the models of X correspond exactly to the possible orders $<$ on M.
- Show that every finite subset of X_0 has a model.
- Use the propositional compactness theorem to conclude that X is satisfiable.

3. Prove the following in the Hilbert calculus.

a) $\vdash \alpha \to \beta \to \alpha$

b) $\vdash \top$